Velocity Moments and the Jeans equations BT 4.2 p 195-198

We usually don't observe the motions of individual stars, but we can observe the average motions, and the spread in velocities (the velocity dispersion). Here we derive equations for the densities, average velocities, and dispersions.

We can derive these *without* taking into account the full distribution function.

Assume a population of objects with density ν and distribution function f in a potential Φ.

Notice that ν is not necessarily the same as ρ, which is the total matter density.

Integrate distribution function $f(\vec{x}, \vec{v})$ over velocities. This gives three velocity moments:

0. Spatial density of stars / 0th moment:

$$\nu(\vec{x}) = \int f(\vec{x}, \vec{v})\, \mathrm{d}^3\vec{v}$$

1. Mean stellar velocity / first moment:

$$\overline{v_i}\,(\vec{x}) \equiv \frac{1}{\nu} \int v_i f(\vec{x}, \vec{v})\, \mathrm{d}^3\vec{v}, \quad i = 1, 2, 3$$

2. Second moments:

$$\overline{v_i v_j}\,(\vec{x}) \equiv \frac{1}{\nu} \int v_i v_j f(\vec{x}, \vec{v})\, \mathrm{d}^3\vec{v}, \quad j = 1, 2, 3$$

Plus:

Velocity dispersion tensor:

$$\sigma_{ij}^2 \equiv \overline{(v_i - \overline{v_i}\,)(v_j - \overline{v_j}\,)} = \overline{v_i v_j} - \overline{v_i}\;\overline{v_j}$$

Much like fluid dynamics, the three moments and the velocity dispersion tensor are constrained by 3 equations: the Jeans equations. These three equations are:

Jeans equation 1 (the Continuity equation):

$$\boxed{\frac{\partial \nu}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu\, \overline{v_i} = 0}$$

Jeans equation 2 (the Force equation):

$$\boxed{\frac{\partial(\nu\, \overline{v_j}\,)}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\nu\, \overline{v_i v_j}\,) + \nu \frac{\partial \Phi}{\partial x_j} = 0}$$

Jeans equation 3 (a common rewrite of Jeans-2):

$$\boxed{\nu \frac{\partial\, \overline{v_j}}{\partial t} + \sum_{i=1}^{3} \nu\, \overline{v_i}\, \frac{\partial\, \overline{v_j}}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \sigma_{ij}^2}$$

Jeans equation 1
(the Continuity equation)

This equation is obtained by taking the 0th moment of the Collisionless Boltzmann Equation (CBE) in v.
Recall the CBE:

$$\frac{\partial f}{\partial t}+\sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i}-\frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i}=0 \qquad \text{(CBE)}.$$

The 0th moment in v of the CBE:

$$\int \text{CBE } \mathrm{d}\vec{v}$$

or:

$$\int \frac{\partial f}{\partial t}\mathrm{d}\vec{v}+\sum_{i=1}^{3}\int v_i \frac{\partial f}{\partial x_i}\mathrm{d}\vec{v}-\sum_{i=1}^{3}\frac{\partial \Phi}{\partial x_i}\int \frac{\partial f}{\partial v_i}\mathrm{d}\vec{v}=0.$$

Using the divergence theorem, we can rewrite the last term as a surface integrale:

$$\int \frac{\partial f}{\partial t}\mathrm{d}\vec{v}+\sum_{i=1}^{3}\int v_i \frac{\partial f}{\partial x_i}\mathrm{d}\vec{v}-\sum_{i=1}^{3}\frac{\partial \Phi}{\partial x_i}\int [f]_{-\infty}^{\infty} d\vec{S}=0.$$

Since $f(\vec{x},\vec{v})=0$ for "infinite" velocities, the last term is zero.

The second term can be simplified by moving the derivative outside the integral:

$$\boxed{\frac{\partial \nu}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \, \overline{v_i} = 0} \qquad \text{Jeans} - 1$$

or:

$$\frac{\partial \nu}{\partial t} + \nabla \cdot (\nu \, \overline{\vec{v}}\,) = 0$$

This is a *continuity equation* for the mean streaming motion $\overline{\vec{v}}$ of the stars in configuration space

Note the similarity with the the continuity equations for fluid mechanics:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

Jeans equation 2 + 3
(the Force equation)

The first moment in v of the CBE:

$$\int \text{ CBE } v\,\mathrm{d}\vec{v}$$

or:

$$\frac{\partial}{\partial t}\int f v_j d\vec{v} + \sum_{i=1}^{3}\int v_i v_j \frac{\partial f}{\partial x_i} d\vec{v} - \sum_{i=1}^{3}\frac{\partial \Phi}{\partial x_i}\int v_j \frac{\partial f}{\partial v_i} d\vec{v} = 0$$

The last term can be simplified. Do the partial integration over dv_i and use the fact that f vanishes for large v:

$$\int v_j \frac{\partial f}{\partial v_i} d\vec{v} =$$

$$\int\int v_j (f(v_i = \infty) - f(v_i = -\infty)) d^2 v_{\neq i} - \int \frac{\partial v_j}{\partial v_i} f d\vec{v} =$$

$$0 - \int \delta_{ij} f d\vec{v} = -\delta_{ij}\nu$$

where $\delta_{ij} = 1$ for $i = j$ and 0 for $i \neq j$.

Hence:

$$\frac{\partial}{\partial t}\int f v_j d\vec{v} + \sum_{i=1}^{3}\int v_i v_j \frac{\partial f}{\partial x_i} d\vec{v} + \sum_{i=1}^{3}\frac{\partial \Phi}{\partial x_i}\delta_{ij}\nu = 0$$

or

$$\boxed{\frac{\partial(\nu\,\overline{v_j}\,)}{\partial t} + \sum_{i=1}^{3}\frac{\partial}{\partial x_i}(\nu\,\overline{v_i v_j}\,) + \nu\frac{\partial \Phi}{\partial x_j} = 0}$$

(Jeans − 2)

Multiply continuity equation (Jeans-1) by $\overline{v_j}$, and subtract from the last equation (Jeans-2):

$$\nu\frac{\partial\,\overline{v_j}}{\partial t} - \sum_{i=1}^{3}\overline{v_j}\,\frac{\partial \nu\,\overline{v_i}}{\partial x_i} + \sum_{i=1}^{3}\frac{\partial \nu\,\overline{v_i v_j}}{\partial x_i} + \nu\frac{\partial \Phi}{\partial x_j} = 0$$

Using

$$\frac{\partial \nu \sigma_{ij}^2}{\partial x_i} = \frac{\partial}{\partial x_i}\nu(\,\overline{v_i v_j} - \overline{v_i}\;\overline{v_j}\,) =$$

$$\frac{\partial(\nu\,\overline{v_i v_j}\,)}{\partial x_i} - \overline{v_j}\,\frac{\partial(\nu\,\overline{v_i}\,)}{\partial x_i} - \nu\,\overline{v_i}\,\frac{\partial\,\overline{v_j}}{\partial x_i}$$

We obtain the more frequently used variant of Jeans-2, Jeans equations 3:

$$\boxed{\nu\frac{\partial\,\overline{v_j}}{\partial t} + \textstyle\sum_{i=1}^{3}\nu\,\overline{v_i}\,\frac{\partial\,\overline{v_j}}{\partial x_i} = -\nu\frac{\partial\Phi}{\partial x_j} - \textstyle\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\nu\sigma_{ij}^2} \qquad (\mathrm{Jeans}-3)$$

Hence we obtain the analogue of the Euler equation:

$$\frac{\rho\partial\vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\rho\vec{\nabla}\Phi - \vec{\nabla}p = 0$$

- Almost the same as Euler equations for fluid, but instead of $\vec{\nabla}p$ we have the summation over the stress tensor $\partial\nu\sigma_{ij}^2\partial x_i$. For a stationary model the left terms disappear completely, and the velocity dispersion tensor counter-acts gravity, just like for a star made of gas. Note that the pressure in a galaxy is anisotropic ! But notice: no equation of state for our "gas" in a galaxy, in contrast to stars !

- Generally 3 equations for 6 unknowns: many solutions!

- Caveat: solutions are not guaranteed to be physical, since no check that $f \geq 0$

<u>Velocity ellipsoid</u>

The tensor σ_{ij}^2 is symmetric $\Rightarrow$ it is diagonal in locally orthogonal coordinates $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$:

$$\begin{pmatrix} \tilde{\sigma}_{11} & 0 & 0 \\ 0 & \tilde{\sigma}_{22} & 0 \\ 0 & 0 & \tilde{\sigma}_{33} \end{pmatrix}$$

The ellipsoid with semi-axes $\tilde{\sigma}_{11}$, $\tilde{\sigma}_{22}$, and $\tilde{\sigma}_{33}$, oriented along the local axes $\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{x}_3$, is called the velocity ellipsoid. It is sometimes used to describe the local velocity distribution

Jeans equations for spherical models
BT 4.2d, page 203-209

Assume a coordinate system (r, θ, ϕ). We assume the system is invariant under rotations about the center. Hence we have

$$\overline{v_r} = \overline{v_\theta} = \overline{v_\phi} = 0$$

$$\overline{v_r v_\theta} = \overline{v_r v_\phi} = \overline{v_\theta v_\phi} = 0$$

$$\overline{v_\theta^2} = \overline{v_\phi^2}$$

so that velocity ellipsoid is everywhere aligned with (r, θ, ϕ) coordinates.

Now the Jeans equation(-2/3) in the stationary case reduces to:

$$\frac{d(\nu \, \overline{v_r^2}\,)}{dr} + \frac{\nu}{r}\left[2\,\overline{v_r^2} - 2\,\overline{v_\theta^2}\right] = -\nu \frac{d\Phi}{dr}$$

Define the anisotropy function:

$$\beta(r) = 1 - \overline{v_\theta^2}\,/\,\overline{v_r^2}\,.$$

Clearly $\beta \leq 1$. We obtain one non-trivial Jeans equation:

$$\frac{1}{\nu}\frac{d}{dr}\nu\,\overline{v_r^2} + 2\frac{\beta}{r}\overline{v_r^2} = -\frac{d\Phi}{dr}$$

Given $\beta(r)$, $\overline{v_r^2}$ and $\nu(r)$ we can derive the potential and mass distribution. Full knowledge of the full distribution function is not necessary to interpret observable parameters such as the velocity dispersion.

Total enclosed mass and rotation curve

For a circular orbit with velocity $v_c(r)$ we have:

$$\frac{d\Phi}{dr} = \frac{GM(< r)}{r^2} = \frac{v_c^2}{r}$$

So the Jeans equation can be written as

$$v_c^2 = \frac{GM(< r)}{r} = -\overline{v_r^2}\left(\frac{d\ln\nu}{d\ln r} + \frac{d\ln\overline{v_r^2}}{d\ln r} + 2\beta\right)$$

Measure: $\nu(r)$, $\overline{v_r^2}$ and β $\Rightarrow$ determine enclosed mass

Isotropic models

Now simplify the equation, and assume $\beta = 0$, or 'isotropic velocity dispersions' - i.e., they are the same in all directions. Below we show how the observed light distribution and observed velocity dispersion can be used to derive the internal mass distribution. In order to do that, we have to calculate the projection effects.

For $\beta = 0$, we have from our previous equations:

$$M(< r) = -\frac{r\,\overline{v_r^2}}{G}\left(\frac{d \ln \nu}{d \ln r} + \frac{d \ln \overline{v_r^2}}{\mathrm{d} \ln r}\right) \qquad \mathrm{BT87:4-56}$$

Now use the geometry of the projection along the line-of-sight to calculate the projected surface brightness $I(R)$, and the projected velocity dispersion σ_p:

2.1 Spherical Systems 39

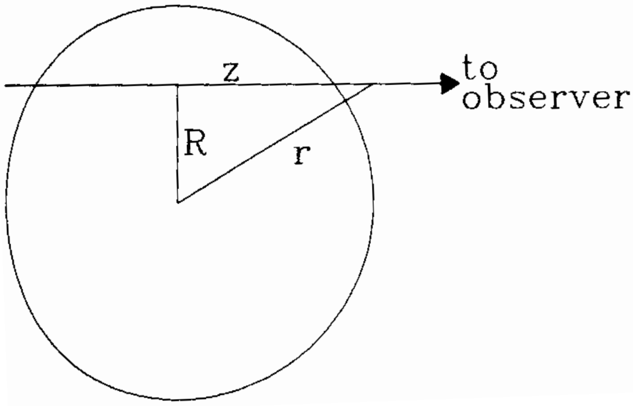


Figure 2-3. Projection of a spherical body along the line of sight.

If I is observed surface brightness, ν is luminosity density, then

$$I(R) = 2\int_0^\infty \nu(\sqrt{z^2+R^2})dz = 2\int_R^\infty \frac{\nu(r)rdr}{\sqrt{r^2-R^2}}$$

Similarly, we derive for the observed dispersion:

$$I(R)\sigma_{\rm p}^2(R) = 2\int_R^\infty \frac{\nu(r)\,\overline{v_r^2}\,(r)rdr}{\sqrt{r^2-R^2}}$$

These are Abel integral equations and can be easily inverted:

$$\nu(r) = -\frac{1}{\pi}\int_r^\infty \frac{dI}{dR}\frac{dR}{\sqrt{R^2-r^2}}$$

$$\nu(r)\,\overline{v_r^2} = -\frac{1}{\pi}\int_r^\infty \frac{d(I\sigma_{\mathrm{p}}^2)}{dR}\frac{dR}{\sqrt{R^2 - r^2}}$$

Hence, the observables σ_p and I can be used directly to infer the intrinsic σ and ν. With the Jeans equation above we can immediately calculate the mass distribution from these two.

An example is M87 (see also BT). Three main questions: a), b), and c).

DYNAMICAL EVIDENCE FOR A CENTRAL MASS CONCENTRATION IN THE GALAXY M87

W. L. W. Sargent* and Peter J Young

Hale Observatories, California Institute of Technology, Carnegie Institution of Washington

A. Boksenberg*† and Keith Shortridge*

Department of Physics and Astronomy, University College London

C. R. Lynds

Kitt Peak National Observatory

and

F. D. A. Hartwick*

Department of Astronomy, University of Victoria

Received 1977 September 26; accepted 1977 November 1

ABSTRACT

The elliptical galaxies NGC 3379 (E1) and M87 (E0) have been observed spectroscopically with the University College London Image Photon Counting System. Analysis of the redshifts and velocity dispersions as a function of radius by a Fourier method has yielded the following results: (*a*) NGC 3379 exhibits slight rotation ($v_\theta = 15\ \mathrm{km\ s^{-1}}$ at $r = 14''$) along the N-S direction (22° from the minor axis). The velocity dispersion is 195 km s^{-1} for $r < 14''$; this shows a small decrease with increasing radius. The data, including the photometric profile, is adequately fitted by a King model with $\log r_T/r_c = 2.20$ and constant $M/\mathscr{L} = 6$ for $0'' < r < 14''$ (with $r_c = 2''.8$). (*b*) M87 shows no rotation ($v_\theta < 10\ \mathrm{km\ s^{-1}}$) for $r < 72''$ in the E-W direction. The velocity dispersion at the edge of the core ($r_c = 9''.6$) is 278 km s^{-1}, but decreases to 230 km s^{-1} when $r = 72''$. Inside the core a sharp increase is observed, up to 350 km s^{-1} at $r = 1''.5$. The photometric profile and velocity dispersion data outside the core are explained by a King model with $M/\mathscr{L} = 6.5$ and $\log r_T/r_c = 2.10$. The data inside the core radius can be explained by a central mass concentration $M = 5 \times 10^9\ M_\odot$ contained within $r = 1''.5$ (=110 pc). For $r < 1''.5$ we find $M/\mathscr{L} = 60$, a factor of 10 higher than that in the outer regions. The observed width (1500 km s^{-1} full width at zero intensity) of the [O II] λ3727 doublet also suggests a central mass of $\sim 5 \times 10^9\ M_\odot$.

We conclude that the observations of M87 are entirely consistent with the presence of a central black hole of $\sim 5 \times 10^9\ M_\odot$.

Subject headings: black holes — galaxies: individual — galaxies: internal motions — galaxies: nuclei — galaxies: redshifts

I. INTRODUCTION

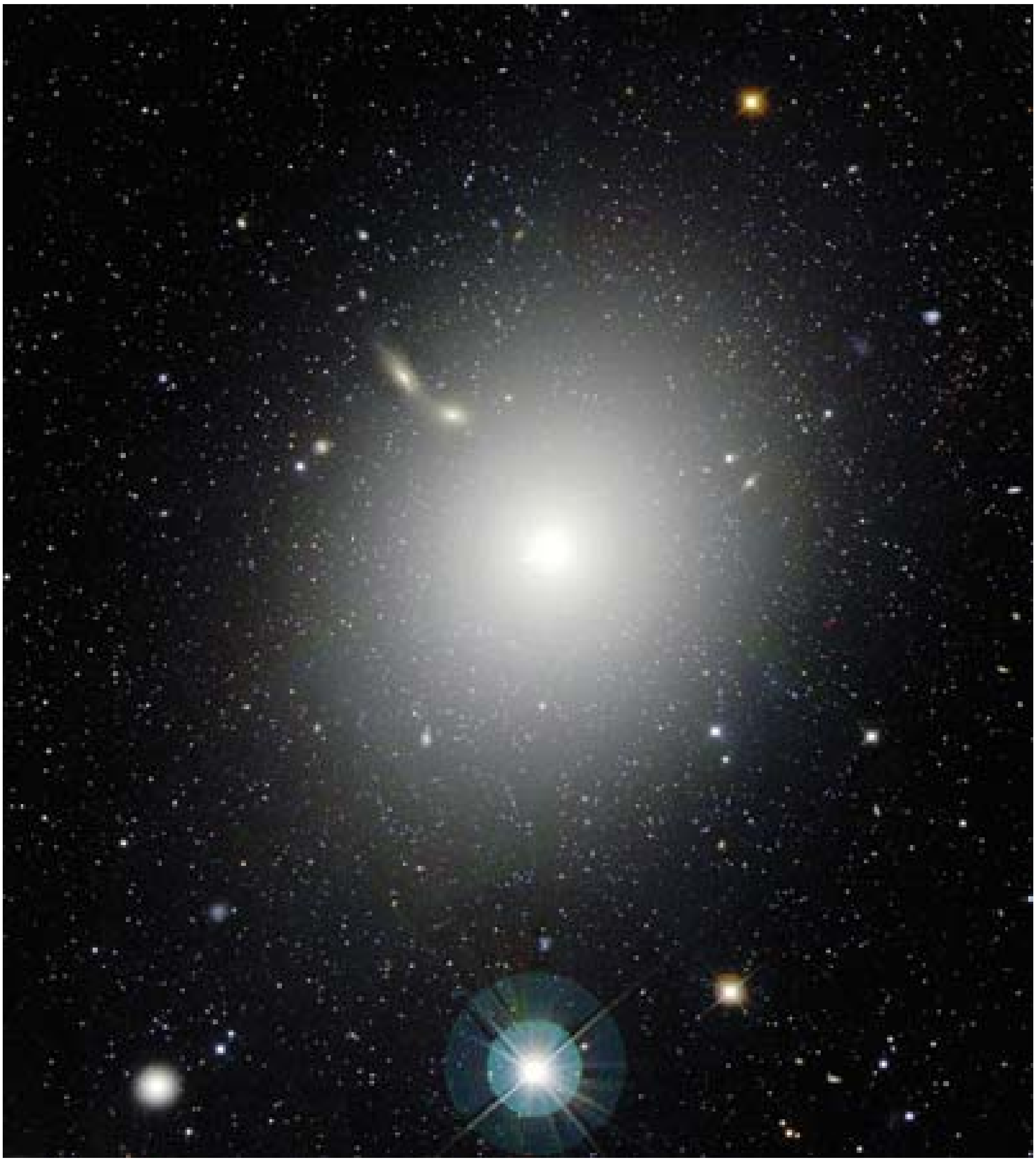
The introduction of linear, two-dimensional detectors has at last made it possible to measure accurate velocity dispersions and redshifts in the outer parts of elliptical galaxies, even in regions where the surface brightness is appreciably below that of the sky. As a result, several topics of current theoretical interest have become amenable to observational study. Perhaps the most important among these are (*a*) the search for evidence for supermassive black holes in the centers of elliptical galaxies, particularly in giant ellipticals in which the presence of such objects has been invoked in order to explain explosive or other nonthermal phenomena, (*b*) studies of the rotation curves and radial variation of velocity dispersion in flattened ellipticals in order to check the hypothesis that such systems are supported by rotation, and (*c*) studies of the radial variation of mass and mass-to-light ratio to investigate whether or not galaxies have massive halos.

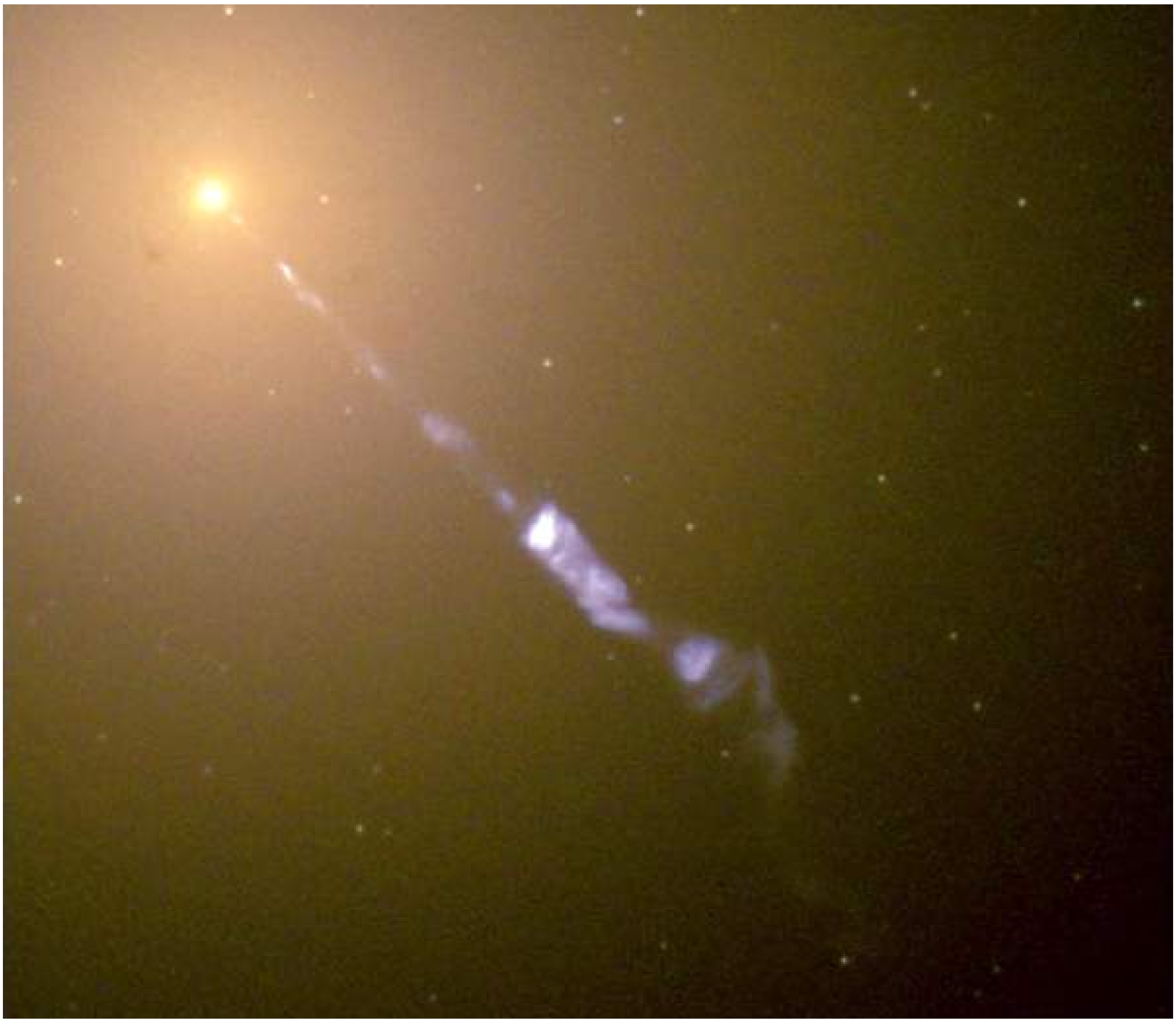
In a previous paper (Young *et al.* 1978*a*) the first results of our observational program on elliptical galaxies with the two-dimensional version of the University College London image photon counting system (IPCS) were discussed. The rotation curve and velocity dispersion of the E5 galaxy NGC 4473 were measured out to a radius of 45″ (3.3 kpc) from the center. Here we describe similar measurements for the E1 galaxy NGC 3379 and the E0 galaxy M87 (NGC 4486).

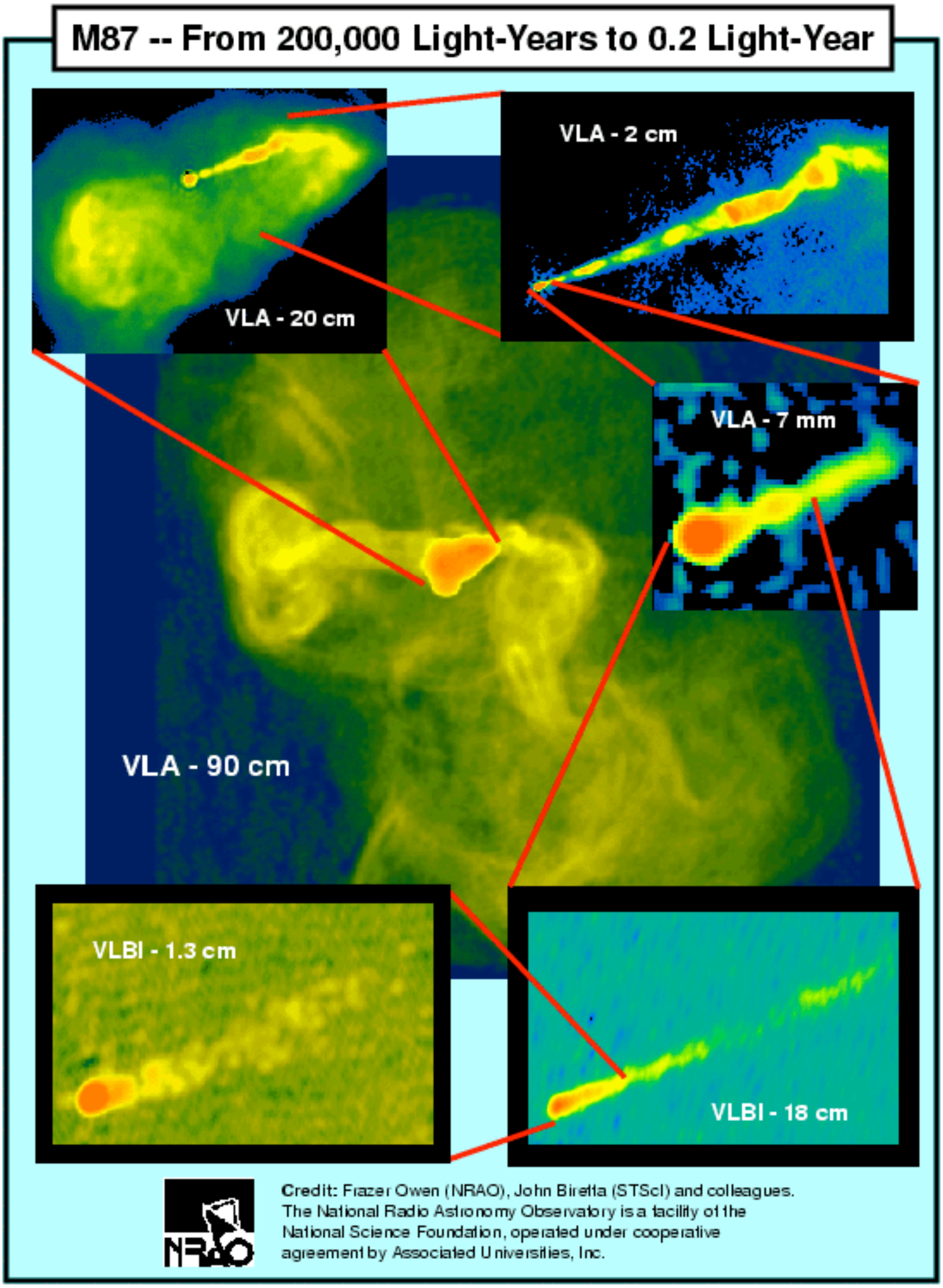
As we shall show in later sections, our results for

* Visiting Astronomers, Kitt Peak National Observatory, which is operated by Associated Universities for Research in Astronomy, Inc., under contract from the National Science Foundation.

† Guest Investigator, Hale Observatories.






M87 -- From 200,000 Light-Years to 0.2 Light-Year
VLA - 2 cm
VLA - 20 cm
VLA - 7 mm
VLA - 90 cm
VLBI - 1.3 cm
VLBI - 18 cm
Credit: Frazer Owen (NRAO), John Biretta (STScI) and colleagues.
The National Radio Astronomy Observatory is a facility of the
National Science Foundation, operated under cooperative
agreement by Associated Universities, Inc.

Sargent et al. 1978: long-slit spectroscopy of M87 yields a galaxy spectrum for positions along the slit.

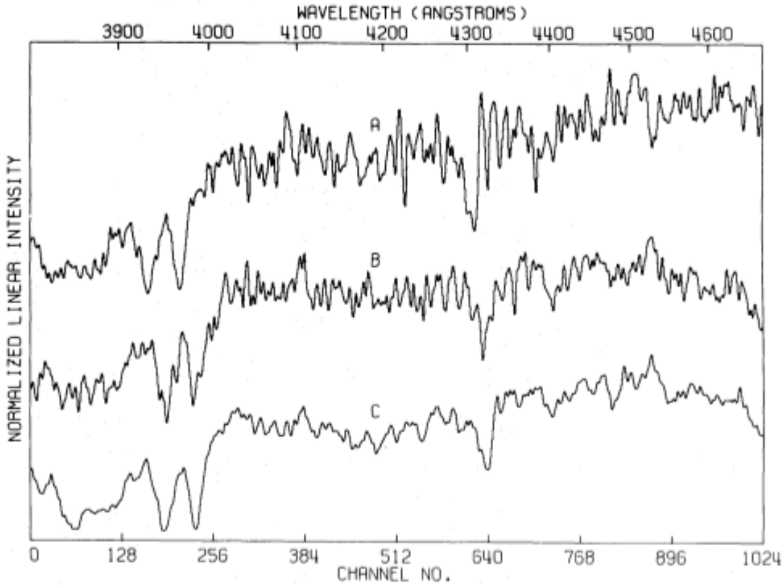


FIG. 1.—IPCS spectra of (*A*) comparison star HR 5709 (K0 III), (*B*) M87 at a distance of 70″ W of the center, (*C*) M87 4″ E of center. Note the broad, shallow lines of spectrum *C* as compared with *B*. The Fourier velocity dispersions were $\sigma_v = 220$ km s^{-1} for *B* and 300 km s^{-1} for *C*.

The strong absorption is due to the sum of the Ca II H and K absorption lines of individual stars in the galaxy. Also is show measurement for a reference star. Comparing the galaxy and stellar spectrum yields the projected velocity dispersion at the observed location of the galaxy.

From this analysis, Sargent et al conclude that the observed velocity dispersion is rising towards the center:

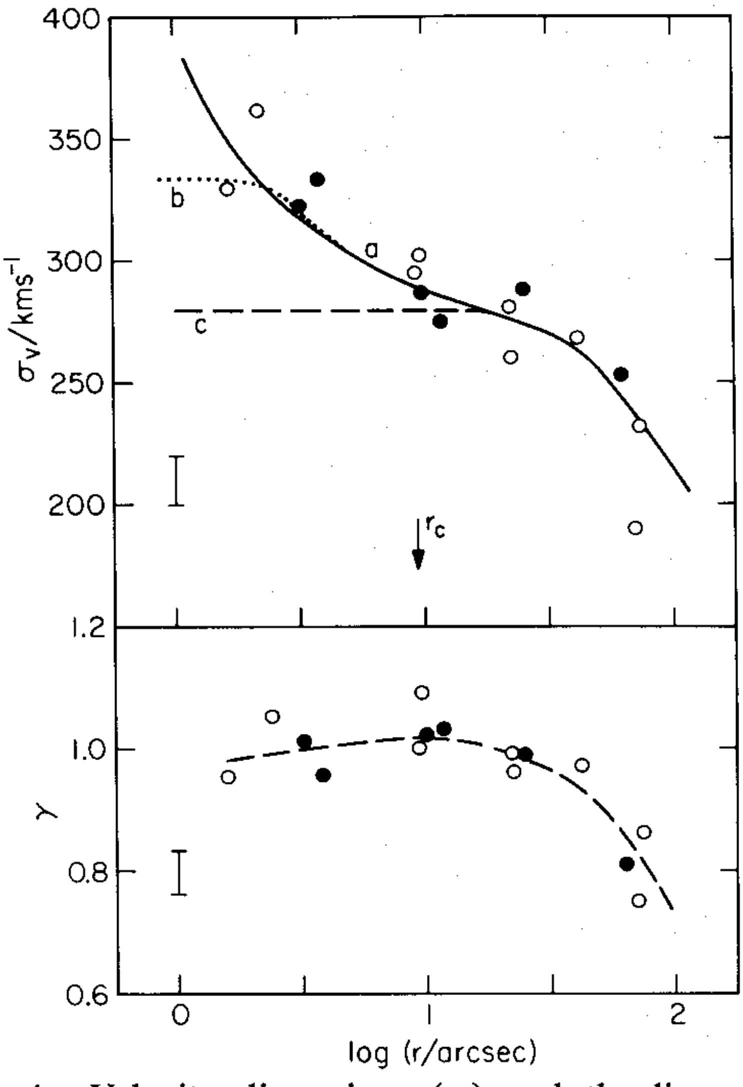


FIG. 4.—Velocity dispersions (σ_v) and the line strengths (γ) for M87. The open circles (○) are points W of the nucleus, and the filled circles (●) are E. The core radius, r_c = 9″.6, of the galaxy is marked. Error bars of length 2 σ are given. Curve (*a*) is the velocity dispersion predicted by the black hole model fitted to the photometric data, (*b*) is the same model convolved with the seeing disk and slit size for the spectroscopic observations, and (*c*) is the King model that would prevail if the black hole were absent.

Based on this velocity dispersion profile, the mass distribution and mass-to-light ratios were calculated. $\Upsilon(r)$ is ratio of mass with radius r to luminosity with radius r

$$\Upsilon(r) = M(< r)/[4\pi \int_0^r \nu r^2 dr]$$

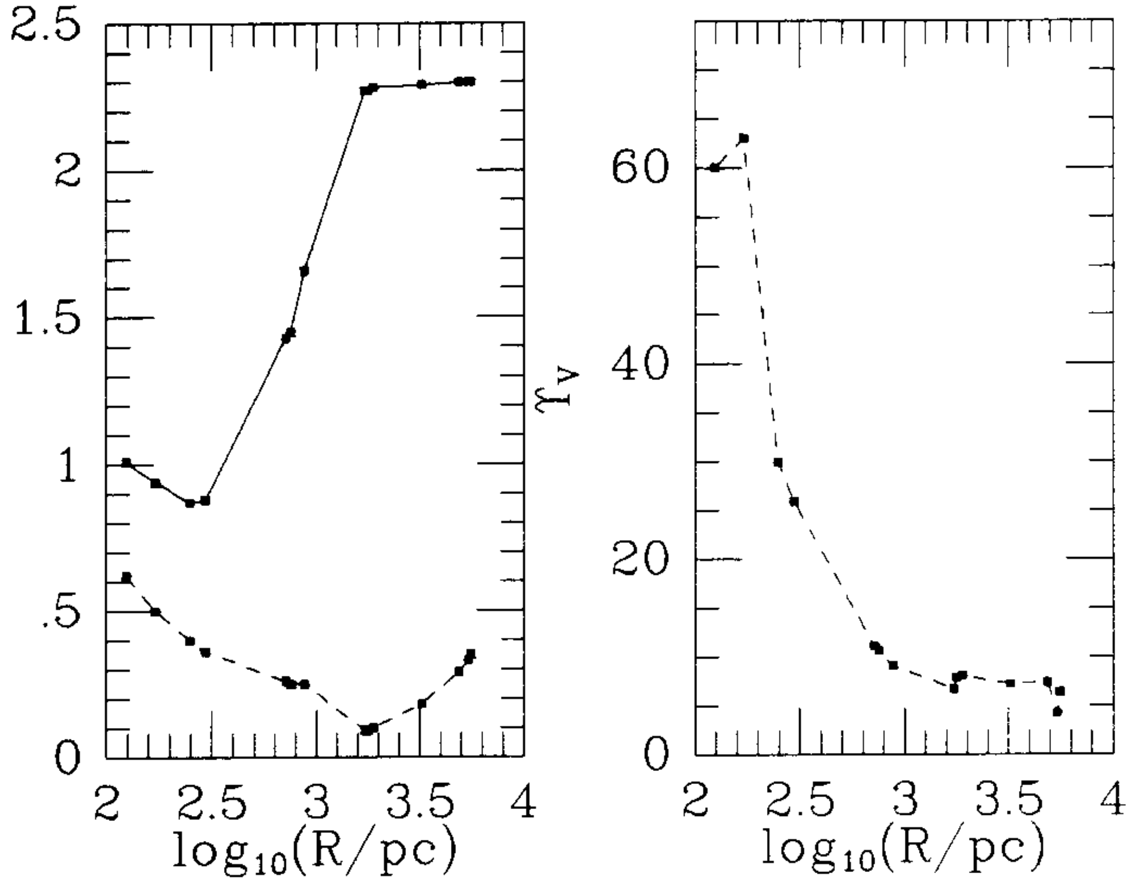


Figure 4-3. Left panel: plots of the logarithmic gradients appearing in equation (4-56) for the giant elliptical galaxy M87: full curve $|d\ln\nu/d\ln r|$; dashed curve $|d\ln\overline{v_r^2}/d\ln r|$. A distance to the galaxy of 16 Mpc has been assumed, and $\overline{v_r^2}$ has been obtained from the observed dispersion under the assumption $\beta = 0$. Right panel: the ratio $M(r)/L(r)$ derived from these data and equation (4-56). (After Sargent et al. 1978.)

The mass-to-light ratio increases strongly towards the center. The authors argued that this proved that a black hole had to be present (with a mass of order 5 10^9 M_{sun}).