1 Problem 1: Displacement due to a passing gravitational wave

A plane gravitational wave described by the small metric disturbance \( h_{\alpha\beta} \) propagates in a flat background spacetime. The wave is propagating in the positive \( z \) direction. We also have 2 test masses, which, in Cartesian coordinates, are located at \((0, 0, 0)\) and at \((X, Y, Z)\). Let \( L_* \) be the unperturbed distance between the 2 test masses. We will denote the straight line path between the two test masses with the symbol \( \gamma \) and define a spatial unit vector \( \vec{n} \) with coordinates \( n^i \), which is a tangent vector to \( \gamma \). Show that the time-dependent change in distance between the masses, produced by the passing gravitational wave, is then given by

\[
\delta L(t) = \frac{1}{2} \int_0^{L_*} d\lambda h_{ij}(t - n^z \lambda)n^in^j.
\]

**Hint:** begin by writing down a parametric form for the path \( \gamma \), involving the quantities defined above, using a parameter \( \lambda \) that varies along the path \( \gamma \), i.e., write down an expression for \( x^i(\gamma)(\lambda) \) where \( x^i(\gamma) \) are the spatial coordinates along the path \( \gamma \).

**Solution**

Since the background metric is flat, and the gravitational wave is a plane wave, moving in the positive \( z \) direction, the metric is given by Eqs. (16.1) and (16.2a) in Hartle, except that we do not have to restrict ourselves to the + polarization. The straight line path \( \gamma \) that connects \((0, 0, 0)\) with \((X, Y, Z)\) has a length \( L_* = \sqrt{X^2 + Y^2 + Z^2} \) and the coordinates of the unit vector \( \vec{n} \) tangent to \( \gamma \) are given by \( \vec{n} = (X/L_*, Y/L_*, Z/L_*) \). A parametric expression for \( \gamma \) is then

\[
x^i(\gamma)(\lambda) = n^i\lambda,
\]

where \( \lambda \) goes from 0 to \( L_* \). The spatial distance along the path \( \gamma \) is

\[
S = \int dS = \int_\gamma \left[ (\delta_{ij} + h_{ij}(t - n^z \lambda)) dx^i dx^j \right]^{\frac{1}{2}},
\]

where \( \delta_{ij} \) is the usual Kronecker delta, which comes from the spatial part of the background flat spacetime metric. To proceed, we will need to use the \( \lambda \)-dependence of \( x^i \) and \( x^j \) as already written down above. Note that \( z \) also has a \( \lambda \)-dependence, since \( z \) may vary along the path \( \gamma \). Therefore we can write

\[
S = \int_0^{L_*} d\lambda \left\{ [\delta_{ij} + h_{ij}(t - n^z \lambda)] \frac{dx^i(\gamma)}{d\lambda} \frac{dx^j(\gamma)}{d\lambda} \right\}^{\frac{1}{2}}.
\]
This expression can be expanded in powers of the amplitude of the gravitational wave. The lowest term (with no perturbation) reproduces the unperturbed distance $L_*$. The change in that distance as the gravitational wave passes through is given by the 1st order term in the perturbation which is

$$
\delta L(t) = \frac{1}{2} \int_0^{L_*} d\lambda \ h_{ij} (t - n^z \lambda)n^i n^j,
$$

which is the result that we were trying to prove.