Spherical accretion

- AGN generates energy by accretion, i.e., capture of ambient matter in gravitational potential of black hole
- Potential energy can be released as radiation, and (some of) this can escape and be observed

- Two cases
  - Capture of stars
  - Accretion of gas

Main results:
- Collisional accretion (gas) much more efficient than collisionless accretion (stars)
- Disk accretion more effective than spherical accretion (next lecture)

Literature
- Black Holes, White Dwarfs, and Neutron Stars
  Shapiro S.A., Teukolsky S.L., 1983, § 14 (ST)
- Galactic Dynamics
- Krolik § 6.1, 6.2
- Khembavi & Narlikar § 5.4

Overview spherical accretion

- Collisionless spherical accretion
  - Capture rate
  - Tidal disruption of stars
  - Relaxation
- Accretion of gas
  - Hydrodynamic spherical accretion
    - Stationary radial flow
    - Determination accretion rate
    - Behaviour of flow at large and small radii
    - Comments on relativistic regime
  - Eddington limit
    - Eddington accretion rate

Lecture notes at:  http://www.strw.leidenuniv.nl/~rottgeri
Collisionless spherical accretion (ST p. 403 - 411)
Consider distribution of stars (collisionless test particles) with phase-space distribution function (DF) \( f(\vec{r}, \vec{v}, t) \)

\[ f(\vec{r}, \vec{v}, t)d^3r d^3v \text{ the number of particles in the phase space volume element } d^3r d^3v \]
centered at \( \vec{r} \) and \( \vec{v} \) at time \( t \)

Particle density \( n(\vec{r}, t) = \int f(\vec{r}, \vec{v}, t)d^3v \) \hfill ST 14.2.2

Velocity dispersion ("temperature")

\[
< v^2(\vec{r}, t) > = \frac{1}{n(\vec{r}, t)} \int v^2 f(\vec{r}, \vec{v}, t)d^3v \hfill \text{(ST 14.2.3)}
\]

The evolution of the distribution is determined by the collisionless Boltzmann equation or Vlasov equation:

\[
\frac{D}{Dt} f(\vec{r}, \vec{v}, t) = \frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_r f + \vec{v} \cdot \nabla_{\vec{v}} f = 0
\]

A simple derivation

http://www.astro.virginia.edu/class/whittle/astb553/Topic08/Lecture_8.html

Look for a continuity equation, since:
- no stars created/destroyed: flow conserves stars
- stars do not jump across the phase space (ie no deflective encounters)

View the DF as a moving fluid of stars in 6-D space (r, v):

ie \( x, y, z, v_x, v_y, v_z \) stars move/flow through the region as their positions and velocities change

Consider a 1-D example using x and \( v_x \), and recall f is a number density
focus on a small element of phase space at x and \( v_x \) with size dx by dv_x, this element contains as the number of particles: \( dx \ dv \ f(x,v_x,t) \)

In interval dt, net flow in x is:

\[
v_x dt \ dv_x \ [f(x,v_x,t) - f(x+dx,v_x,t)] = - v_x \ dt \ dv_x \ \frac{\partial f}{\partial x} \ dx
\]
• the net flow due to the velocity gradient is
\[ dx \frac{dv_x}{dt} \left[ f(x,v_x,t) - f(x,v_x + dv_x,t) \right] = -dx \frac{dv_x}{dt} \frac{\partial f}{\partial v_x} \frac{dv_x}{dt} \]
the sum of these equals the net change to \( f \) in the region, i.e. at \( x, v_x \) of size \( dx \ dv_x \)
\[ dx \ dv_x \frac{\partial f}{\partial t} \frac{dt}{dx} = -dv_x \ dv_x \frac{\partial f}{\partial v_x} \ dv_x - dx \ dv_x \ \frac{dv_x}{\partial t} \ \frac{\partial f}{\partial v_x} \ dv_x \]
or, dividing by \( dx \ dv_x \ dt \), we get
\[ \frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + \frac{dv_x}{dt} \frac{\partial f}{\partial v_x} = 0 \]
adding the \( y \) and \( z \) dimensions, which are independent, we get Vlasov equation:
\[ \frac{\partial f}{\partial t} + v \cdot \nabla f - \nabla \phi \cdot \frac{\partial f}{\partial v} = 0 \]
The Vlasov describes how the DF changes in time. It is a direct consequence of:

1. conservation of stars
2. stars follow smooth orbits
3. flow of stars through \( r \) defines implicitly the location \( \mathbf{v} = \left( \frac{dr}{dt} \right) \)
4. flow of stars through \( \mathbf{v} \) is given explicitly by \( -\nabla \phi \)

Clearly, the phase space density \( f \) along the star's orbit is constant; i.e., the flow is "incompressible" in phase-space.

- If a region gets more dense, the velocity dispersion will increase.
- If a region expands, the velocity dispersion will decrease.

A simple example of an incompressible flow in phase space is provided by an idealized marathon race in which all runners travel at constant speed. At the start of the course, the density of runners is large, but they travel at a wide variety of speeds; at the finish, the density is low, but at any given time all runners passing the post have nearly the same speed (Binney and Tremaine, page 192).

Stationary spherical geometry, non-relativistic treatment, the DF and potential \( \Phi \)

\[ f = f(r, \mathbf{v}), \quad \Phi = GM/r \]

Orbits are planar, with two integrals of motion, energy

\[ E = \frac{1}{2} v^2 + \Phi(r) = \frac{1}{2} v_r^2 + \frac{L^2}{2r^2} - \frac{GM}{r} \]

and angular momentum per unit mass: \( J = rv_l \)

Given \( E \) and \( J \), trajectories are fully determined, hence \( f = f(E,J) \).

If the velocity distribution is everywhere isotropic, then \( f = f(E) \).

No particles with \( E < \Phi(r) \).

The particle density (ST4.2.2) reduces to

\[ n(r) = 4\pi \int v^2 f dv = 4\pi \int_{E=\Phi}^{\infty} \left[ 2(E-\Phi) \right]^{1/2} f(E) dE \quad \text{(ST 14.2.9)} \]

And the velocity dispersion ("temperature") (ST 14.2.3)

\[ \left\langle v^2(r) \right\rangle = \frac{4\pi}{n(r)} \int_{E=\Phi}^{\infty} \left[ 2(E-\Phi) \right]^{3/2} f(E) dE \]
Accretion of non interaction particles onto the BH. Particles with an angular momentum less than $J_{\text{min}}(E)$ will be captured:

$$J_{\text{min}} = \begin{cases} 
\frac{4GM}{c} & \text{for BH (cf previous lecture) (ST 14.2.12)} \\
\left[2\left(E + \frac{GM}{R}\right)\right]^{1/2}R & \text{Newtonian (ST 14.2.11)}
\end{cases}$$

The ‘loss cone’ $J < J_{\text{min}}(E)$ defines a region in DF within which particles are captured.

It is now convenient to write:

$$d^3v = 2\pi v_t dv_t dv_r = \frac{4\pi J dJ dE}{r^2|v_r|}$$

and:

$$d^3r = 4\pi r^2 dr$$

**Calculation of the capture rate**

Let $N_{\text{in}}(r, E, L)$ be the number of stars with energy $E$, angular momentum $L$ that flow inward through radius $r$. Then

$$N_{\text{in}}(r, E, J) dE dJ = \frac{1}{2} f(E, J) d^3v d^2r$$

$$= 8\pi^2 \frac{J}{|v_r|} f dE dJ$$

Total capture rate $\dot{N}_{\text{tot}}$ for particles

$$\dot{N}_{\text{tot}} = \int_{J_{\text{min}}(E)}^{\infty} \int_{\Phi(r)}^{\infty} dE dJ |v_r| N_{\text{in}}(r, E, J)_{r=R}$$

$$= 8\pi^2 \int_{\Phi(r)}^{\infty} dE dJ J f(E, L) \quad ST14.2.15$$

And the mass accretion rate:

$$\dot{M} = m \dot{N}_{\text{tot}}$$
Now consider an infinite collection of stars with an isotropic and mono-energetic distribution. Far from the black hole, the particle density is uniform and equal to \( n_\infty \). Then:

\[
f(E) = \frac{n_\infty}{4\pi \sqrt{2E_\infty}} \delta(E - E_\infty)
\]

(ST 14.2.16)

Where \( E_\infty = \frac{v_\infty^2}{2} \) (ST 14.2.17)

Note: the renormalization is given through eq. ST 14.2.9/page 8 using \( \Phi = 0 \) at \( r = \infty \)

\[
\dot{N}(E > 0) = 8\pi^2 \int_0^\infty dE f(E) \int_0^{J_{\min}} dJ J = 4\pi^2 \int_0^\infty dE f(E) J_{\min}^2(E)
\]

(ST 14.2.18)

Combing Eq. 14.2.15, 14.2.11, 14.12, 14.2.16-18; with \( \rho_\infty = mn_\infty \)

\[
\dot{M}(E > 0) = m\dot{N}(E > 0) = \begin{cases} 2\pi GM^2 \rho_\infty \frac{R}{v_\infty \dot{M}} \left(1 + \frac{v_\infty^2 R}{2MG}\right) & \text{Newtonian} \\ 16\pi (GM)^2 \frac{\rho_\infty}{v_\infty c^2} & \text{for BH} \end{cases}
\]

Resulting accretion rate on central black hole of AGN, using conditions appropriate to the ionized component of our Galaxy:

\[
\dot{M} \sim 5 \times 10^{-9} \left(\frac{\rho_\infty}{10^{-24} \text{g cm}^{-3}}\right) \left(\frac{M}{10^8 M_\odot}\right)^2 \left(\frac{v_\infty}{300 \text{km s}^{-1}}\right)^{-1} M_\odot \text{yr}^{-1}
\]

Eddington accretion rate

\[
\dot{M}_E = \frac{L_E}{\eta c^2} \sim 2.2 M_8 M_\odot \text{yr}^{-1}
\]

(page 9, AGN-4)

The stellar accretion rate not high enough to feed AGN activity.
Asymptotic behavior for distances $r$ larger than capture radius $R$, $r \gg R$
(ignoring removal of particles):

Evaluate (ST 14.2.9) using (ST 14.2.16):

Particle density $n_{E>0}(r) = n_\infty \left(1 + \frac{2GM}{v_\infty^2 r}\right)^{1/2}$

And (ST 14.2.10) the velocity dispersion:

$$\langle v^2(r) \rangle_{(E>0)} \equiv v^2(r) = v_\infty^2 \left(1 + \frac{2GM}{v_\infty^2 r}\right)$$

Derive $T$ by use of $\frac{3}{2}kT = \frac{1}{2}mv^2$

Let $r_a \equiv \frac{2GM}{v_\infty^2}$ be the accretion radius (kinetic energy is equal to its potential energy) then

- For $r \gg r_a$, density and temperature similar to asymptotic value
- For $r \ll r_a$, density and temperature much increased, due to gravitational focusing by mass $M$

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Relaxation

- Region of phase space with $0 \leq L \leq L_{\text{min}}(E)$ is called the loss-cone
- The stars in the galaxy surrounding the black hole are bound, and after loss-cone is depleted, it can only be replenished by two-body relaxation in the dense stellar core
- Fokker-Planck treatment of the Boltzmann equation for the evolution of $f(E, L, t)$ leads to an equilibrium solution with $n(r) \propto r^{-7/4}$ near the black hole
- For typical number density of stars, amount of mass that is lost to the hole is too low for sustained fuelling

**Tidal disruption of stars (K. P120 - 121)**

Tidal field of black hole of mass $M$ disrupts a star of mass $M_*$ and radius $R_*$ inside the *tidal radius* $r_t$:

$$r_t \sim R_* \left( \frac{M}{M_*} \right)^{1/3}$$

Simple derivation assumes disruption occurs when mean density $\rho_*$ of star becomes equal to mean density of BH inside volume with radius $r_t$, so that

$$3M/4\pi r_t^3 \sim 3M_*/4\pi R_*^3.$$ See BT § 7.3 for detailed derivation

In terms of the Schwarzschild radius

$$r_+ = 2GM/c^2$$

$$\frac{r_t}{r_+} \sim \left( \frac{10^8M_\odot}{M} \right)^{2/3} \left( \frac{\rho_\odot}{\rho_*} \right)^{1/3}$$

For solar type stars: tidal radius within event horizon for massive black holes

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**Implications:**

- Main sequence stars have $\rho_* \sim \rho_\odot$, so can be swallowed *whole* when $M > 10^8M_\odot$: no good for luminous quasars

- Giants have $\rho_* \sim 10^{-6}\rho_\odot$, so can be tidally disrupted even for the largest black hole masses relevant for AGN

-Fuelling of AGN requires *accretion of low-density material*. This material could come from
  - Mass loss from nearby stars
  - Radial inflow from further away
Main conclusions related to stellar accretion

• Needed (Eddington) accretion rate:

\[ \dot{M}_E = \frac{L_E}{\eta c^2} \sim 2.2 M_8 M_\odot \text{yr}^{-1} \]

• Stellar accretion rate is too low:

\[ \dot{M} \sim 5 \times 10^{-9} \left( \frac{\rho_\infty}{10^{-24} \text{g cm}^{-3}} \right) \left( \frac{M}{10^8 M_\odot} \right)^2 \left( \frac{v_\infty}{300 \text{km s}^{-1}} \right)^{-1} M_\odot \text{yr}^{-1} \]

• And since the tidal radius is within the Schwarzschild radius

\[ \frac{r_t}{r_+} \sim \left( \frac{10^8 M_\odot}{M} \right)^{2/3} \left( \frac{\rho_\odot}{\rho_*} \right)^{1/3} \]

swallowing of stars does produce much light
Accretion of gas

Accretion is a complex time-varying process:
- Hydrodynamical flows (1,2,3 dimensional)
  - Angular momentum losses
- Heating and cooling mechanisms
- Optical depth effects
- Magnetic fields
- Radiation pressure
- Boundary conditions at the outer and inner edge

Requires solving time-dependent relativistic magnetohydrodynamic equations with coupled radiative transfer

Here summarize hydrodynamic flow for two geometries:

**Spherical**
- Useful for obtaining insight
- Relevant for advection dominated accretion flows
- Applies to accretion of hot gas surrounding giant ellipticals

**Disk-like (next lecture)**
- Standard model
  - Natural geometry for infall of gas with angular momentum

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**Hydrodynamic spherical accretion (ST 412 - 422)**

Properties of gas are governed by the equations of fluid mechanics:

- **Continuity equation**, i.e. the equation enforcing mass conservation (no sinks or sources)
  \[
  \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
  \]
  where \( \rho \) is the density and \( \vec{v} \) the velocity

- **Euler (or force) equation**:
  \[
  \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P + \rho \vec{g}
  \]
  with \( P \) the pressure and \( \vec{g} \) the acceleration due to gravity (no forces due to radiation)

**Consider a stationary radial flow in spherical geometry** (Bondi 1952, MNRAS, 112, 95)

- Assume a **polytropic** equation of state:
  \[
  P = K \rho^\gamma
  \]
  with \( K \) a constant, and \( \gamma \) the adiabatic index \( (r, \theta, \phi) \)
Transform the fluid equations to spherical coordinates (use expression for $\nabla$ in spherical coordinates (BT eq. 1B-46)) and write $u=v_r$, and take $u>0$ inward:

(*) Continuity eq.: 
$$ \frac{1}{r^2} \frac{d}{dr} (r^2 \rho u) = 0 \quad \text{(ST 14.3.2)} $$

(**) Force eq.: 
$$ u \frac{du}{dr} = -\frac{1}{\rho} \frac{dP}{dr} - \frac{GM}{r^2} \quad \text{(ST 14.3.3)} $$

- Integrate equation (*): 
$$ 4\pi r^2 \rho u = \dot{M} = \text{constant (independent of } r) $$

This defines the mass accretion rate $\dot{M}$, valid at all $r$

-Solving equation (**)

$$ \frac{1}{\rho} \frac{dP}{dr} = \gamma K \rho^{\gamma - 2} \frac{d\rho}{dr} = \frac{\gamma}{\gamma - 1} \frac{1}{dr} \left( \frac{P}{\rho} \right) = \frac{1}{\gamma - 1} \frac{da^2}{dr} $$

with a defined sounds speed: 
$$ a^2 = \left( \frac{dP}{d\rho} \right) = \frac{\gamma P}{\rho} = \gamma K \rho^{\gamma - 1} $$

Then:
$$ \frac{1}{2} u^2 + \frac{a^2}{\gamma - 1} - \frac{GM}{r} = \text{constant} $$

Boundary conditions at $r\rightarrow \infty$

$$ \frac{GM}{r} = 0 \quad u = 0 \quad a = a_\infty \quad \rho = \rho_\infty \quad P = P_\infty $$

Bernoulli equation (#):
$$ \frac{1}{2} u^2 + \frac{a^2}{\gamma - 1} - \frac{GM}{r} = \text{constant} = \frac{a_\infty^2}{\gamma - 1} $$
Rewrite the continuity equation (*) and the force equation (**) as follows
\[
\frac{2}{r} \frac{d}{dr} \left( \frac{u'}{\rho} \right) + u' \frac{d'}{u} + u' = 0
\]
\[
u u' + a^2 \frac{\rho'}{\rho} + \frac{GM}{r^2} = 0
\]
with \( \frac{d}{dr} \)

Two linear equations for \( u' \) and \( \rho' \), with solution
\[
u' = \frac{D_1}{D} \quad \rho' = -\frac{D_2}{D}
\]
\[
D = \frac{1}{u\rho} (u^2 - a^2) \quad D_1 = \frac{1}{r \rho} (2a^2 - \frac{GM}{r}) \quad D_2 = \frac{1}{ru} (2u^2 - \frac{GM}{r})
\]
We want the solution to be smooth at all radii \( r \), so must be careful at radius \( r_s \) where \( D = 0 \), i.e., \( u^2 = a^2 \). Here also \( D_1 \) and \( D_2 \) must vanish. At the transonic radius the flow speed equals the sound speed, or Mach number \( \mathcal{M} = u/a = 1 \)

\[
u_s^2 = a_s^2 = \frac{1}{2} \frac{GM}{r_s}
\]
(###)

Relating (###) with Bernouilli equation (#):
\[
u_s^2 = u_s^2 = \frac{2}{5 - 3\gamma} a_s^2
\]
\[
r_s = \frac{5 - 3\gamma}{4} \frac{GM}{a_s^2}
\]

Thus at the transonic radius, the gravitational potential \( GM/r_s \) is comparable to the internal ambient thermal energy per unit mass

Then relate \( \dot{M} \) to the boundary conditions, and the value of \( \gamma \):
\[
\dot{M} = 4\pi r_s^2 \rho(r_s)u_s = 4\pi \lambda_s \left( \frac{GM}{a_s^2} \right)^{2/3} \rho_s a_s
\]
where
\[
\lambda_s = \left( \frac{1}{2} \right)^{2(\gamma+1)/(\gamma-1)} \left( \frac{5 - 3\gamma}{4} \right)^{-3\gamma/(2(\gamma-1))}
\]

Taking the sound speed \( a_s \) to be comparable to the mean particle speed \( v_s \), then this \( \dot{M} \)

\[
\dot{M} \text{ is } c_s^2/a_s^{\frac{2}{3}} \text{ times the collisionless accretion rate (page 11) !}
\]

For typical ionised interstellar gas with \( a_s \sim 10 \text{ km/s} \), the accretion rate is a factor \( 10^9 \) more efficient.

Physical reason is that the particle collisions restricts tangential motions and funnels motions in the radial direction
Treat $\gamma = \frac{5}{3}$ separately. Combined result:

One can show that:
- Limits: $\lambda_s = \frac{1}{4}$ for adiabatic index $\gamma = \frac{5}{3}$ and $\lambda_s = \frac{1}{4}e^{3/4}$ for $\gamma = 1$
- The transonic accretion rate is also the maximum possible accretion rate for a given $\gamma$
- No steady solutions for $\gamma > \frac{5}{3}$ (ST p. 416)

Now assume the gas is ideal:

\[
P \approx \frac{\rho k T}{\mu m_w} \quad \quad \quad \quad a^2 = \frac{\gamma k T}{\mu m_w} \quad \quad \quad \quad T = T_\infty \left( \frac{\rho}{\rho_\infty} \right)^{\gamma - 1}\]

and take $\gamma = 5/3$. Then

\[
M \sim 3 \times 10^{-3} \left( \frac{M}{10^8 M_\odot} \right)^2 \left( \frac{\rho_\infty}{10^{-24} \text{g cm}^{-3}} \right) \left( \frac{a_\infty}{300 \text{km s}^{-1}} \right)^{-3} M_\odot \text{yr}^{-1}
\]

This is $10^6$ times larger than collisionless accretion rate!
Behavior of flow at large and small radii

Newtonian solution has following asymptotic behavior: \(1 \leq \gamma < \frac{5}{3}\),
- When \(r \gg r_s\), influence gravitational potential small:
  \[\rho \sim \rho_\infty, \quad T \sim T_\infty, \quad a \sim a_\infty\]
- Using ST 14.3.4, 14.3.16, 14.3.21, we find for the velocity
  \[u \sim a_\infty \lambda_s \left(\frac{GM}{a_\infty^2}\right) r^{-2}\]
- When \(r < r_s\), the flow velocity approaches free fall and gas pressure becomes negligible:
  \[u \sim \left(\frac{2GM}{r}\right)^{1/2}\]
- In this case the density and temperature are:
  \[\rho = \rho_\infty 2^{1/2} \left(\frac{GM}{a_\infty^2}\right)^{3/2} r^{-3/2}, \quad T = T_\infty \left[\frac{\lambda_s}{2^{3/2}} \left(\frac{GM}{a_\infty^2}\right)^{3/2}\right]^{-1} r^{-3(\gamma-1)/2}\]
- When \(\gamma = \frac{5}{3}\), then \(r_s = 0\). Approximations can now be derived for \(r \ll GM/a_\infty^2\)

Assignment

1. Verify that for the special case of \(\Gamma = 5/3\), and for the critical solutions satisfies:
   \[a \approx u \approx \left(\frac{GM}{2r}\right)^{1/2}\]
   and that the corresponding density and temperature profiles are:
   \[\rho \approx \rho_\infty 2^{1/4} \left(\frac{GM}{a_\infty^2}\right)^{3/4} r^{-3/2}, \quad T \approx \frac{T_\infty GM}{2a_\infty^2} r^{-1}\]
   Hint: follow the suggestions on page 418 of ST.
2. Give numerical expressions \(u(r)\), \(\rho(r)\) and \(T(r)\) for reasonable values for the ambient temperature, density and sound speeds (for example: page 415 of ST)
3. What temperature and density does the gas obtain near the event horizon? Do you think those are realistic values? What radiative process makes this gas shine?
Relativistic regime

The above approximations for \(1 \leq \gamma < \frac{5}{3}\) are valid also for the Schwarzschild black hole, when we take
- \(r\) the radial coordinate in the Schwarzschild metric
- \(u\) the radial component of the velocity four vector
- \(\rho\) the proper rest-mass density

Details in Appendix G of ST, including case \(\gamma = \frac{5}{3}\)

Other solutions

Bondi’s equations (*) and (**) also allowed

- Subsonic motion everywhere
  - One solution for each \(\lambda\) with \(0 \leq \lambda < \lambda_s\)
  - Behavior at large \(r\) same as for critical solution
  - Flow is choked off at small radii by back pressure

- Outflow solution
  - Subsonic for \(r < r_s\) and supersonic for \(r > r_s\)
  - Relevant for stellar winds

- Supersonic outflow
  - Requires acceleration mechanism near the surface

Which regime is applicable depends on the boundary conditions of the accretion objects. For a hard surface (white dwarf or neutron star) steady subsonic flow is allowed. For BHs, flows must be transonic (ST App G).
The Eddington limit (K 6.2)

Bondi calculation ignores radiation produced in accretion flow, which provides outward force (radiation pressure) $\vec{F}_{\text{rad}}$:

$$\vec{F}_{\text{rad}} = \frac{1}{c} \int d\nu \ F_\nu \kappa_\nu \rho$$

Here $F_\nu$ is radiative flux per unit frequency, and $\kappa_\nu$ is the total opacity (scattering + absorption) at frequency $\nu$ per unit mass.

Assumptions:
- Spherical geometry
- Fully ionized non-relativistic gas: $\kappa$ provided by electron scattering, independent of $\nu$ and given by

$$\kappa = \frac{\sigma_T n_e}{\rho} \text{ with } \sigma_T = \frac{8 \pi e^4}{3 m_e^2 c^4}$$

where $\sigma_T$ is the Thomson cross section, and $n_e$ is the electron number density.

Define radiative and gravity acceleration:

$$g_{\text{rad}} = \frac{\sigma_T}{\mu_e} \frac{L}{4 \pi r^2}$$

$$g_{\text{grav}} = \frac{GM}{r^2}$$

where $\mu_e$ is mean molecular weight per electron.
No accretion possible if $g_{\text{grav}} < g_{\text{rad}}$, i.e., $L > L_E$, with
\[ L_E = \frac{4\pi c^2 \mu_e G M}{\sigma_T} = 1.5 \times 10^{38} \frac{M}{M_\odot} \text{erg s}^{-1} \]
This is the Eddington limit
See The Internal Constitution of the Stars (Eddington 1926):
\[ L/L_\odot < 4 \times 10^5 M/M_\odot \]
for massive stars dominated by radiation pressure
AGN have $L \sim 10^{43}$ to $10^{47}$ erg s$^{-1}$, so condition $L < L_E$ puts lower limit on $M$ in range $10^5$ to $10^9 M_\odot$

Eddington accretion rate
\[ \dot{M}_E = \frac{L_E}{c^2 \epsilon} = 3 \left( \frac{M}{10^8 M_\odot} \right) \left( \frac{0.1}{\epsilon} \right) M_\odot \text{yr}^{-1} \]
Growth time
\[ \frac{dM}{dt} = \frac{L}{c^2 \epsilon} = \frac{L_E}{4\pi G \mu_e M} \]
so that at fixed $L/L_E$, $M$ grows exponentially, with characteristic time
\[ t_{\text{growth}} = \frac{\epsilon c \sigma_T}{4\pi G \mu_e} \frac{L_E}{L} \sim 3.7 \times 10^7 \left( \frac{\epsilon}{0.1} \right) \sim 10^8 L_E L \text{yr} \]

Main conclusions related to stellar accretion

- Needed (Eddington) accretion rate:
  \[ \dot{M}_E = \frac{L_E}{c^2 \epsilon} = 3 \left( \frac{M}{10^8 M_\odot} \right) \left( \frac{0.1}{\epsilon} \right) M_\odot \text{yr}^{-1} \]

- Stellar accretion rate is too low:
  \[ \dot{M} \sim 5 \times 10^{-9} \left( \frac{\rho_\infty}{10^{-24} \text{g cm}^{-3}} \right)^{\frac{2}{3}} \left( \frac{M}{10^8 M_\odot} \right)^{\frac{3}{2}} \left( \frac{v_\infty}{300 \text{km s}^{-1}} \right)^{-1} M_\odot \text{yr}^{-1} \]

- And spherical gas accretion can be $10^8$ times more efficient
  \[ \dot{M} \sim 3 \times 10^{-3} \left( \frac{M}{10^8 M_\odot} \right)^{2} \left( \frac{\rho_\infty}{10^{-24} \text{g cm}^{-3}} \right) \left( \frac{a_\infty}{300 \text{km s}^{-1}} \right)^{-3} M_\odot \text{yr}^{-1} \]