Problem 1: Particles falling into a black hole

Two particles fall in radially from infinity in a Schwarzschild geometry. Particle 1 starts with $e = 1$, particle 2 with $e = 2$. A stationary observer at $r = 6M$ measures their velocities as they pass by. Show that the ratio of the velocities measured by the observer is $V_2/V_1 \approx 1.58$.

Solution

An observer with 4-velocity $u_{\text{obs}}$ will measure the energy of a particle as

$$E = -p \cdot u_{\text{obs}}.$$  \hspace{1cm} (1.1)

For a stationary observer the 3 spacelike components of the velocity 4-vector are 0. Using the condition $u_{\text{obs}} \cdot u_{\text{obs}} = -1$ in Schwarzschild geometry then gives

$$u^t_{\text{obs}} = \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}},$$  \hspace{1cm} (1.2)

so we can write

$$E = m \gamma = -p \cdot u_{\text{obs}} = -m (u \cdot u_{\text{obs}}) = -mg_{\alpha\beta}u^\alpha u^\beta_{\text{obs}}$$

$$= m \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} u^t,$$  \hspace{1cm} (1.3)

(1.4)

where $p$ and $u$ are the momentum 4-vector and the 4-velocity of the infalling particle. Since $e = (1 - 2M/r) u^t$ is a conserved quantity (see Hartle Chapter 9), we can now write

$$\frac{1}{\sqrt{1 - V^2}} = \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} e.$$  \hspace{1cm} (1.5)

From this we can solve for $V$ and get

$$V = \frac{1}{e} \left(e^2 - 1 + \frac{2M}{r}\right)^{\frac{1}{2}},$$  \hspace{1cm} (1.6)

which, using $r = 6M$, yields

$$\frac{V_2}{V_1} = \frac{\sqrt{10}}{2} \approx 1.58.$$  \hspace{1cm} (1.7)
2 Problem 2: Christoffel symbols in a 3-dimensional Schwarzschild geometry

We consider a 3-dimensional version of the Schwarzschild geometry, with the line element

\[ ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\phi^2. \]  

(2.1)

Use the principle of extremal proper time to show that the non-zero Christoffel symbols for this geometry are

\[ \Gamma^r_{tt} = \left( 1 - \frac{2M}{r} \right)^{-1} \frac{M}{r^2} = \Gamma^t_{rt} \]  

(2.2)

\[ \Gamma^r_{rr} = - \left( 1 - \frac{2M}{r} \right)^{-1} \frac{M}{r^2} \]  

(2.3)

\[ \Gamma^r_{r\phi} = - (r - 2M) \]  

(2.4)

\[ \Gamma^\phi_{r\phi} = \frac{1}{r} \]  

(2.5)

\[ \Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi}. \]  

(2.6)

Hint: Start by writing down the Lagrangian to be used in the variational principle for extremal proper time of a timelike world line starting at A and ending at B. Express every coordinate as a function of a parameter \( \sigma \), which runs from 0 (at A) to 1 (at B). Eliminate the parameter \( \sigma \) when writing out the Euler-Lagrange equations, using the method described by Hartle in Chapter 8.

Solution

The solution has three parts:

(a) Write down the Lagrangian to be used in the variational principle for extremal proper time.

(b) Work out the Euler-Lagrange equations for the coordinates, to arrive at the geodesic equations.

(c) Deriving the Christoffel symbols from this.

As mentioned in the problem, we let every coordinate be a function of a parameter \( \sigma \), which runs from 0 to 1. Writing \( \dot{x}^\alpha \equiv dx^\alpha / d\sigma \), the Lagrangian can be written down as

\[ L(\dot{x}^\alpha, x^\alpha) = \left[ (1 - \frac{2M}{r}) \dot{t}^2 - \left( 1 - \frac{2M}{r} \right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2 \right]^{1/2}. \]  

(2.7)

The second part consists of working out the Euler-Lagrange equations

\[ -\frac{d}{d\sigma} \left( \frac{\partial L}{\partial \dot{x}^\alpha} \right) + \frac{\partial L}{\partial x^\alpha} = 0, \]  

(2.8)
for every coordinate. This is a lot of algebra. It is essential to use here the method used on Hartle page 172, in particular the “trick” below Eq. (8.12), realizing that differentiating the square root produces a factor $1/L$, which can be replaced by $d\sigma/d\tau$, so that the parameter $\sigma$ can be eliminated. This results in the geodesic equations

$$\frac{d}{d\tau} \left[ \left( 1 - \frac{2M}{r} \right) \frac{dt}{d\tau} \right] = 0,$$

(2.9)

$$-\frac{d}{d\tau} \left[ -r^2 \frac{d\phi}{d\tau} \right] = 0,$$

(2.10)

and

$$-\frac{d}{d\tau} \left[ - \left( 1 - \frac{2M}{r} \right)^{-1} \frac{dr}{d\tau} \right] + M \frac{dt}{d\tau}^2 + \left( 1 - \frac{2M}{r} \right)^{-2} \frac{2M}{r^2} \left( \frac{dr}{d\tau} \right)^2 - r \left( \frac{d\phi}{d\tau} \right)^2 = 0.$$

(2.11)

The non-zero Christoffel symbols can then simply be worked out. For instance, take the second component of the geodesic equation, which reads

$$-\frac{d}{d\tau} \left[ -r^2 \frac{d\phi}{d\tau} \right] = 0.$$

(2.12)

This works out to

$$\frac{d^2 \phi}{d\tau^2} = -\frac{2}{r} \frac{dr}{d\tau} \frac{d\phi}{d\tau},$$

(2.13)

which implies that

$$\Gamma^\phi_{\phi r} = \frac{1}{r} = \Gamma^\phi_{r\phi},$$

(2.14)

since the minus sign disappears (because of the definition of the Christoffel symbols), and the 2 disappears because these 2 Christoffel symbols get summed in the geodesic equation.