1 Problem 1: The worldline with longest proper time

Alice, Bob & Carol are standing close to each other on the surface of the Earth, each holding an accurate atomic clock. They make sure that their clocks are all synchronized at \( t = t_0 \). Assume for the purpose of this problem that the Earth’s surface constitutes an inertial reference frame (in reality this is not correct). At a certain time after \( t = t_0 \), Alice throws her clock vertically up. It returns after a time \( T \) as shown on Bob’s clock, which he holds in his hand all the time. Carol does something different: she carries her clock up to the maximum height reached by Alice’s clock, and then back down again, moving with constant speed, both going up and going down. She also returns after a time \( T \), as measured on Bob’s clock. In this problem, include the speed of light explicitly, i.e., do not use a \( c = 1 \) coordinate system.

(a) Calculate the total elapsed time measured on each clock, assuming that the maximum height reached is much smaller than the radius of the Earth. Include relativistic effects but use Newtonian trajectories, and calculate only to order \( 1/c^2 \).

(b) Which clock shows the longest elapsed time? Explain why this is.

2 Problem 2: No way out

In black holes one encounters the situation that you can cross a certain boundary, but cannot go back. Here we will study a simple 2-dimensional spacetime that exhibits the same behaviour. We consider a 2-dimensional spacetime with coordinates \((t, x)\) and line element

\[
ds^2 = -x \, dt^2 + 2 \, dt \, dx. \tag{2.1}
\]

(a) Give the equations defining the lightcone at \((t, x)\).

(b) Draw a \((t, x)\) spacetime diagram showing how the lightcones change with \(x\).

(c) Show that a particle can move from positive to negative \(x\) but cannot cross from negative to positive \(x\).