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(BT 2 - 2.1)

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Material

Binney and Tremaine, 1987, Galactic Dynamics:
Section 2: 2.1+2.2, Appendix B.3

1. Gravitational forces and potentials

A galaxy contains $\sim 10^{11}$ stars (plus gas, dark matter etc.) and is kept from falling apart by gravity.

Before we study the motions of individual particles, we show how we can calculate the gravitation force and potential from a smoothed and extended density distribution.

The gravitational force $\vec{F}(\vec{x})$ on particle m_s at position $\vec{x}$ is due to the mass distribution $\rho(\vec{x}')$. According to Newton's inverse-square law, the force $\delta\vec{F}(\vec{x})$ on the particle m_s at location $\vec{x}$ due to a mass $\delta m(\vec{x}')$ at location $\vec{x}'$ is:

$$\delta\vec{F}(\vec{x}) = G m_s \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \delta m(\vec{x}')$$

$$= G m_s \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \rho(\vec{x}') \mathrm{d}^3\vec{x}'$$

The total force on particle m_s is now:

$$\vec{F}(\vec{x}) = m_s \vec{g}(\vec{x})$$

$$\vec{g}(\vec{x}) \equiv G \int \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \rho(\vec{x}') \mathrm{d}^3\vec{x}' \qquad (1)$$

where $\vec{g}(\vec{x})$ is the *gravitational field*, the force per unit mass.

Define the *gravitational potential*:

$$\Phi(\vec{x}) \equiv -G \int \frac{\rho(\vec{x}')\mathrm{d}^3\vec{x}'}{|\vec{x}' - \vec{x}|} \qquad (2)$$

Now use (exercise one)

$$\nabla_{\vec{x}} \left(\frac{1}{|\vec{x}' - \vec{x}|} \right) = \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \qquad (3)$$

and find:

$$\vec{g}(\vec{x}) = \nabla_{\vec{x}} \int G \frac{\rho(\vec{x}')\mathrm{d}^3\vec{x}'}{|\vec{x}' - \vec{x}|}$$

$$= -\nabla\Phi$$

So the gravitational vector field is the gradient of the potential.

The potential is a scalar field: “easy” to visualize and to use in calculations.

However, the triple integration is often expensive.

Therefore we often consider simple and symmetric geometries, such as:

- Sphere $\rho = \rho(r)$

- Classical ellipsoid
  $\rho = \rho(m^2)$ where $m^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$

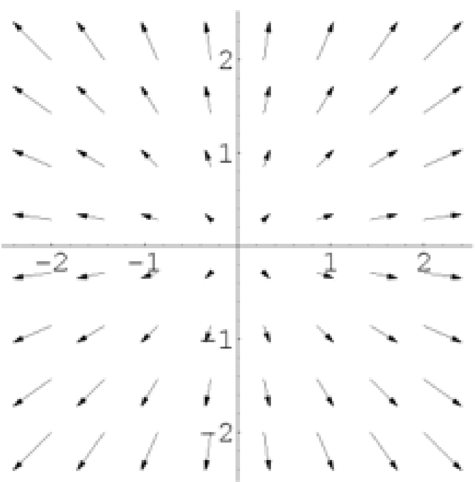
- Thin disk

Intermezzo: divergence and divergence theorem (BT: B.3)

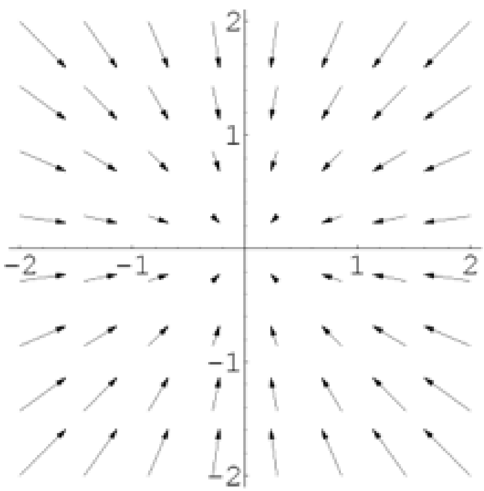
The divergence of a vector field $\vec{F}(\vec{x})$ is a scalar field. In Cartesian coordinates:

$$\vec{\nabla} \cdot \vec{F} \equiv \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

If $\vec{F}$ is the velocity field of a fluid flow, the value of $\nabla \cdot \vec{F}$ at a point (x, y, z) is the rate at which fluid is being piped in or drained away at (x, y, z). If $\nabla \cdot \vec{F} = 0$, then all that comes into the infinitesimal box, goes out: nothing is either added or taken away from the flow through the box



Divergence > 0

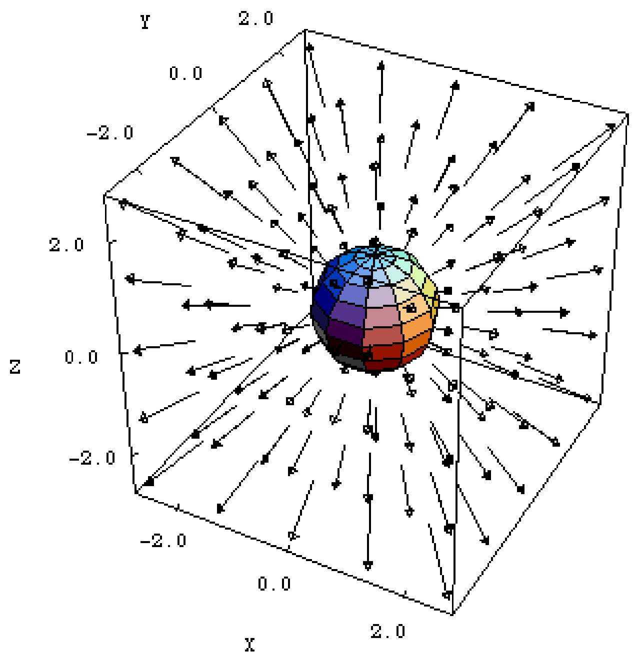


Divergence < 0

The divergence theorem

$$\int_V \vec{\nabla} \cdot \vec{F} = \oint_S d^2\vec{S} \cdot \vec{F}$$

The integrated rate at which fluid is being piped or drained away within a given volume V is equal to the total flux through the surface S enclosing the volume. This total flux is the surface integral of the flux normal to the each surface element.



(see also:

http://www.math.umn.edu/~ nykamp/m2374/readings/divcurl/)

Poisson's equation

If we take the divergence of the gravitational field (equation (1)):

$$\vec{\nabla} \cdot \vec{g}(\vec{x}) = G \int \vec{\nabla}_{\vec{x}} \cdot \left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) \rho(\vec{x}') \mathrm{d}^3\vec{x}'$$

or

$$\vec{\nabla}^2 \Phi(\vec{x}) = G \int \vec{\nabla}_{\vec{x}} \cdot \left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) \rho(\vec{x}') \mathrm{d}^3\vec{x}' \quad (4)$$

Using the divergence theorem, the right side can be evaluated (exercise 2) to find *Poisson's equation*:

$$\vec{\nabla}^2 \Phi(\vec{x}) = 4\pi G \rho(\vec{x})$$

This equation couples the local density and gravitational potential.

Mass

The mass in some volume can easily be derived from the force field: Integrate both sides of Poisson's equation over the volume enclosing a total mass M. For the right hand side we obtain:

$$4\pi G \int_V \rho \, \mathrm{d}\vec{x} = 4\pi G M$$

Using the divergence theorem, we obtain for the left hand side:

$$\int_V \nabla^2 \Phi \mathrm{d}\vec{x} = \int_S \vec{\nabla}\Phi \cdot \mathrm{d}^2\vec{S}$$

Combining both sides gives
<u>*Gauss's theorem*</u>:

$$4\pi G M = \int \vec{\nabla}\Phi \cdot \mathrm{d}^2\vec{S}$$

In words: the integral of the normal component of $\vec{\nabla}\Phi$ over any closed surface equals $4\pi G$ times the total mass contained within that surface

Potential energy

The *potential energy* can be shown to be:

$$W = 1/2 \int \rho(\vec{x})\Phi(\vec{x})\mathrm{d}^3\vec{x}$$

"Proof" Assume that we "build" up the galaxy slowly. We have a galaxy with a density $f\rho$, with $0 < f < 1$. If we add a small amount of mass δm from infinity to position $\vec{x}$, the work done is $\delta m\Phi(\vec{x})$. (Note that $\Phi(\vec{x}) = 0$ at infinity).

Ignoring the change in the potential due to the mass added, this costs an energy

$$\int \delta f\rho(\vec{x})\ f\Phi(\vec{x})\mathrm{d}^3\vec{x}$$

where $f\Phi$ is simply the potential of density $f\rho$, and the integral is the integral over the full galaxy volume.

We now have to add all the contributions together to derive the full energy needed to "build" the full galaxy

$$W = \int_0^1 \int \rho(\vec{x})\ f\Phi(\vec{x})\mathrm{d}^3\vec{x}\ df$$

$$= \int \rho(\vec{x})\Phi(\vec{x})\mathrm{d}^3\vec{x} \int_0^1 f df$$

$$= 1/2 \int \rho(\vec{x})\Phi(\vec{x})\mathrm{d}^3\vec{x}$$

For a more precise and elaborate derivation of the same result, see BT, p. 33+34

2. Spherical systems (BT 2.1, 2.2)

Newton's Theorems

First Theorem: A body inside an infinitesimally thin spherical shell of matter experiences no net gravitational force from that shell

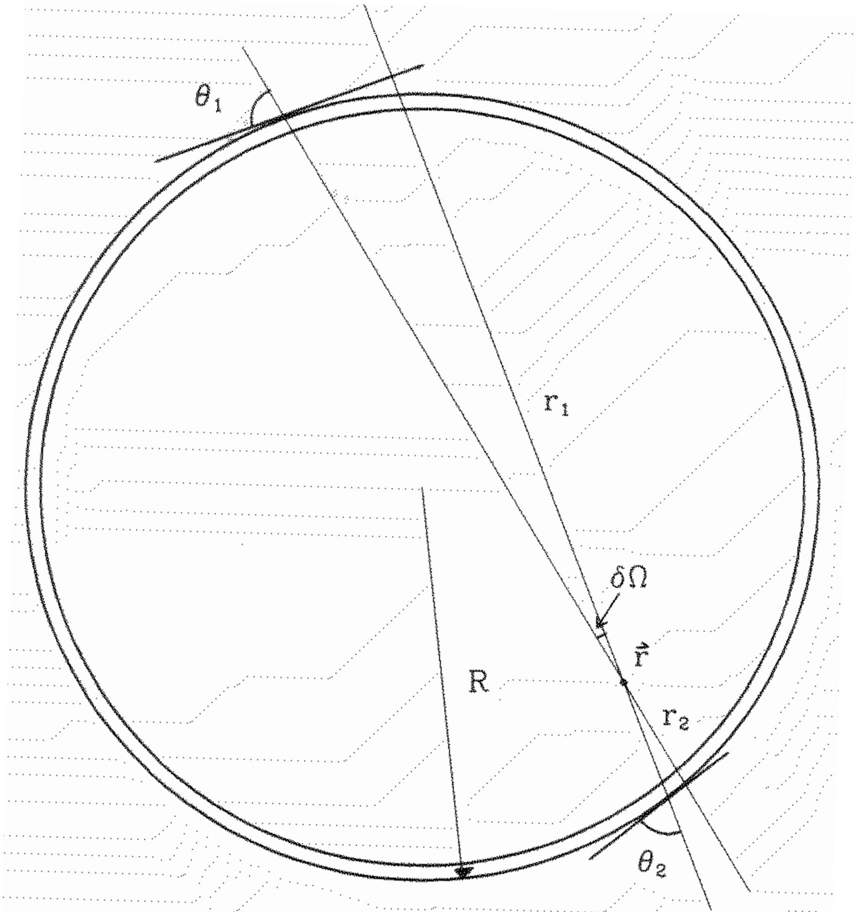


Figure 2-1. Proof of Newton's first theorem.

"Proof" Consider contributions to the force at point $\vec{r}$, due to the matter in the shell in a very narrow cone $d\Omega$. The intersection angles at 1 and 2, θ_1 and θ_2, are equal for infinitely small $d\Omega$. The relative masses in the cone δm_1 and δm_2 satisfy $\delta m_1/\delta m_2 = (r_1/r_2)^2$. The gravitational forces are proportional to $\delta m_1/r_1^2$ and $\delta m_2/r_2^2$, and therefore equal, but of opposite sign. Hence the matter in the cone does not contribute any net force at the location $\vec{r}$. If we sum over all cones, we find no net force !

Potential within a shell of mass M

Since there is no net force $\vec{g} = -\vec{\nabla}\Phi = 0$, the potential is a constant.

Using the gravitational potential as already defined:

$$\Phi(\vec{x}) \equiv -G \int \frac{\rho(\vec{x}')\mathrm{d}^3\vec{x}'}{|\vec{x}' - \vec{x}|} \tag{5}$$

and evaluating the potential at the center of the shell, where all points on the shell are at the same distance R, one finds:

$$\Phi = -\frac{GM}{R}$$

Second Theorem The gravitational force on a body outside a closed spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its center.

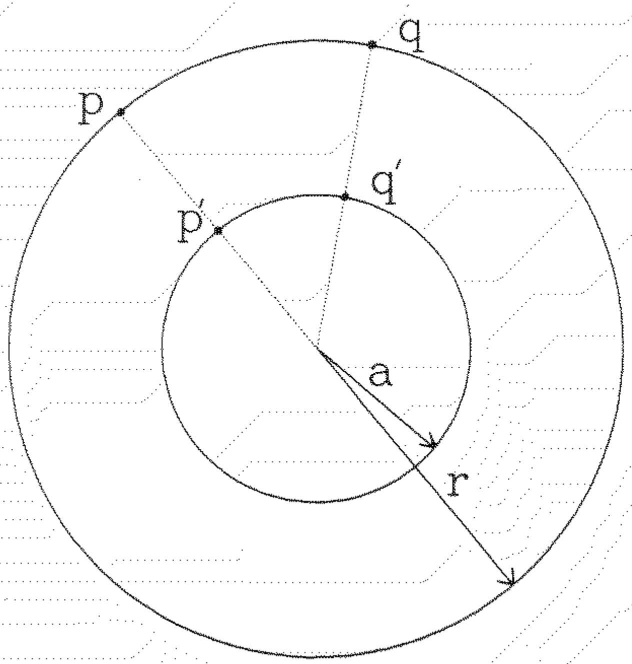


Figure 2-2. Proof of Newton's second theorem.

"Proof" Calculate the potential at point $\vec{p}$ at radius r from the center of an infinitesimally thin shell with mass M and radius a. Consider the contribution from the portion of the sphere with solid angle $\delta\Omega$ at q':

$$\delta\Phi_p = -\frac{GM}{|\vec{p} - \vec{q}'|}\frac{\delta\Omega}{4\pi}$$

Now take an infinitesimally thin shell with the same mass M, but radius r.

Calculate the potential at $\vec{p}'$. The contribution of the matter near $\vec{q}$ with the same solid angle $\delta\Omega$ is:

$$\delta\Phi_{p'} = -\frac{GM}{|\vec{p}' - \vec{q}|}\frac{\delta\Omega}{4\pi}$$

Since $|\vec{p} - \vec{q}'| = |\vec{p}' - \vec{q}|$, $\delta\Phi_p = \delta\Phi_{p'}$. Sum over all solid angles to obtain

$$\Phi_p = \Phi_{p'}$$

Since $\Phi_{p'}$ is the potential inside a sphere with mass M and radius r, it is equal to $\Phi_{p'} = -GM/r$, and this is equal to Φ_p. This is the same as the potential at r if all the mass is concentrated at the center.

Forces in a spherical system

We can now calculate forces exerted by a spherical system with density $\rho(r)$. From Newton's first and second theorem, it follows that the force on the unit mass at radius r is determined by mass interior to r:

$$\vec{F}(r) = -\frac{d\Phi}{dr}\vec{e}_r = -\frac{GM(r)}{r^2}\vec{e}_r,$$

where

$$M(r) = 4\pi \int_0^r \rho(r')r'^2 dr'.$$

Potential of a spherical system

To calculate the potential, divide system up into shells, and add contribution from each shell. Distinguish between shells with radius $r' < r$ and shells with $r' > r$:

$r' < r\ :\ \delta\Phi(r) = -G\delta M/r$
$r' > r\ :\ \delta\Phi(r) = -G\delta M/r'$

Hence total potential:

$$\Phi = -\frac{G}{r}\int_0^r \mathrm{d}M(r') - G\int_r^\infty \frac{\mathrm{d}M(r')}{r'}$$

$$= -4\pi G\left[\frac{1}{r}\int_0^r \rho(r')r'^2 dr' + \int_r^\infty \rho(r')r' dr'\right]. \tag{6}$$

Circular velocity and escape speed

The circular speed $v_c(r)$ is defined as the speed of a test particle with unit mass in a circular orbit around the center, with radius r. Equate gravitational force to centripetal acceleration v_c^2/r. We derive

$$v_c^2(r) = r\frac{d\Phi}{dr} = rF = \frac{GM(r)}{r}.$$

The circular speed measures the mass inside r. It is independent of the mass outside r.

The escape speed v_e is the speed needed to escape from the system, for a star at radius r. It is given by

$$v_e(r) = \sqrt{2|\Phi(r)|}$$

Only if a star has a speed greater than that, it can escape. It is dependent on the full mass distribution.

3. Simple potentials (BT 2.2.2)

Pointmass

$$\Phi(r) = -\frac{GM}{r}, \quad v_c(r) = \sqrt{\frac{GM}{r}}, \quad v_e(r) = \sqrt{\frac{2GM}{r}}$$

If the circular speed declines like $\frac{1}{\sqrt{r}}$ we call it "Keplerian". The first application was the solar system.

Homogeneous Sphere

Density ρ is constant within radius a, outside $\rho = 0$. For $r < a$:

$$M(r) = \tfrac{4}{3}\pi r^3 \rho, \quad v_c = r\sqrt{\tfrac{4}{3}\pi G\rho}$$

The circular velocity is proportional to the radius of the orbit. Hence the orbital period is:

$$T = \frac{2\pi r}{v_c} = \sqrt{\frac{3\pi}{G\rho}}$$

independent of radius !

Dynamical time

Equation of motion for a test mass released from rest at position r:

$$\frac{d^2r}{dt^2} = -\frac{GM(r)}{r^2} = -\tfrac{4}{3}\pi G\rho r$$

This is equation of motion of harmonic oscillator of angular frequency $2\pi/T$. The test mass will reach the center in a fixed time, independent of r. This time is given by

$$t_{dyn} = \frac{T}{4} = \sqrt{\frac{3\pi}{16G\rho}}$$

which we call the dynamical time. Even for systems with variable density we apply this formula (but then take the mean density).

Using eq. (6) we find for the Potential:

$\Phi(r) =$

$$r < a: \quad -2\pi G\rho(a^2 - 1/3r^2)$$

$$r > a: \quad -\frac{4\pi G\rho a^3}{3r}$$

Logarithmic Potential
(for Singular Isothermal Sphere)

assume $\rho = \rho_o/r^2$. This density distribution is called the "Singular Isothermal Sphere". (We will see later that this is because the structure of the resulting equations are similar to an isothermal self-gravitating sphere).

It is often used to approximate galaxies. Calculate the mass inside r:

$$M(r) = 4\pi \int_0^r \rho r'^2 dr' = 4\pi \int_0^r \rho_o dr' =$$

$$= [4\pi\rho_o r]_0^r = 4\pi\rho_o r$$

Hence the total mass is infinite. Now calculate the potential by comparing to potential at $r = 1$

$$\Phi(r) = \Phi(1) - \int_1^r F dr' = \Phi(1) - \int_1^r \frac{-GM(r')}{r'^2} dr' =$$

$$\Phi(1) + \int_1^r G 4\pi\rho_o \frac{1}{r'} dr' = \Phi(1) + 4\pi G\rho_o \left[\ln r'\right]_1^r =$$

$$\Phi(1) + 4\pi G\rho_o \ln r$$

This model is therefore called the "logarithmic potential".

We have a special relation for the circular velocity:

$$v_c^2 = rF = r4\pi G\rho_o/r = 4\pi G\rho_o$$

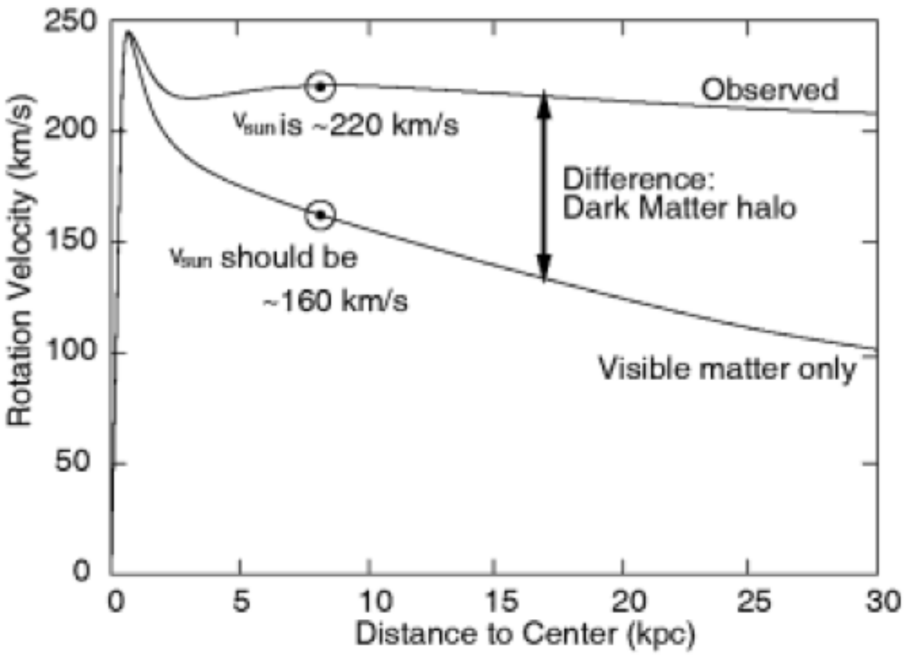
$$v_c = \sqrt{4\pi G\rho_o}$$

The circular velocity is constant as a function of radius ! We can also express the potential and density in terms of v_c, instead of ρ_o:

$$\Phi(r) = v_c^2 \ln r$$

$$\rho(r) = \frac{v_c^2}{4\pi G}\frac{1}{r^2}$$

With the circular velocity constant as a function of radius, the logarithmic potential gives a description of the circular velocities in the outer regions of spiral galaxies: In the first lecture we discussed the rotation curve of the Milky way:



Axisymmetric models

Generally much more complex, but always easy to get ρ from Φ:

Miyamoto & Nagai model

$$\Phi(R,z) = -\frac{GM}{\sqrt{R^2+(a+\sqrt{z^2+b^2})^2}}$$

Special cases:

$a = 0$ Plummer sphere: density constant at center, goes to zero at infinity

$b = 0$:Kuzmin disk: $\rho(R,z) = \Sigma(R)\delta(z)$ with $\Sigma(R) = \frac{Ma}{2\pi}\frac{1}{(R^2+b^2)^{3/2}}$

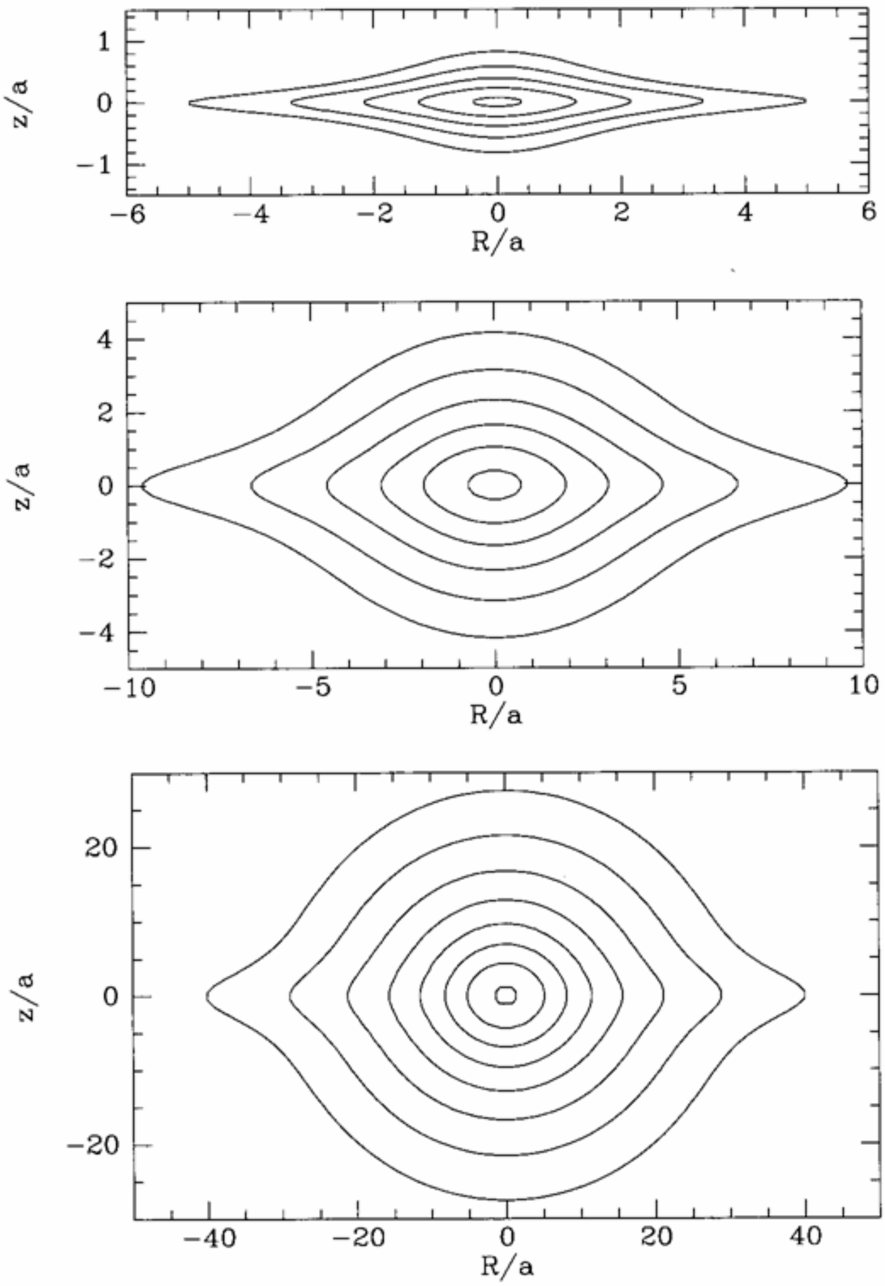


Figure 2.7 Contours of equal density in the (R, z) plane for the Miyamoto–Nagai density distribution (2.69b) when: $b/a = 0.2$ (top); $b/a = 1$ (middle); $b/a = 5$ (bottom). There are two contours per decade, and the highest contour levels are $0.3M/a^3$ (top), $0.03M/a^3$ (middle), and $0.001M/a^3$ (bottom).

When $b/a \sim 0.2$, light distribution similar to disk galaxies.

Axisymmetric logarithmic potential

Extension from spherical symmetric logarithmic potential that gives a flat rotation curve at large radius.

Potential:

$$\Phi(R,z) = \tfrac{1}{2}v_0^2 \ln(R_c^2 + R^2 + \frac{z^2}{q^2}) \qquad (7)$$

Circular velocity: $v_c = \frac{v_0 R}{\sqrt{R_c^2 + R^2}}$

Density distribution:

$$\rho(R,z) = \frac{v_0^2}{4\pi G q^2} \frac{(1+2q^2)R_c^2 + R^2 + (2-1/q^2)z^2}{(R_c^2 + R^2 + z^2/q^2)^2}$$

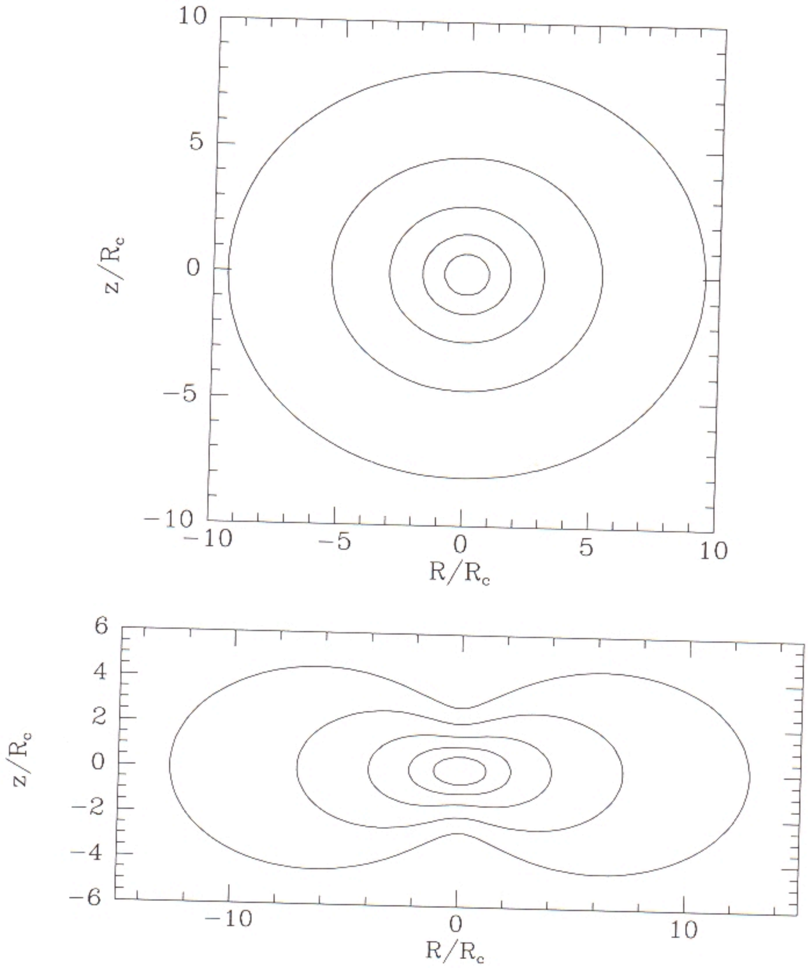


Figure 2.9 Contours of equal density in the (R, z) plane for $\rho_{\rm L}$ (eq. 2.71c) when $q_\Phi = 0.95$ (top), $q_\Phi = 0.7$ (bottom). There are two contours per decade and the highest contour level is $0.1 v_0^2/(G R_c^2)$. When $q_\Phi = 0.7$ the models are unphysical because the density is negative near the z axis for $|z| \gtrsim 7 R_c$.

Homework assignments

1. Proof the equality in eq. (3)

2. Derive Poisson's equation starting from eq. (1). (hint: follow instructions in BT, page 30-31)

3. Derive the potential from the density for the point mass, homogeneous sphere, and logarithmic potential, using equation (6).

4. The model given by $\rho = 1/(1 + r^2)^{2.5}$ is a Plummer model. Derive the potential of this model. What is the total mass ?

5. Give the derivation of the density related to the axisymmetric logarithmic potential given in equation 7.