Astronomical Observing Techniques 2019

Lecture 2: Monsieur Fourier and his Elegant Transform

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1. Introduction
Fourier Transformation

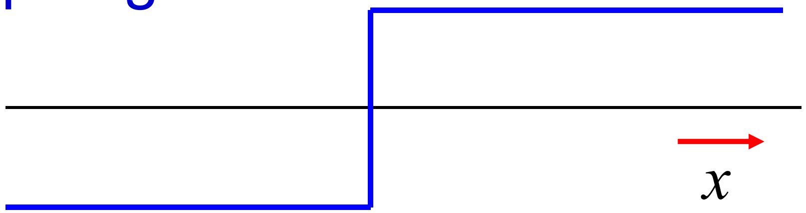
Functions *f(x)* and *F(s)* are Fourier pairs

$$\boxed{\begin{aligned} F(s) &= \int_{-\infty}^{+\infty} f(x)\cdot e^{-i2\pi xs}\,dx \\ f(x) &= \int_{-\infty}^{+\infty} F(s)\cdot e^{i2\pi xs}\,ds \end{aligned}}$$

A discrete sum of sines

Spatial frequency analysis of a step edge

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{otherwise} \end{cases}$$



Fourier decomposition

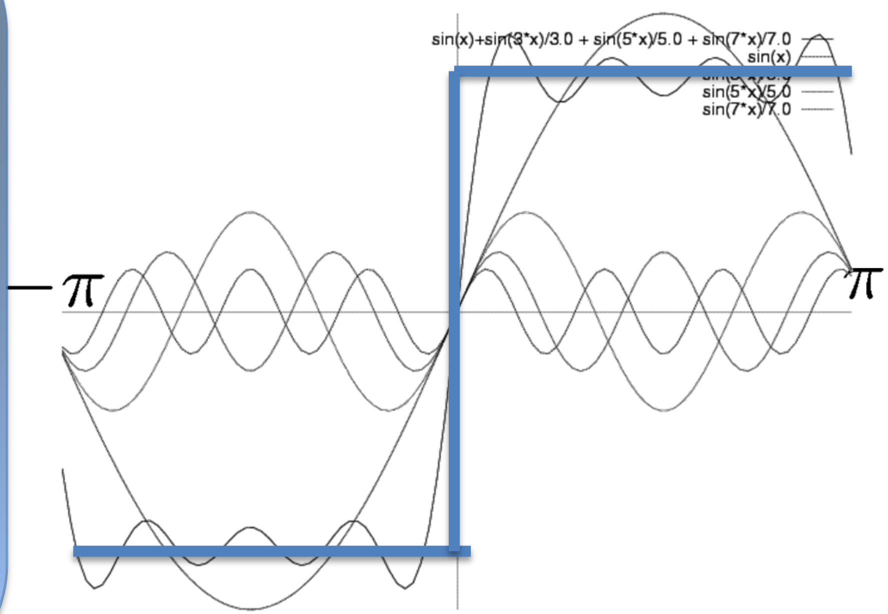
Fourier Series

$$f(x) = \sum_n a_n \sin nx$$

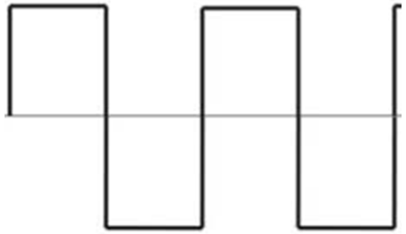
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin nx \, dx = \begin{cases} \frac{4}{n\pi} & \text{if n odd} \\ 0 & \text{otherwise} \end{cases}$$

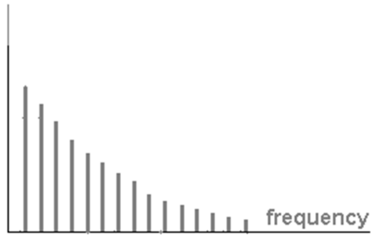
$$f(x) = \sum_{n=1} \frac{4}{(2n-1)\pi} \sin(2n-1)x$$

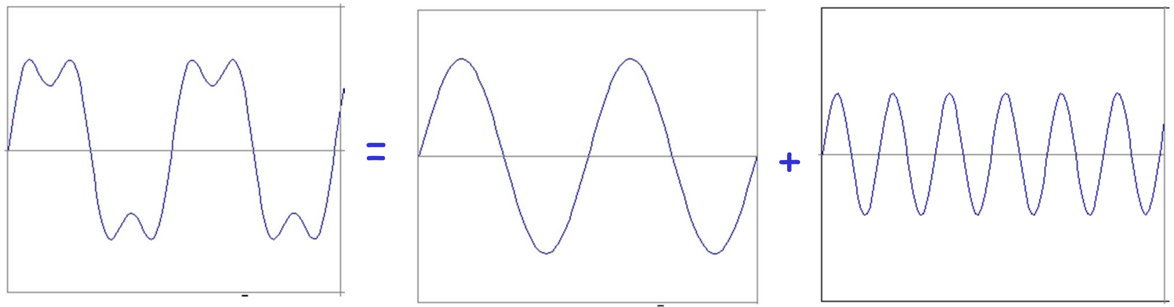


Fourier series for a square wave



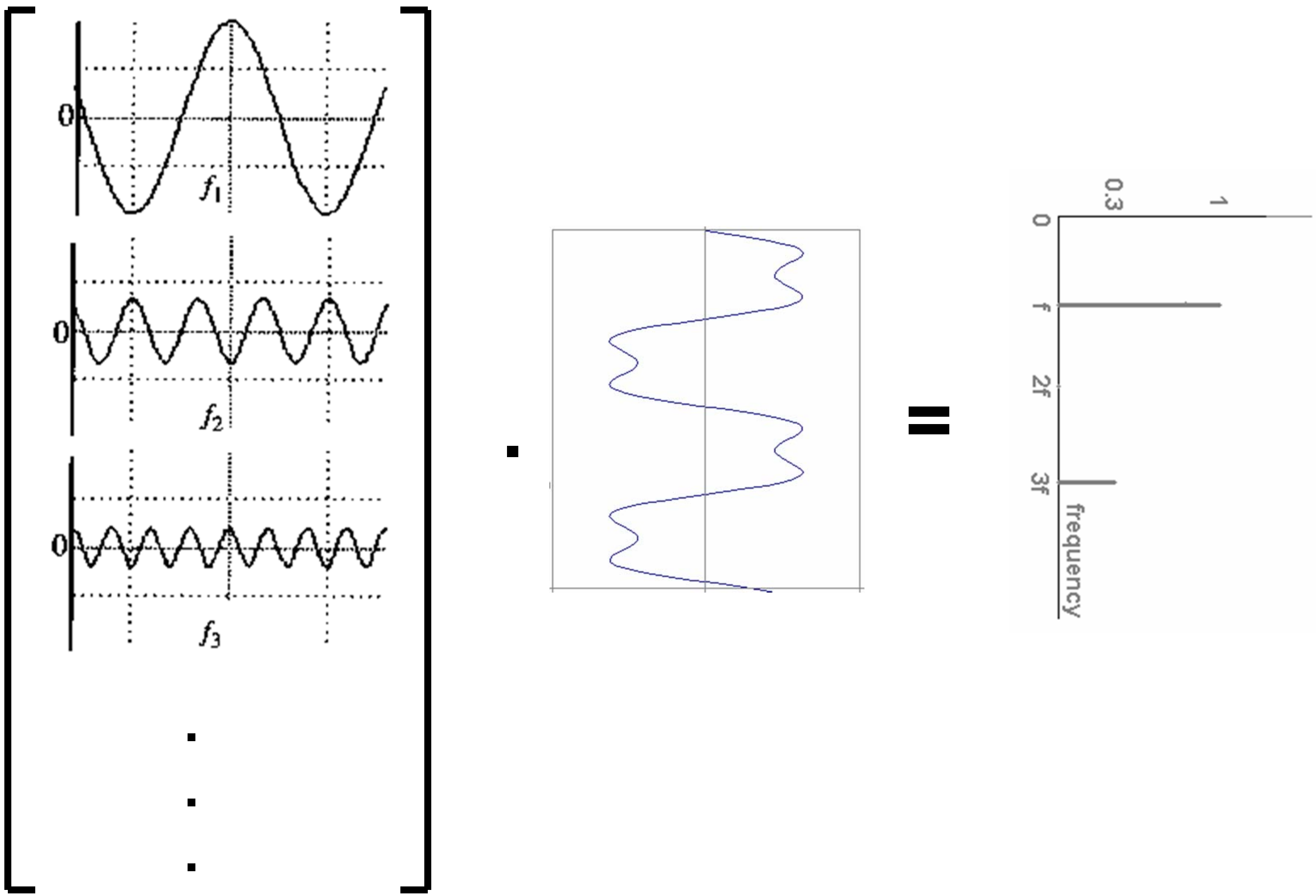
$$= \quad f(x) = \sum_{n=1,3,5,\ldots} \frac{1}{n} \sin nx$$



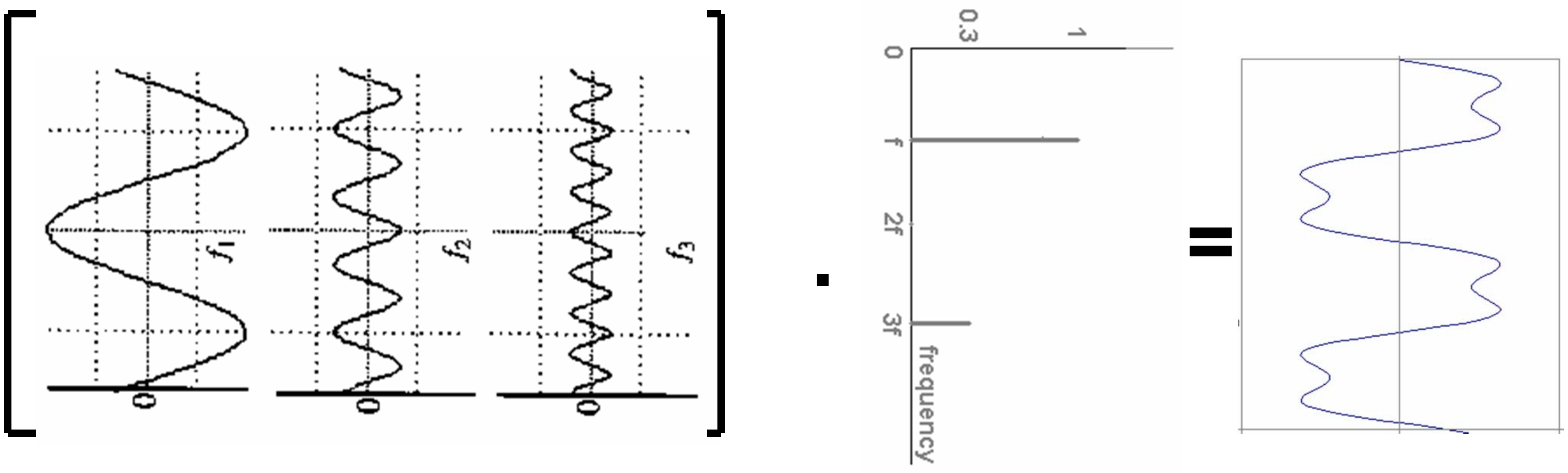


$$f(x) = \sin x + \frac{1}{3} \sin 3x + \ldots$$

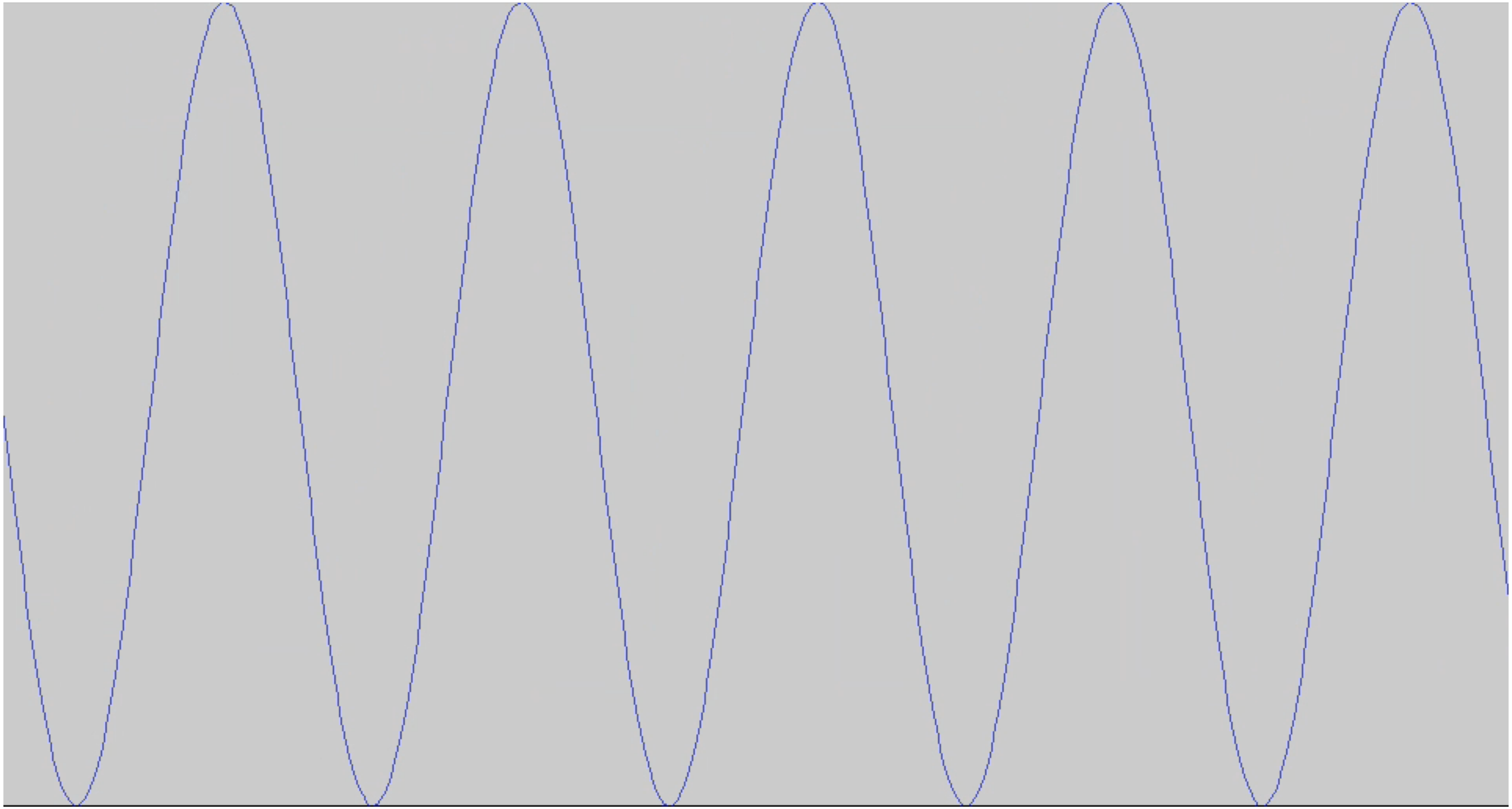
Fourier transform: just a change of basis



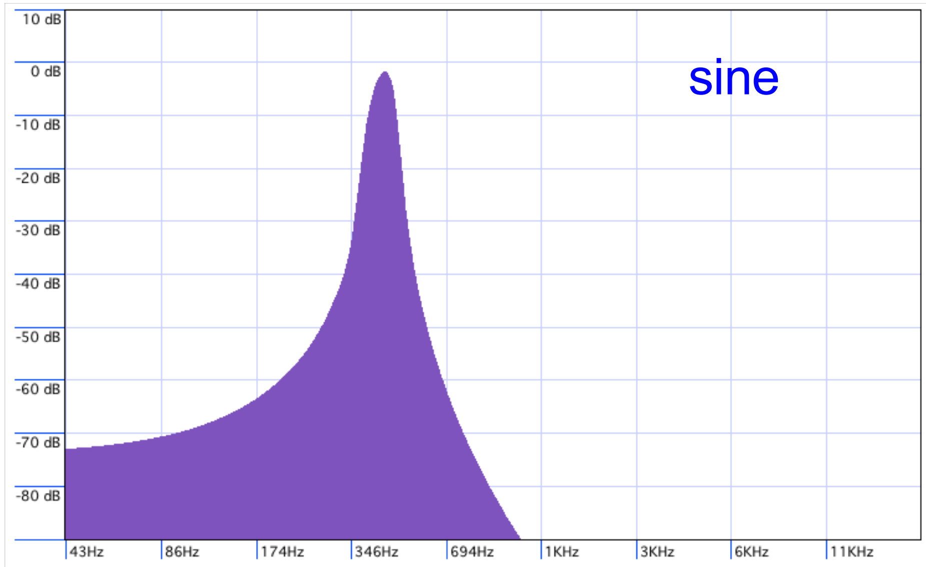
Inverse Fourier transform: just a change of basis

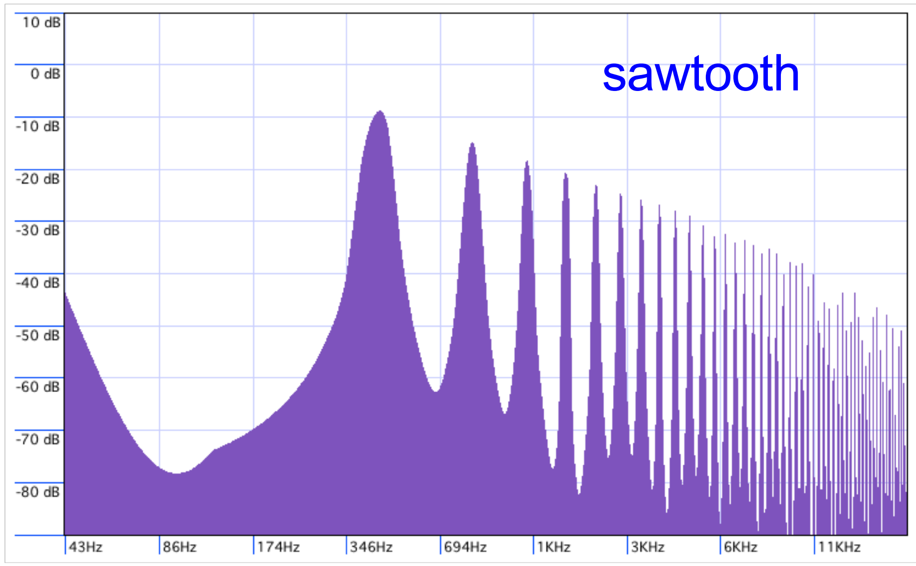


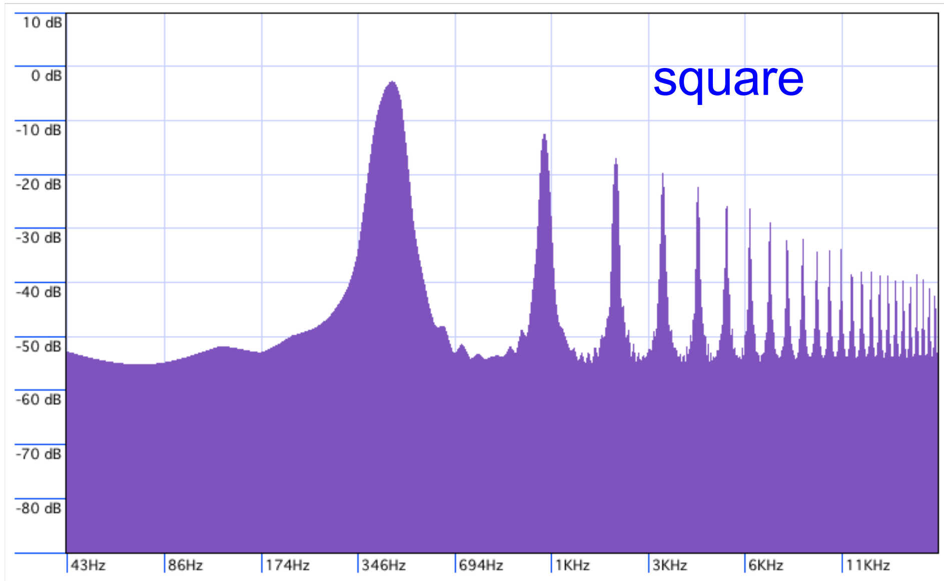
Hear the Difference



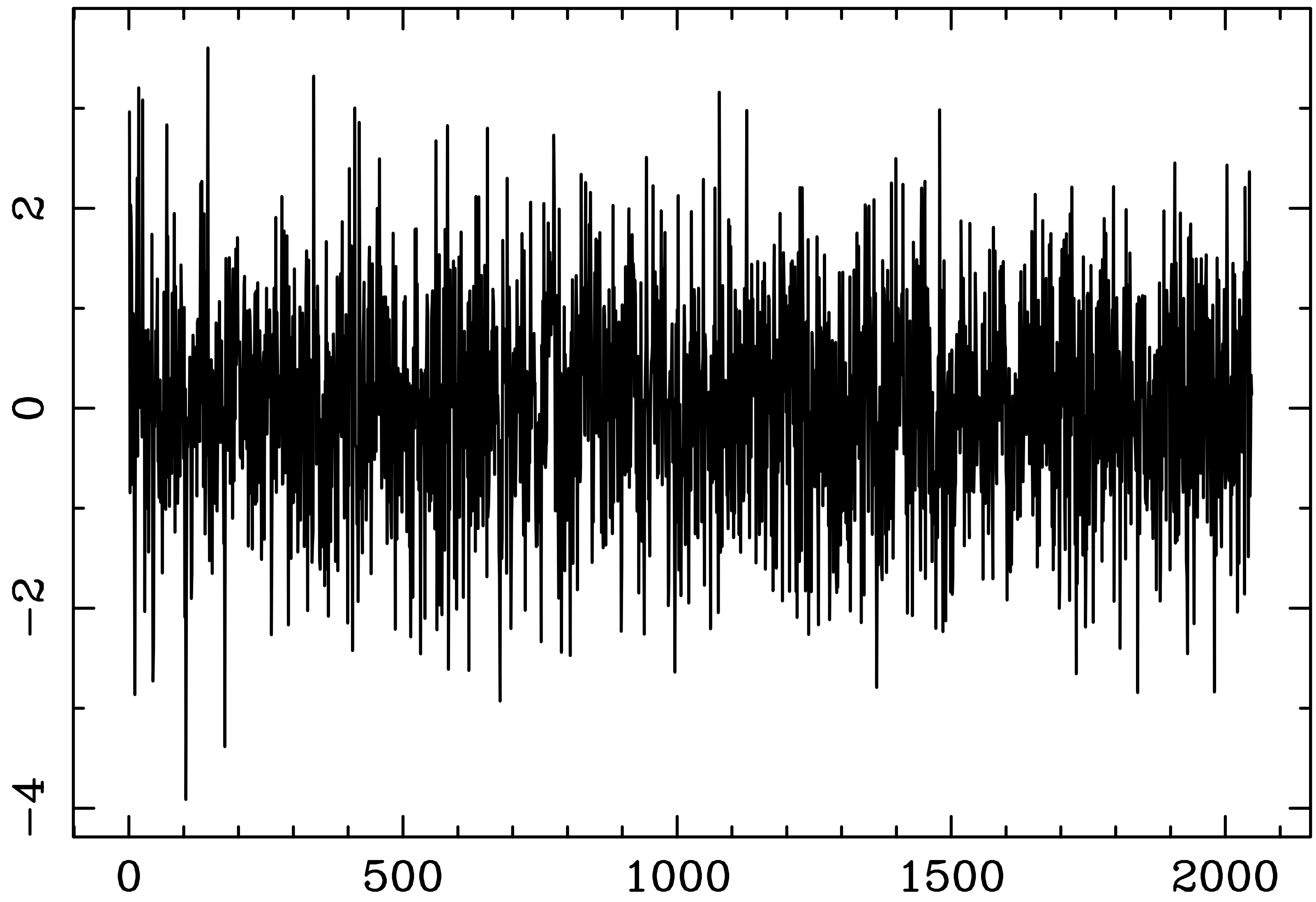
See the Difference



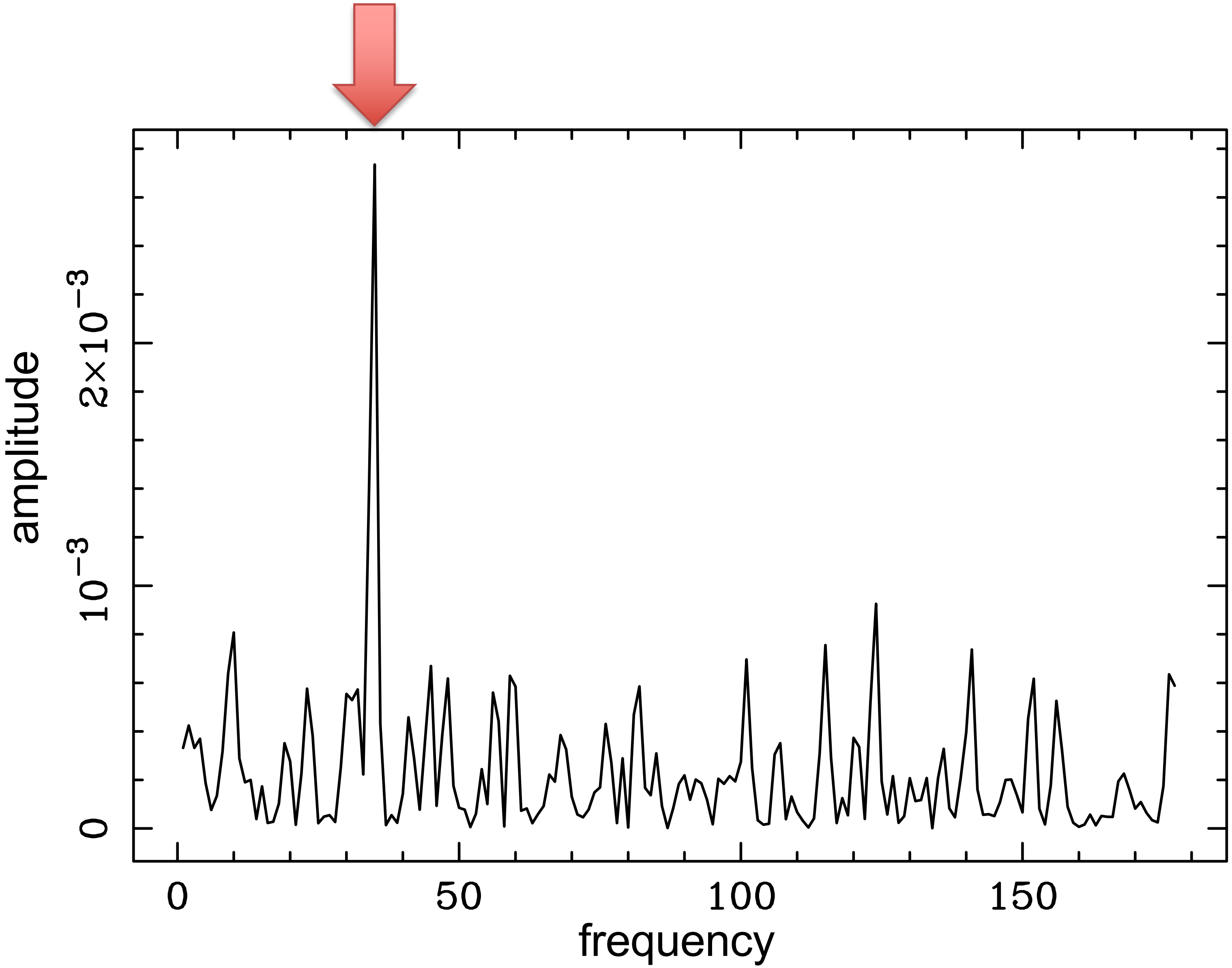




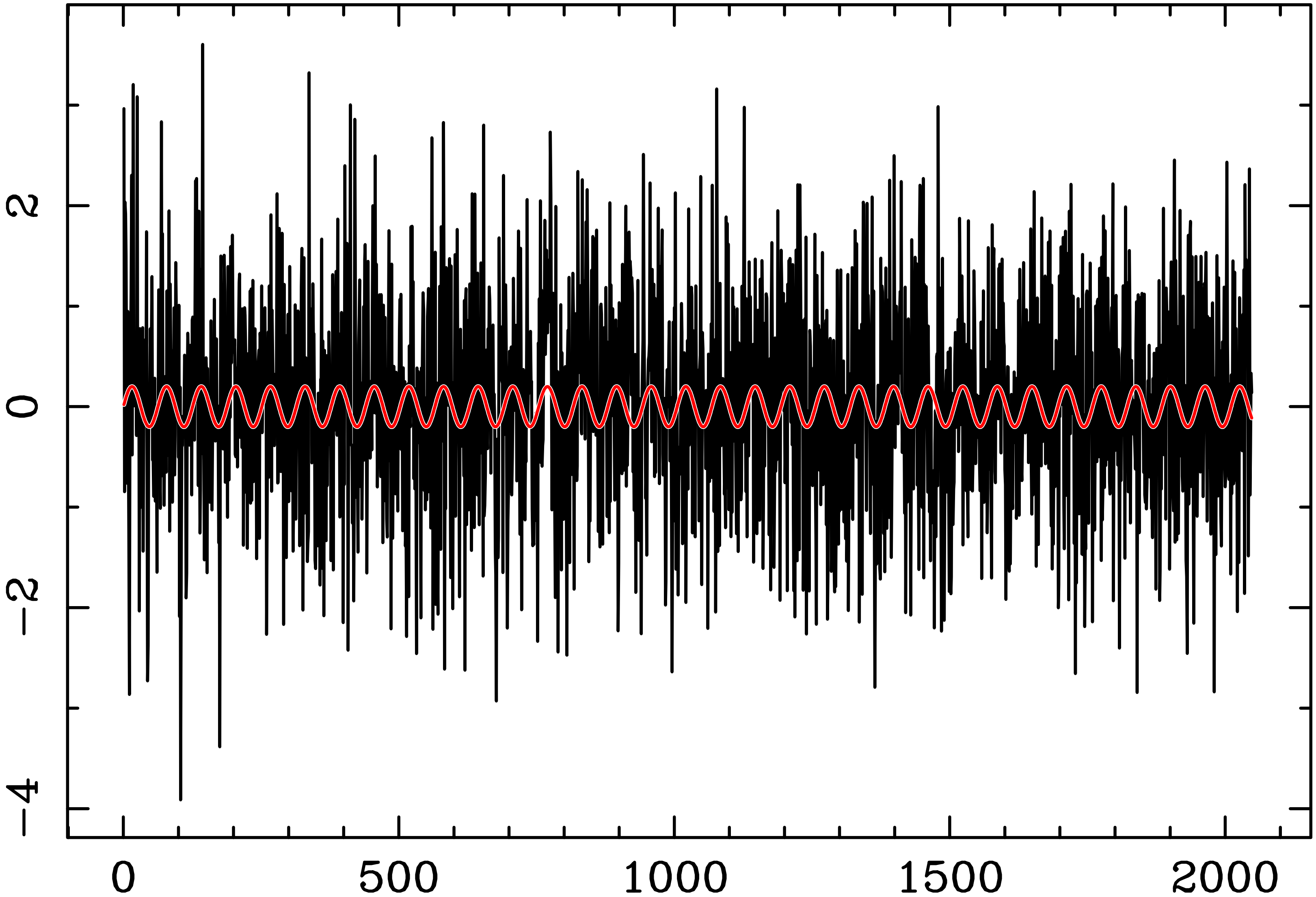
Find the Signal



Fourier Frequency Analysis



See the Periodic Signal



See the difference

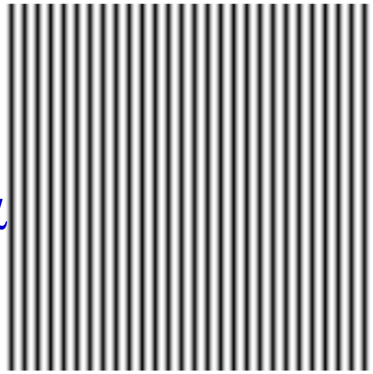
The spatial function $f(x, y)$

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux+vy)} \, du \, dv$$

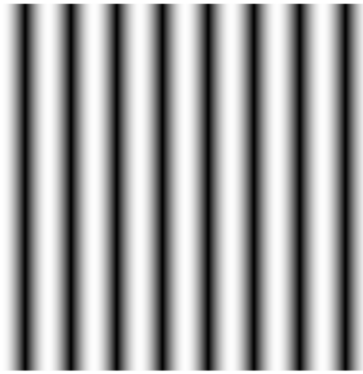
is decomposed into a weighted sum of 2D orthogonal basis functions in a similar manner to decomposing a vector onto a basis using scalar products.

f(x,y)

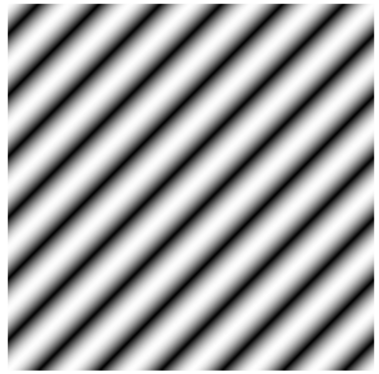
$= \alpha$



$+\beta$

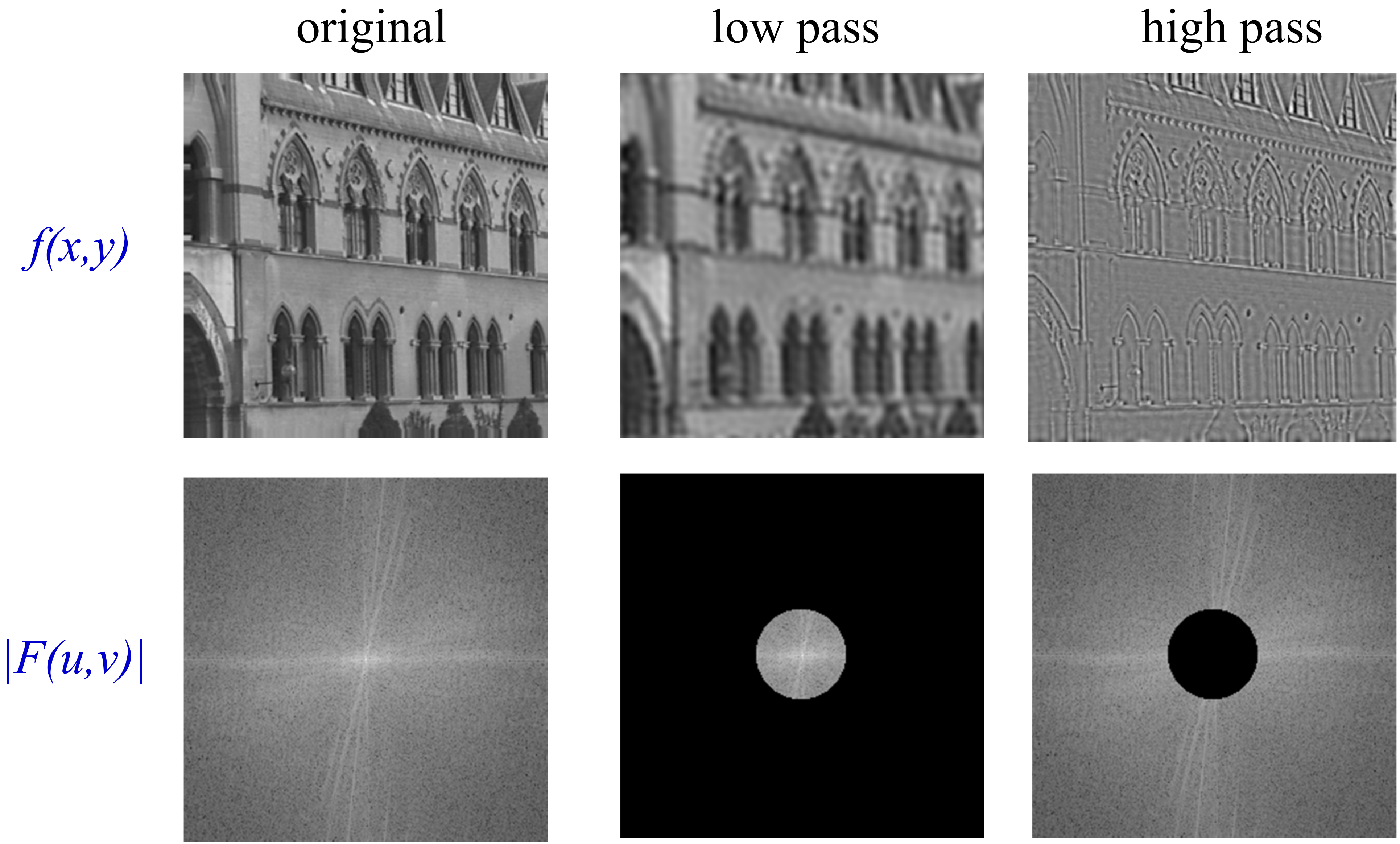


$+\gamma$



$+ \ldots$

Credits: http://www.robots.ox.ac.uk/~az/lectures/ia/lect2.pdf



``Signal at the pupil ('mirror') of a telescope''

2. Fourier Series of Periodic Functions

Decomposition using sines and cosines as orthonormal basis set

Periodic function: $f(x) = f(x+P)$

Fourier series:
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n x}{P}\right) + b_n \sin\left(\frac{2\pi n x}{P}\right)\right]$$

Fourier coefficients:
$$a_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\cos\left(\frac{2\pi n x}{P}\right)dx$$
$$b_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\sin\left(\frac{2\pi n x}{P}\right)dx$$

Period: P

Frequency: $\nu = 1/P$

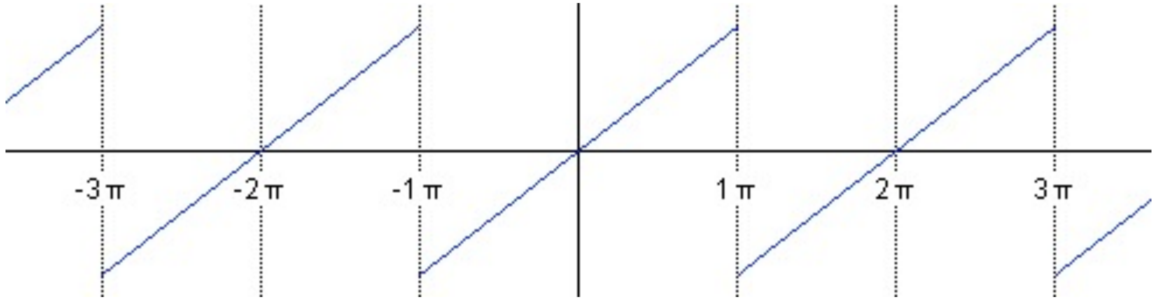
Angular frequency: $\omega = 2\pi/P$

Example: Sawtooth Function

Sawtooth function:

$$f(x) = x \qquad \text{for} \quad -\pi < x < \pi$$

$$f(x+2\pi) = f(x)$$



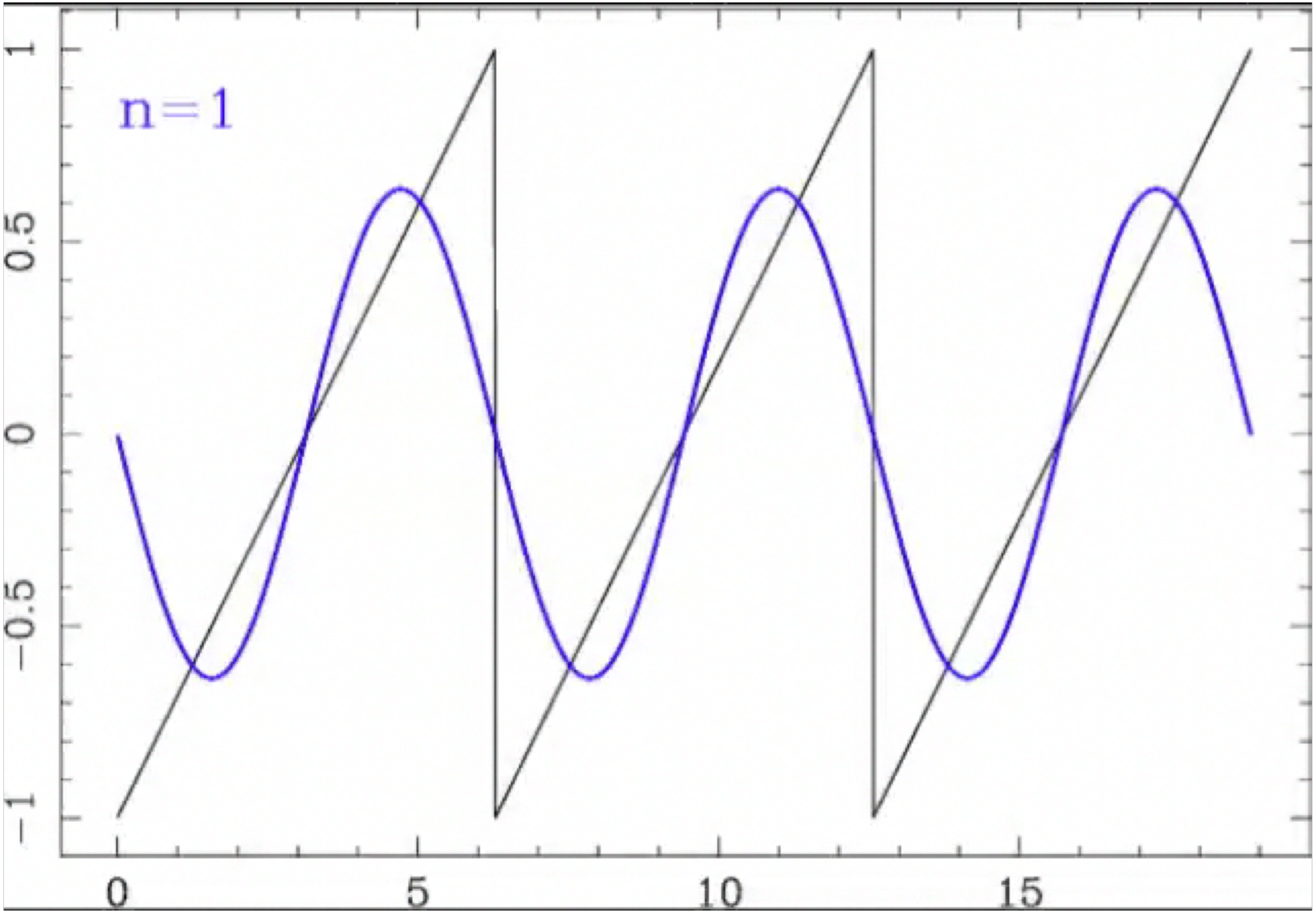
Fourier coefficients are:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx \overset{!}{=} 0 \qquad \text{(cos() is symmetric around 0)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = 2\frac{(-1)^{n+1}}{n}$$

and hence: $f(x) = \cancel{\frac{a_0}{2}} + \sum_{n=1}^{\infty}\left[\cancel{a_n\cos(nx)} + b_n\sin(nx)\right] = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$

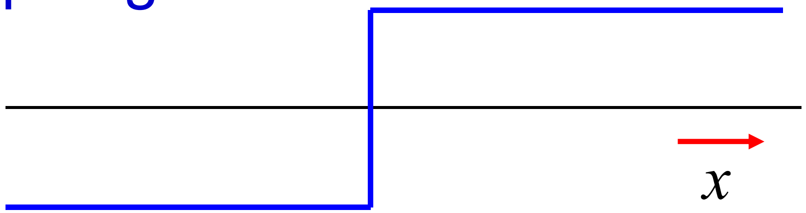
Sawtooth Approximation $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$



Second example

Spatial frequency analysis of a step edge

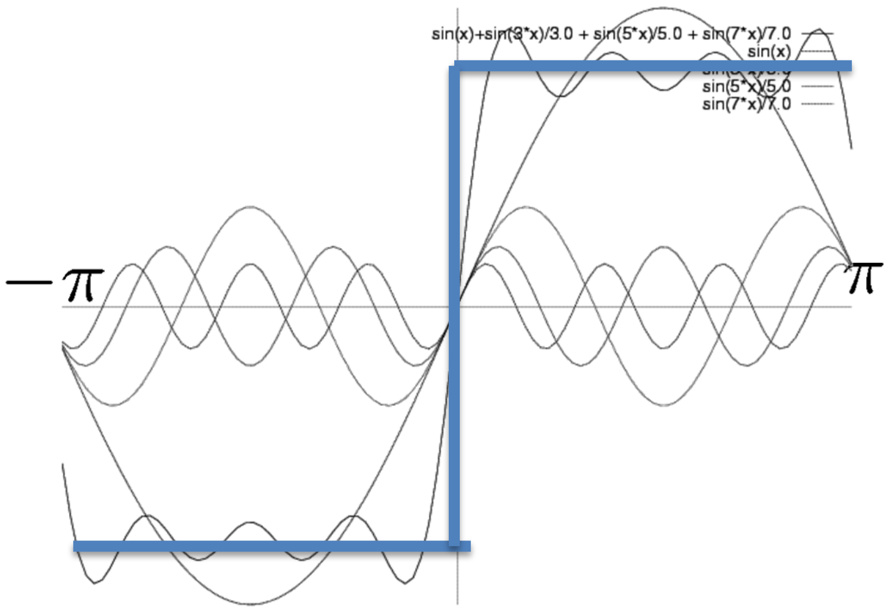
$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{otherwise} \end{cases}$$



Fourier decomposition

Fourier Series

$$\begin{aligned} f(x) &= \sum_n a_n \sin nx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ &= \frac{2}{\pi}\int_0^{\pi} \sin nx \, dx = \begin{cases} \frac{4}{n\pi} & \text{if n odd} \\ 0 & \text{otherwise} \end{cases} \\ f(x) &= \sum_{n=1} \frac{4}{(2n-1)\pi} \sin(2n-1)x \end{aligned}$$



Fourier Transformation

Functions *f(x)* and *F(s)* are Fourier pairs

$$F(s) = \int_{-\infty}^{+\infty} f(x) \cdot e^{-i2\pi xs} dx$$

$$f(x) = \int_{-\infty}^{+\infty} F(s) \cdot e^{i2\pi xs} ds$$

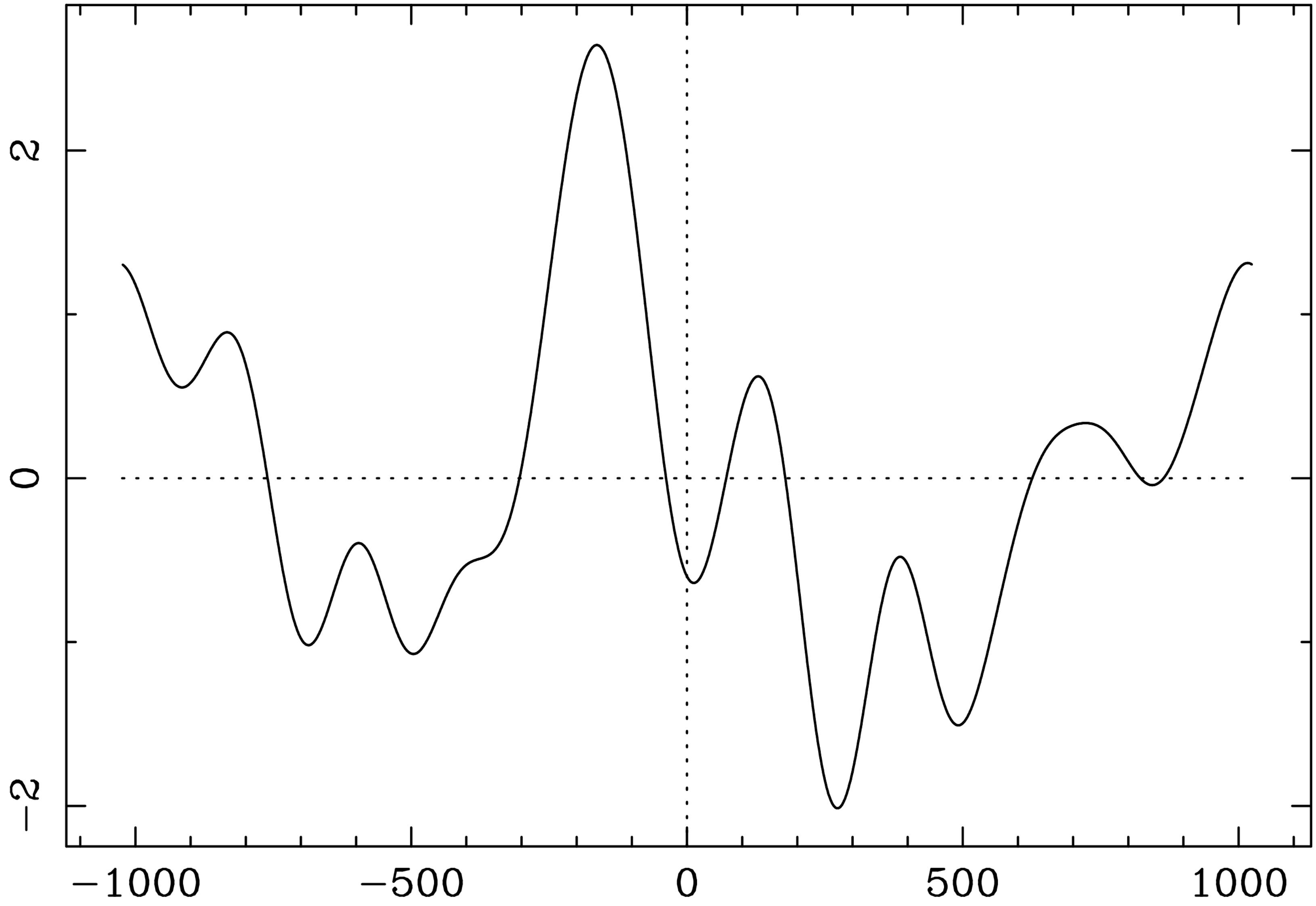
- *x*, *s* can be scalar or vector (*x·s* becomes scalar product)
- Fourier transform is reciprocal (exponent sign changes)
- exponent sign and factor 2π not well defined
- various normalization factors are used

Symmetries and Fourier Transforms

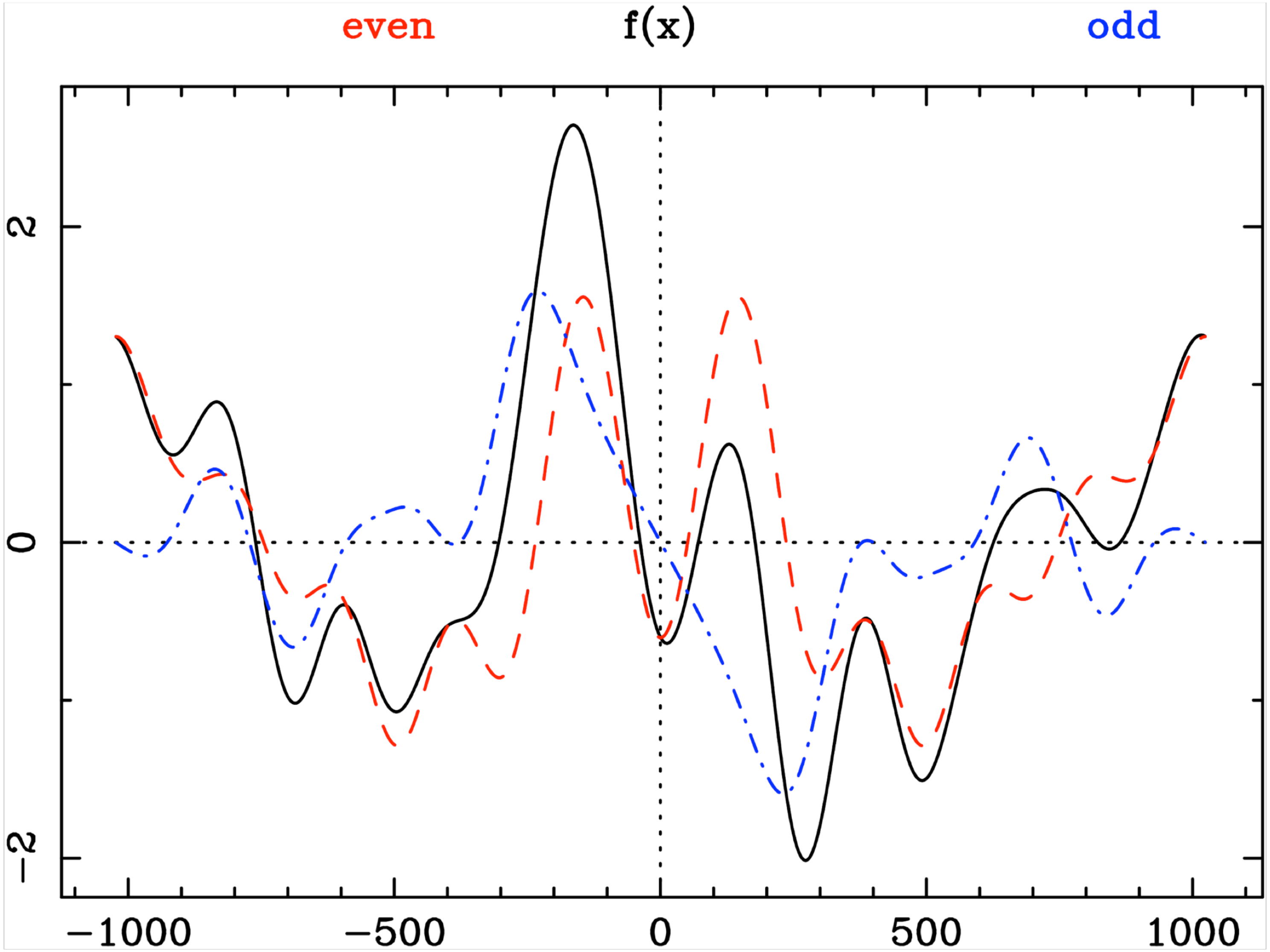
symmetry properties of Fourier transforms have many applications

- Define
  - even function: $f_{even}(-x) = f_{even}(x)$
  - odd function: $f_{odd}(-x) = -f_{odd}(x)$
- Hence
  - even part of f(x): $f_{even}(x) = ½[f(x)+f(-x)]$
  - odd part of f(x): $f_{odd}(x) = ½[f(x)-f(-x)]$
  - arbitrary function: $f(x) = f_{even}(x) + f_{odd}(x)$

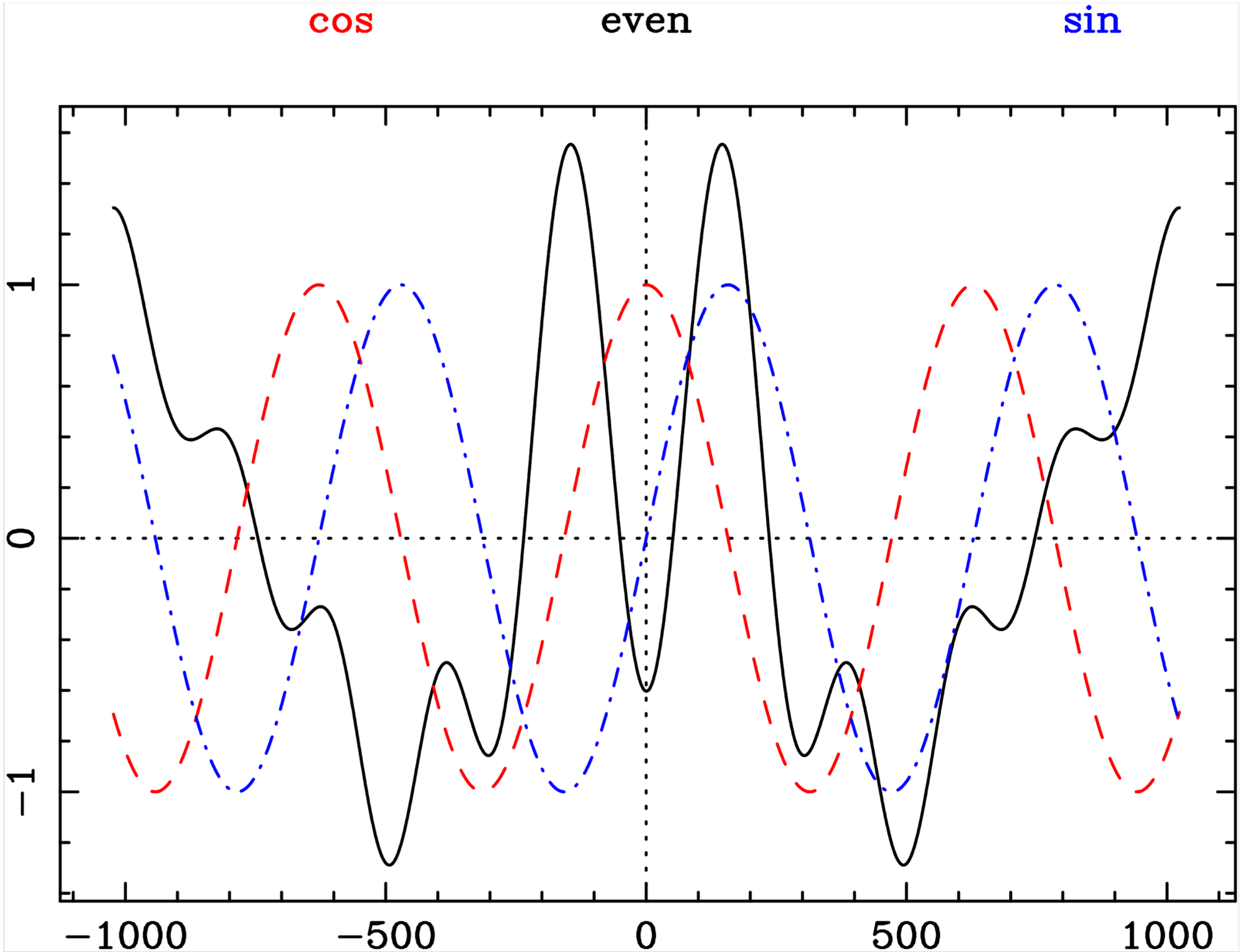
Arbitrary Function



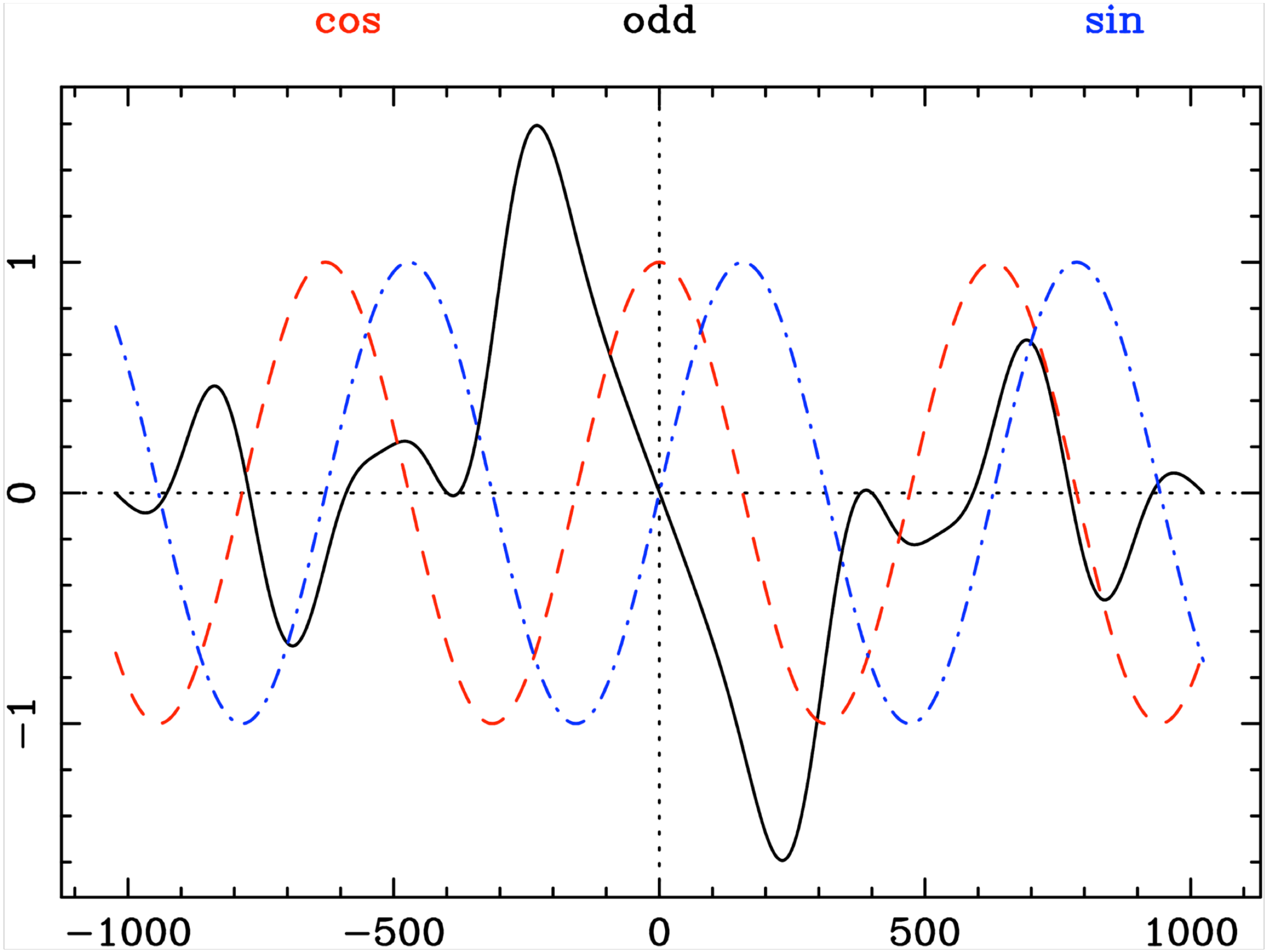
Even & Odd Decomposition



Even Function



Odd Function



Fourier Transform Symmetries

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x)$$

$$f_{\text{even}}(-x) = f_{\text{even}}(x) \quad f_{\text{odd}}(-x) = -f_{\text{odd}}(x)$$

$$e^{-i2\pi xs} = \cos(2\pi xs) - i\sin(2\pi xs)$$

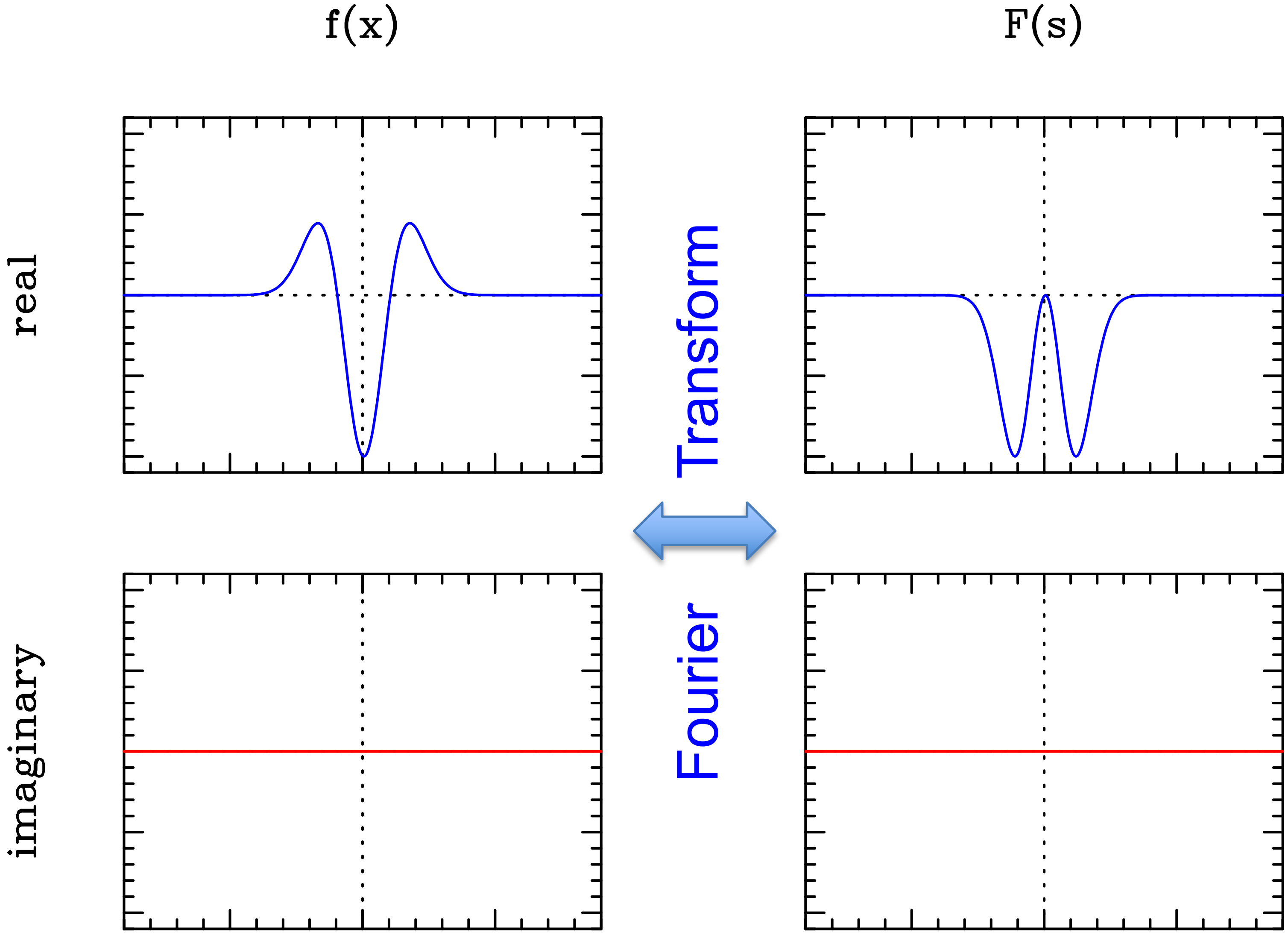
$$\Rightarrow F(s) = 2\int_0^{+\infty} f_{\text{even}}(x)\cos(2\pi xs)dx$$

$$-i\ 2\int_0^{+\infty} f_{\text{odd}}(x)\sin(2\pi xs)dx$$

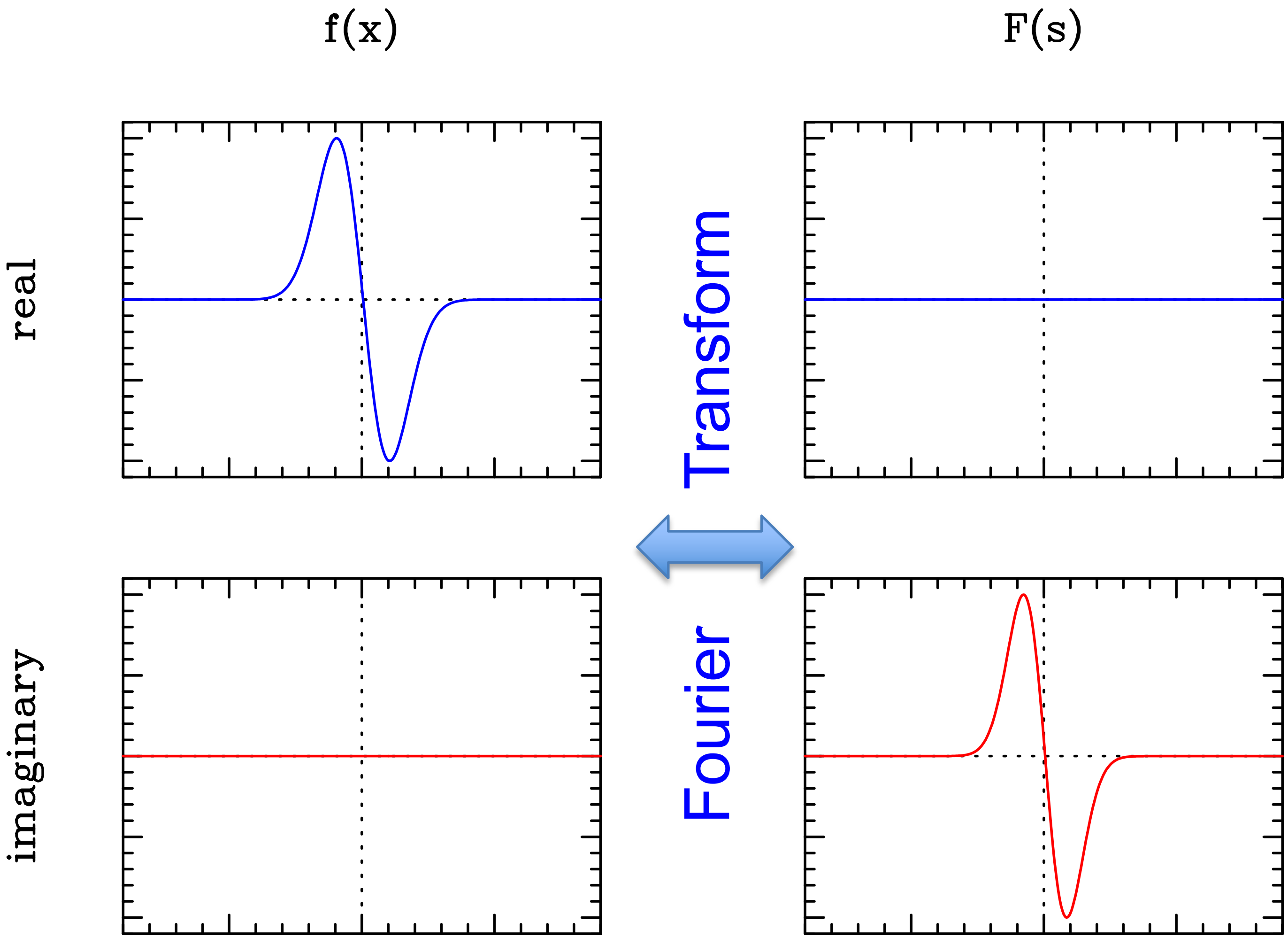
f(x) real: f_{even}(x) transforms to (even) real part of F(s),

f_{odd}(x) transforms to (odd) imaginary part of F(s).

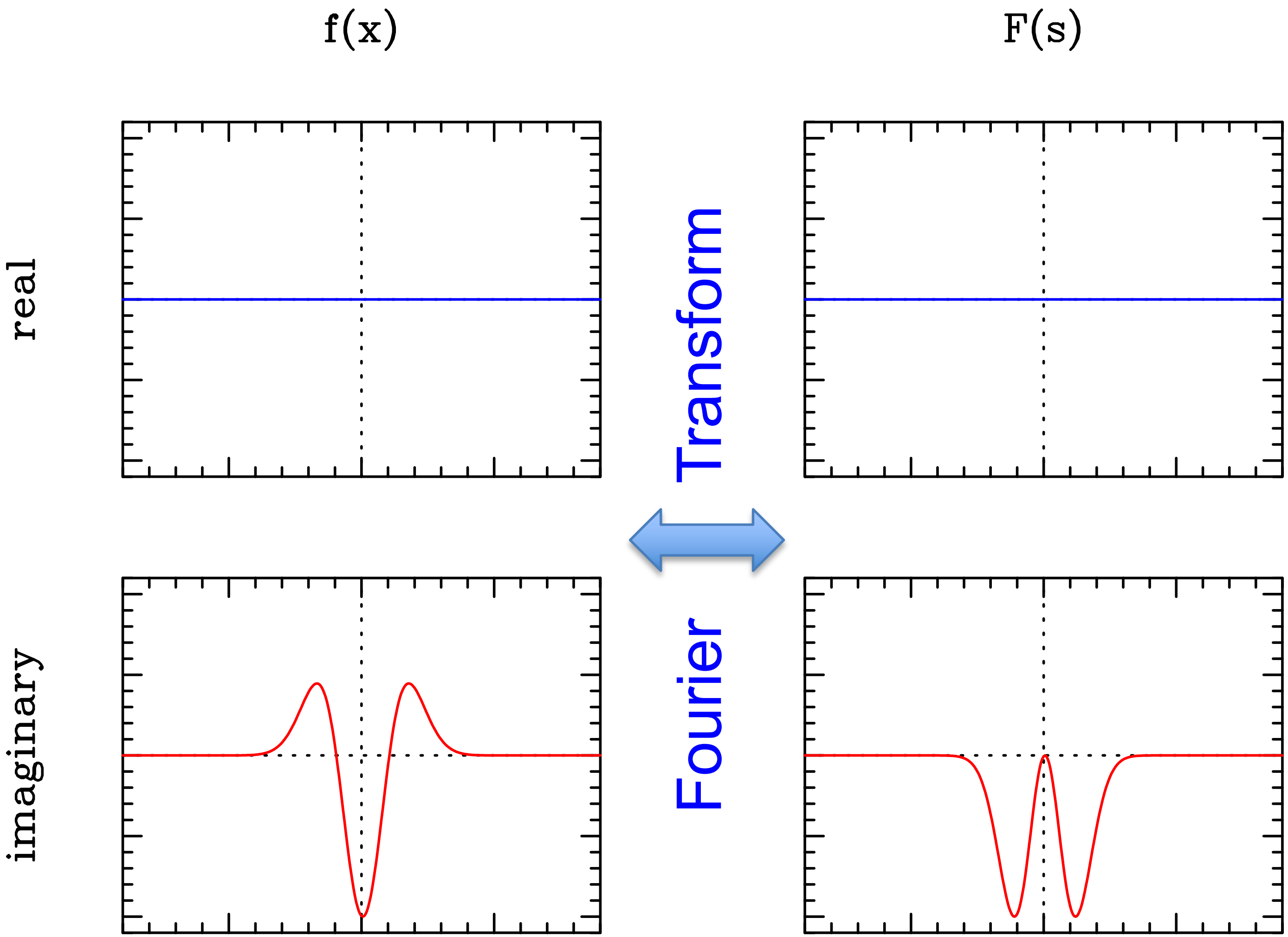
Real, Even



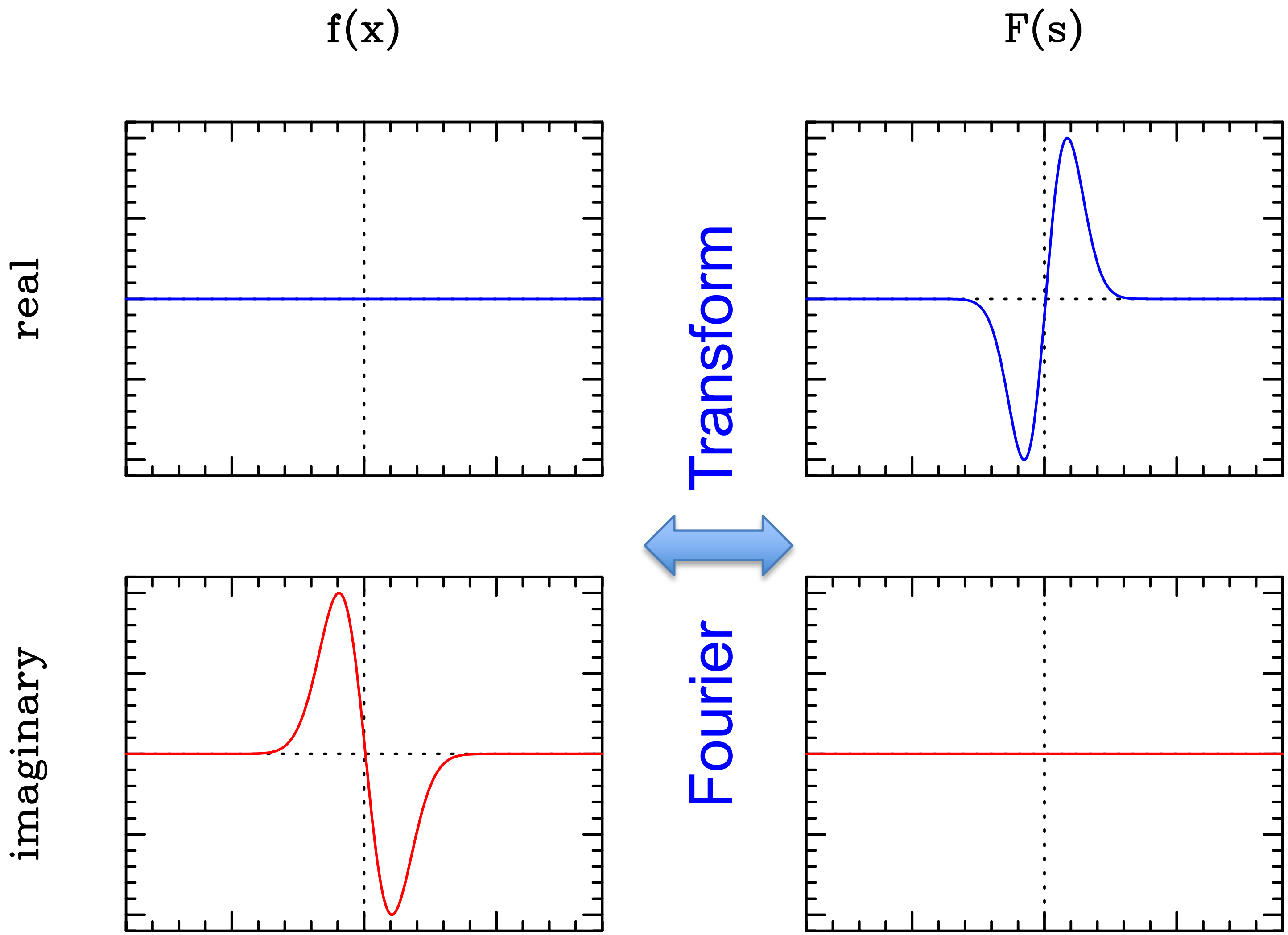
Real, Odd



Imaginary, Even

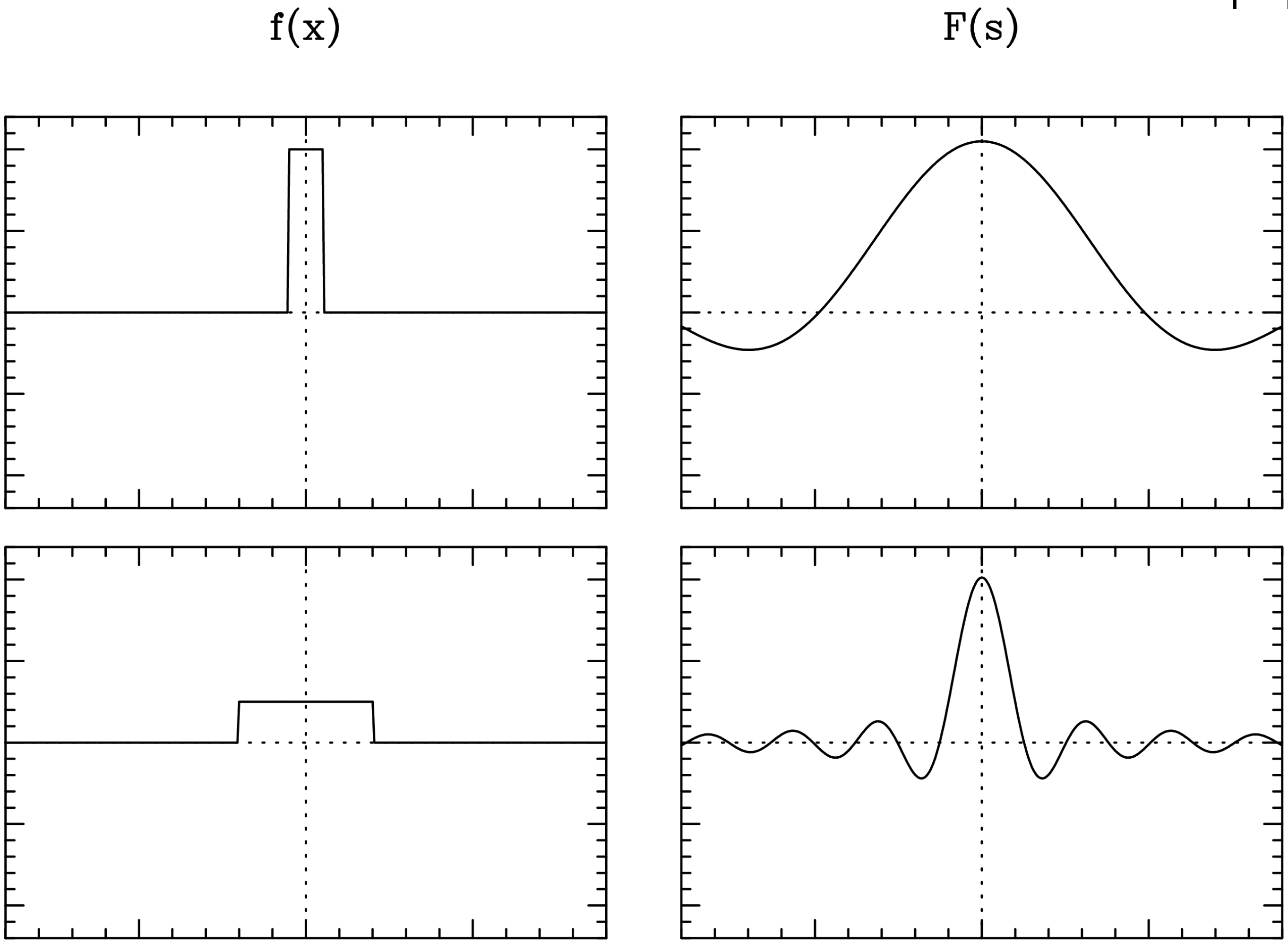


Imaginary, Odd



Fourier Transform Similarity

Expansion of *f(x)* contracts F*(s)*: $$f(x) \rightarrow f(ax) \Leftrightarrow \frac{1}{|a|} F\left(\frac{s}{a}\right)$$



Fourier Transform Properties

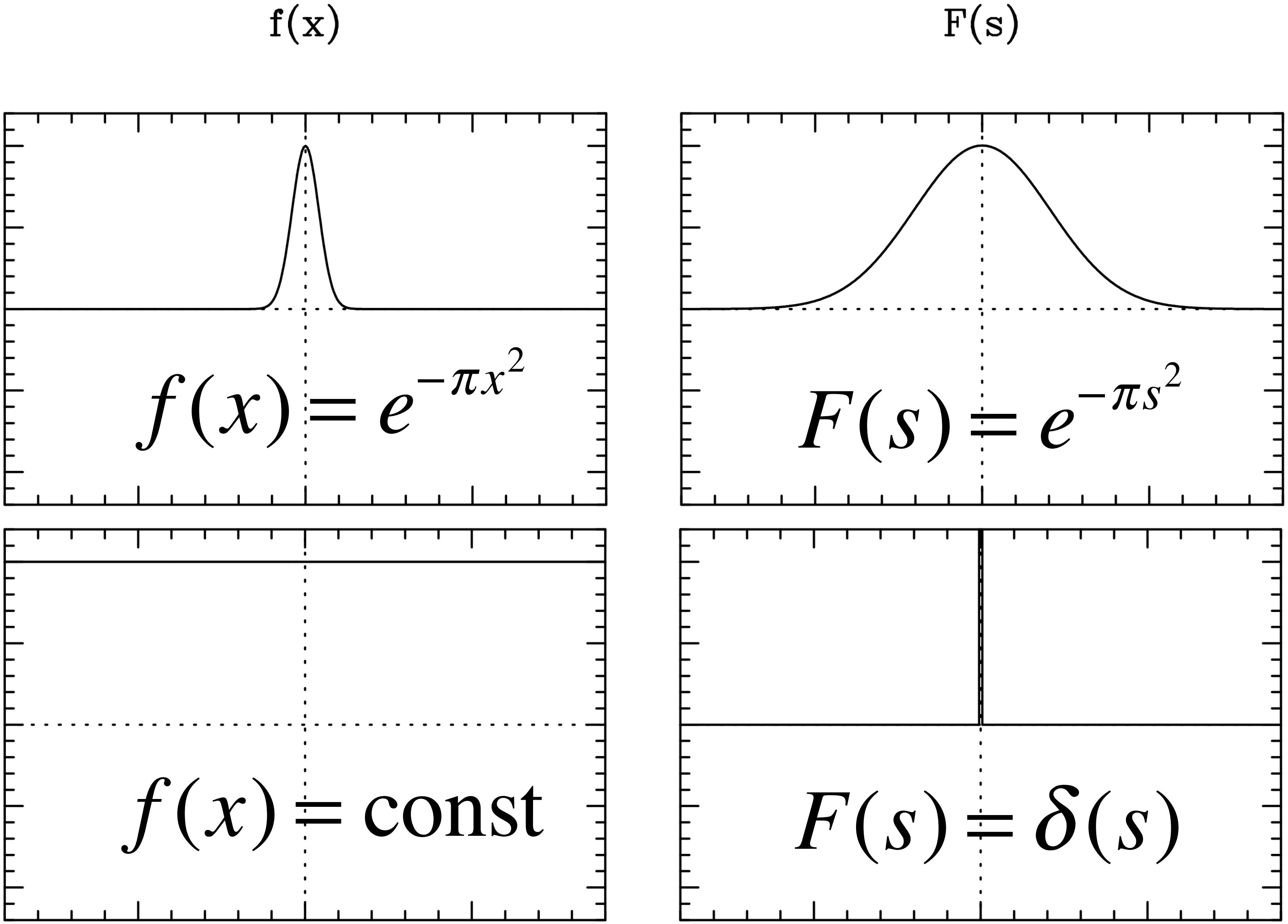
LINEARITY: $a \cdot f(x) + b \cdot g(x) \Leftrightarrow a \cdot F(s) + b \cdot G(s)$

TRANSLATION: $f(x-a) \Leftrightarrow e^{-i2\pi as} F(s)$

DERIVATIVE: $\dfrac{\partial^n f(x)}{\partial x^n} \Leftrightarrow (i2\pi s)^n F(s)$

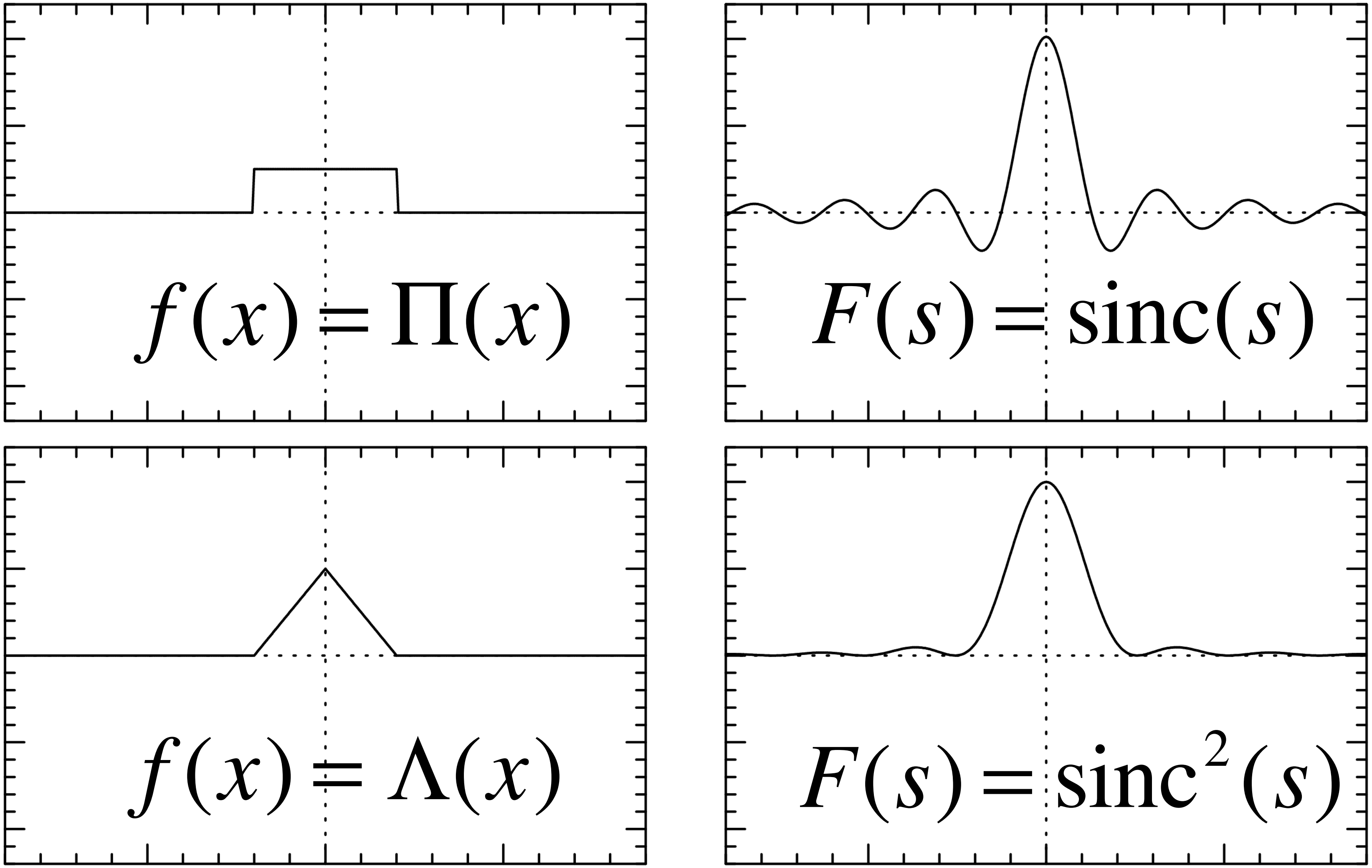
INTEGRAL: $\int f(x) \partial x \Leftrightarrow (i2\pi s)^{-1} F(s) + c\delta(s)$

Important 1-D Fourier Pairs 1

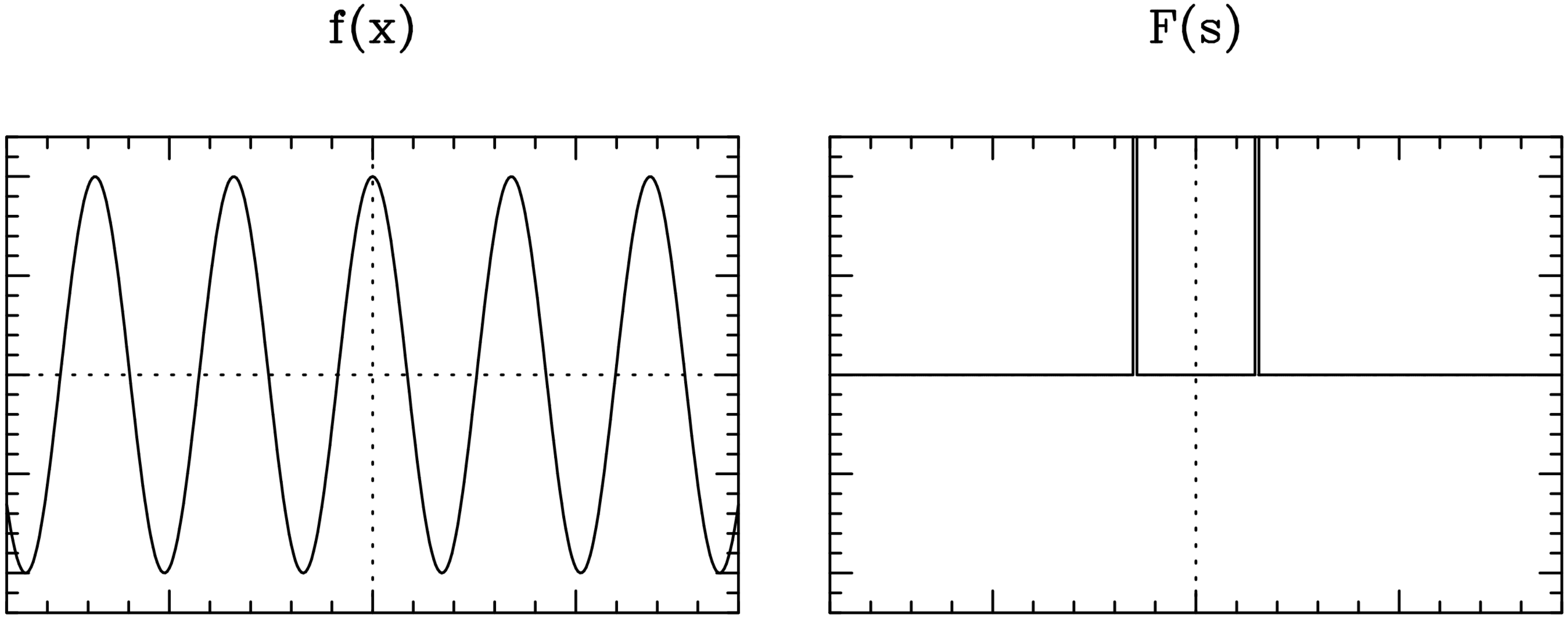


Important 1-D Fourier Pairs 2

f(x) F(s)



Important 1-D Fourier Pairs 3



$$f(x) = \cos(\pi x) \qquad F(s) = \delta(s \pm \frac{1}{2})$$

Numerical Fourier Transforms

- Problems with Fourier Transform
  - signal is only know for finite time/space/…
  - cannot integrate over $\pm\infty$
  - only know signal at discrete points (samples)
- Assumptions
  - signal is periodic beyond known interval
  - first and last data point become adjacent!
  - signal is sampled at discrete, evenly spaced points

Often used: fast Fourier transform: compute time ~NlogN

4. Convolution, cross- and auto correlation

Convolution

Convolution of two functions, $f*g$, is integral of product of functions after one is reversed and shifted:

$$h(x) = f(x)*g(x) = \int_{-\infty}^{+\infty} f(u)\cdot g(x-u)\,du$$

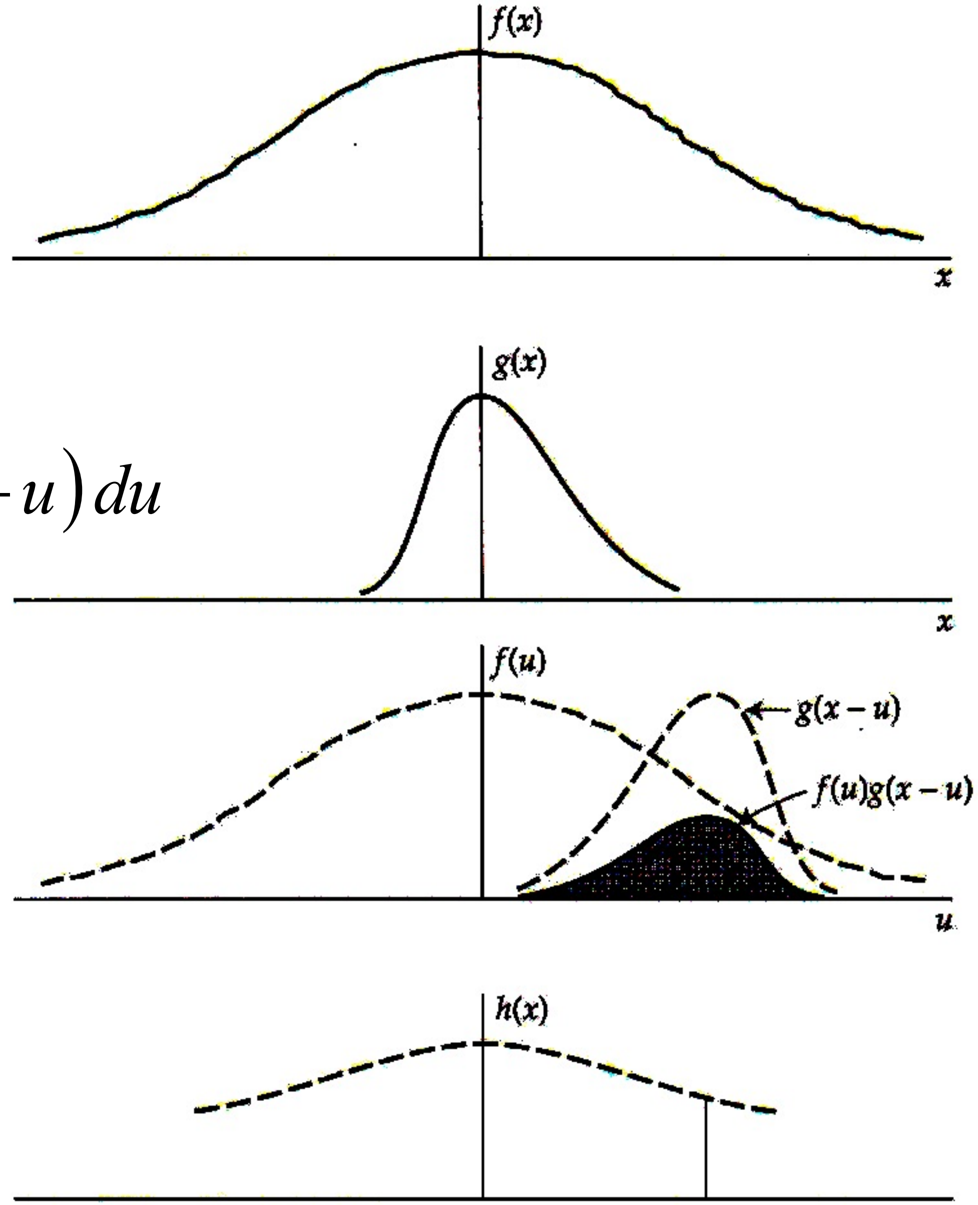
$$f(x) \Leftrightarrow F(s)$$

$$g(x) \Leftrightarrow G(s)$$

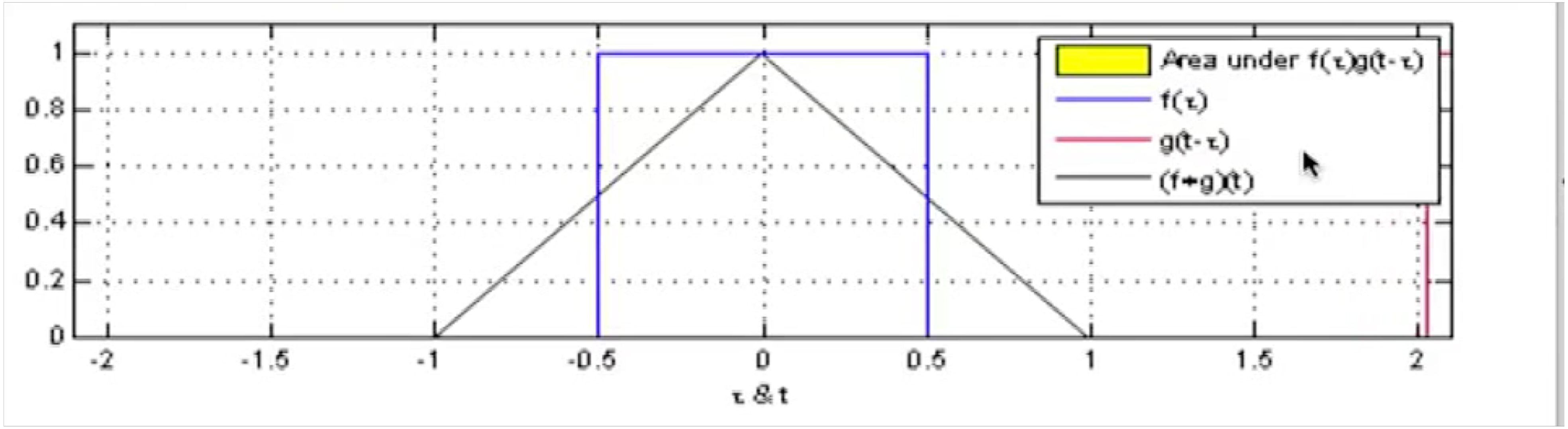
$$h(x) = f(x)*g(x)$$

$$\Leftrightarrow$$

$$F(s)\cdot G(s) = H(s)$$



Convolution: The Movie



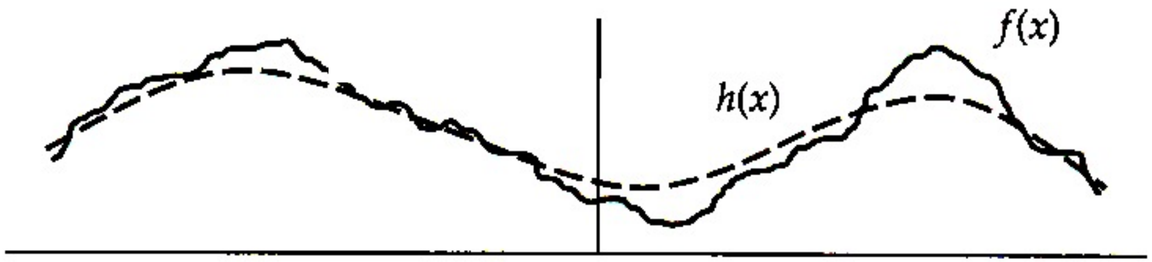
https://www.youtube.com/watch?v=a0IdGLczoAA

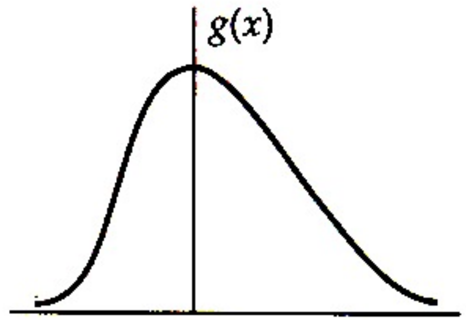
Convolution: Applications

$$f(x) * g(x) = h(x)$$

<u>Example:</u>
f(x) : object in sky
g(x): point spread function of telescope
h(x): observed image

<u>Example:</u>
Convolution of *f(x)* with a smooth kernel g(x) can be used to smoothen *f(x)*





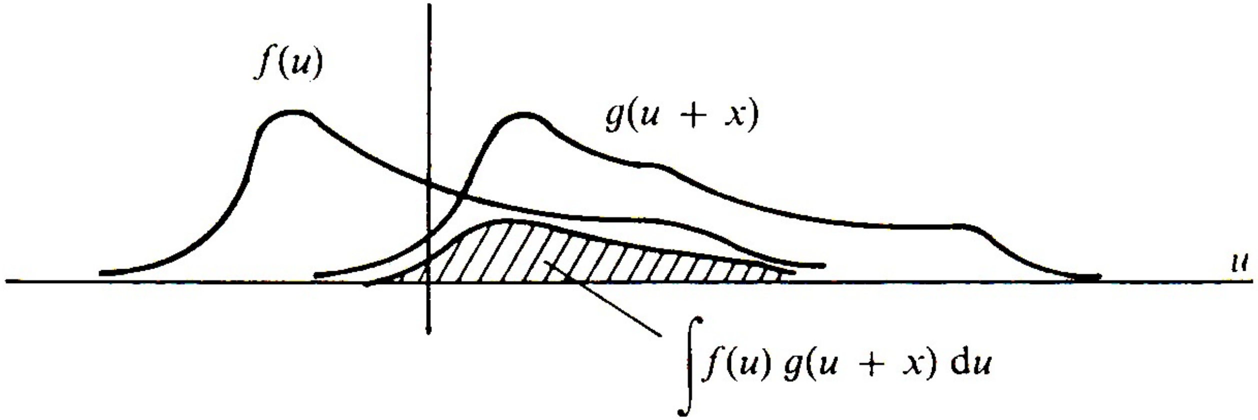
Cross-Correlation

Cross-correlation (or covariance) is measure of similarity of two waveforms as function of time-lag between them.

$$k(x) = f(x) \otimes g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x+u)\, du$$

Difference between cross-correlation and convolution:

- Convolution reverses the signal ('-' sign)
- Cross-correlation shifts the signal and multiplies it with another



Interpretation: By how much (x) must $g(u)$ be shifted to match $f(u)$? Answer given by maximum of $k(x)$

Cross-Correlation in Fourier Space

$$f(x) \Leftrightarrow F(s)$$

$$g(x) \Leftrightarrow G(s)$$

$$h(x) = f(x) \otimes g(x) \Leftrightarrow F(s) \cdot G^*(s) = H(s)$$

In contrast to convolution, in general

$$f \otimes g \neq g \otimes f$$

Interpretation of Cross-Correlation

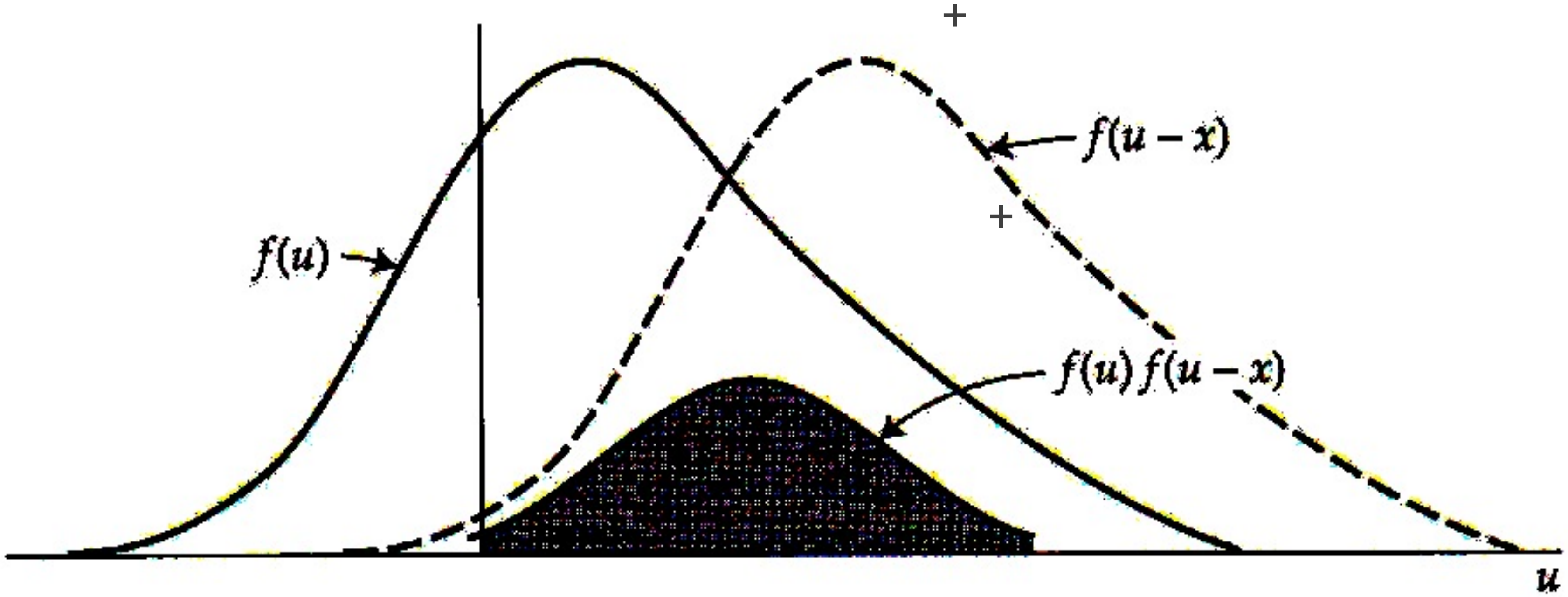
$$\min_{\Delta x,\Delta y} \sum_{x,y} \left[S_1(x,y) - S_2(x+\Delta x, y+\Delta y) \right]^2 =$$

$$\max_{\Delta x,\Delta y} \sum_{x,y} S_1(x,y) S_2(x+\Delta x, y+\Delta y)$$

Auto-Correlation Theorem

Auto-correlation is cross-correlation of function with itself:

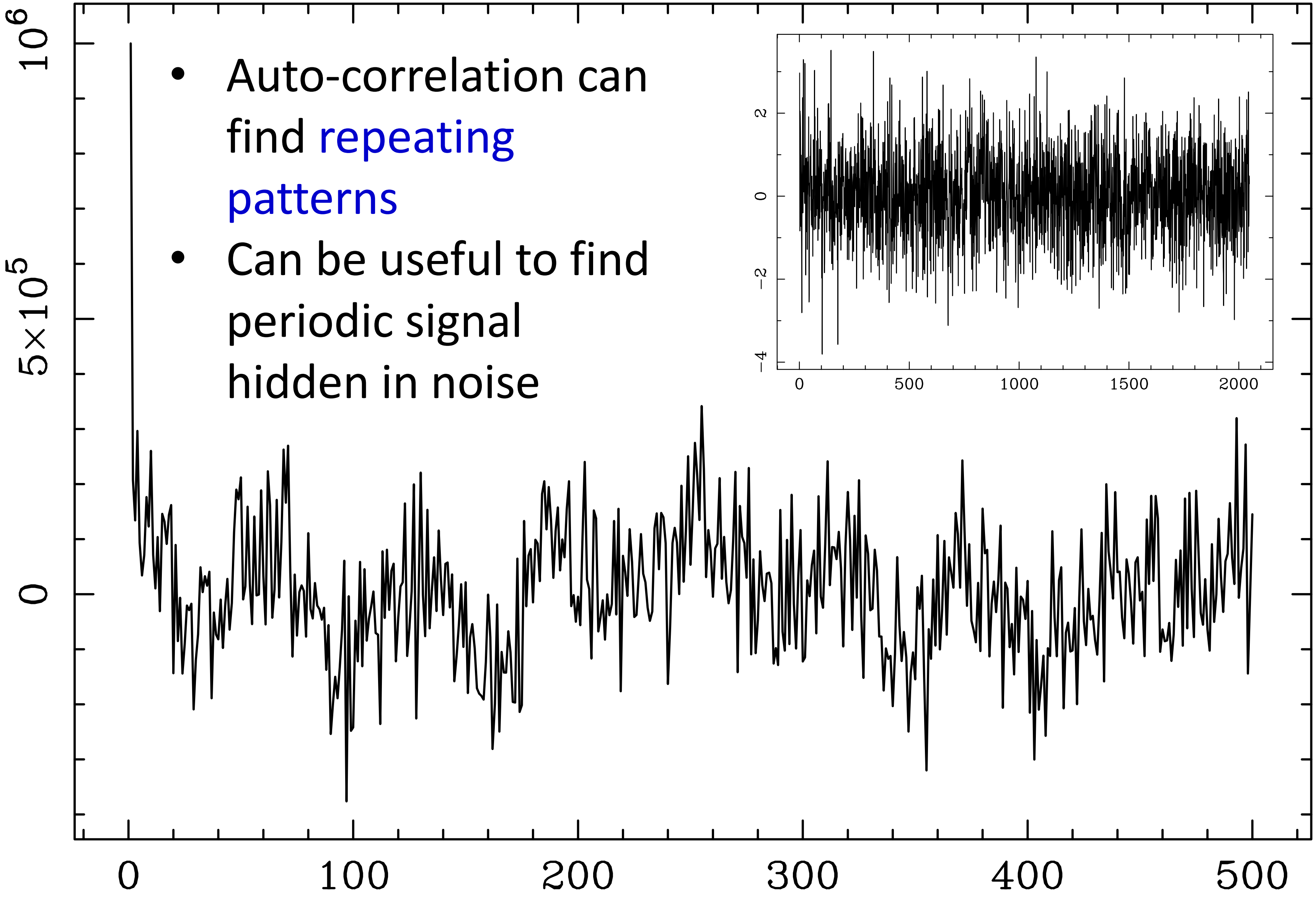
$$k(x) = f(x) \otimes f(x) = \int_{-\infty}^{+\infty} f(u) \cdot f(x+u)\, du$$



$$f(x) \otimes f(x) \Leftrightarrow F(s) F^*(s) = |F(s)|^2$$

Auto-Correlation: Application

- Auto-correlation can find repeating patterns
- Can be useful to find periodic signal hidden in noise



Power Spectrum

Power Spectrum S_f of $f(x)$ (or the Power Spectral Density, PSD) describes how the power of a signal is distributed with frequency.

Power is often defined as squared value of signal:

$$S_f(s) = |F(s)|^2$$

Power spectrum is Fourier transform of autocorrelation and indicates what frequencies carry most of the energy.

Total energy of a signal is: $\int_{-\infty}^{+\infty} S_f(s)\,ds$

Applications: spectrum analyzers, calorimeters of light sources, …

Parseval's Theorem

Parseval's theorem (or Rayleigh's Energy Theorem):
Sum of square of a function is the same as sum of square of the Fourier transform:

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |F(s)|^2 ds$$

<u>Interpretation:</u> Total energy contained in signal *f(x)*, summed over all x is equal to total energy of signal's Fourier transform *F(s)* summed over all frequencies *s*.

Wiener-Khinchin Theorem

Wiener–Khinchin theorem states that the power spectral density S_f of a function *f(x)* is the Fourier transform of its auto-correlation function:

$$|F(s)|^2 = FT\{f(x)\otimes f(x)\}$$

$$\updownarrow$$

$$F(s)\cdot F^*(s)$$

Applications: E.g. in the analysis of linear time-invariant systems, when the inputs and outputs are not square integrable, i.e. their Fourier transforms do not exist.

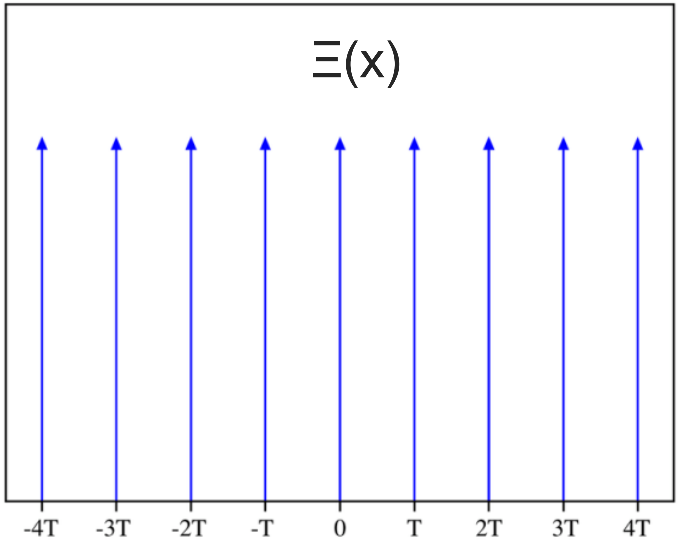
Equation Summary

Convolution	$h(x) = f(x) * g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x-u)\, du$
Cross-correlation	$k(x) = f(x) \otimes g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x+u)\, du$
Auto-correlation	$k(x) = f(x) \otimes f(x) = \int_{-\infty}^{+\infty} f(u) \cdot f(x+u)\, du$
Power spectrum	$S_f(s) = \lvert F(s) \rvert^2$
Parseval's theorem	$\int_{-\infty}^{+\infty} \lvert f(x) \rvert^2 dx = \int_{-\infty}^{+\infty} \lvert F(s) \rvert^2 ds$
Wiener-Khinchin theorem	$\lvert F(s) \rvert^2 = FT\{f(x) \otimes f(x)\} = F(s) \cdot F^*(s)$

Dirac Comb

Dirac's delta "function":

$$f(x) = \delta(x) = \int_{-\infty}^{+\infty} e^{i2\pi sx}\, ds \;\rightarrow\; FT\{\delta(x)\} =$$



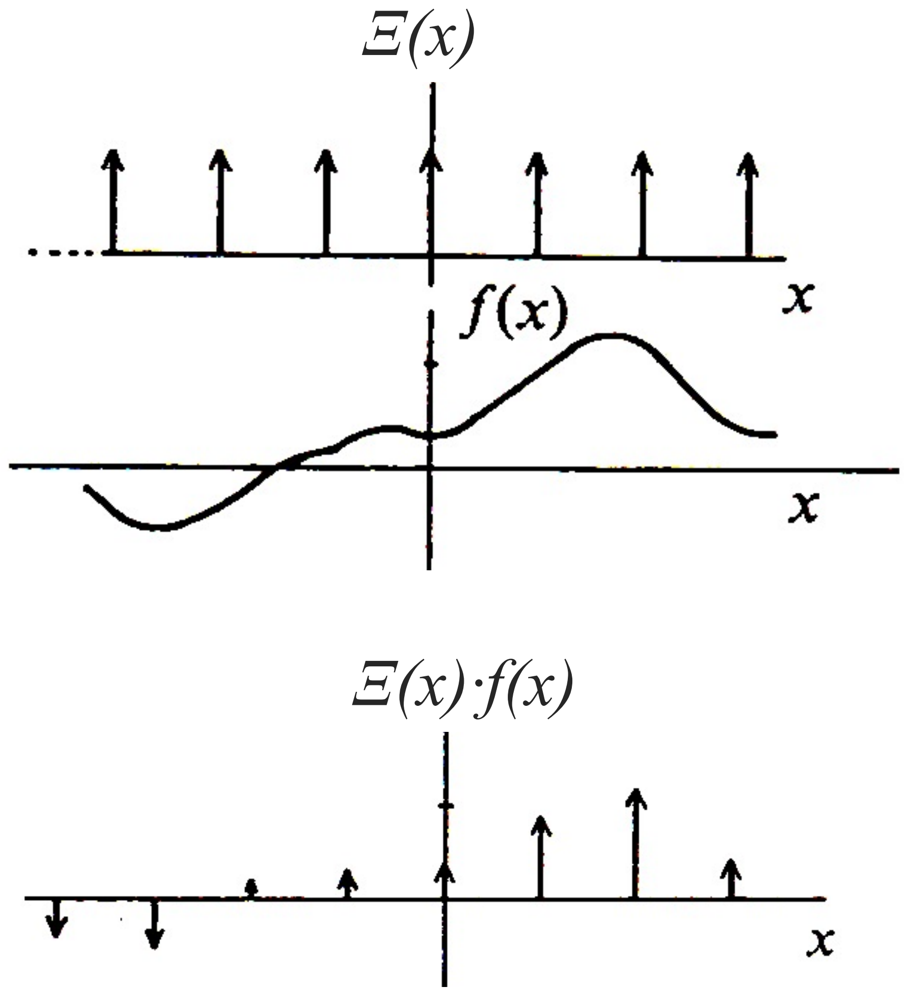
Dirac comb: infinite series of delta functions spaced at intervals of T:

$$\Xi_T(x) = \sum_{k=-\infty}^{\infty} \delta(x - kT) \underset{series}{\overset{Fourier}{=}} \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i2\pi nx/T}$$

- Fourier transform of Dirac comb is also a Dirac comb
- Dirac comb is also called impulse train or sampling function

5. Sampling

- Signal only sampled at discrete points in time
- Often constant sampling interval
- Sampling can be described as multiplication of true signal with Dirac comb
- Fourier transform of sampled signal is sampled Fourier transform of true signal



Nyquist Theorem

Sampling: signal at discrete values of x: $$f(x) \to f(x) \cdot \Xi\left(\frac{x}{\Delta x}\right)$$

Interval between two successive readings is sampling rate

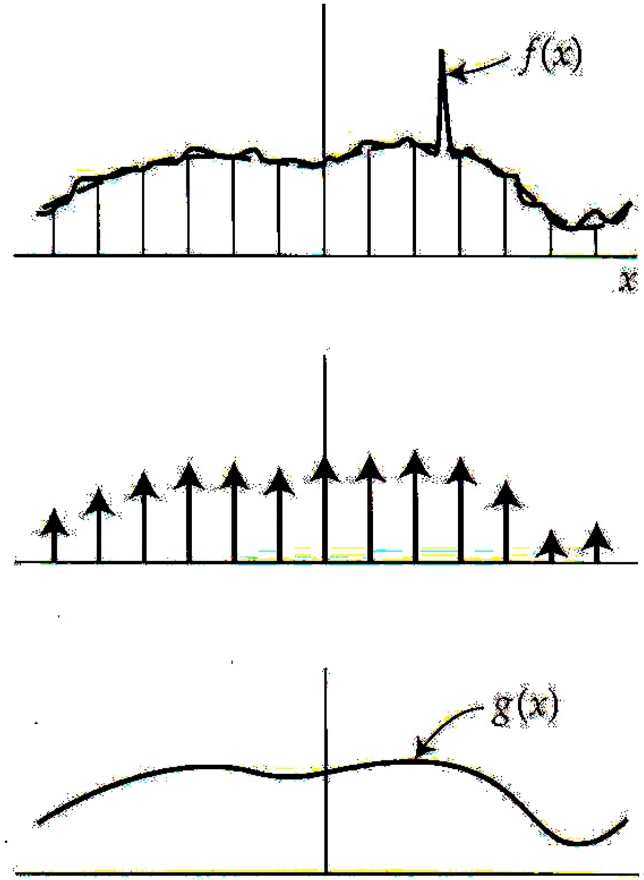
Critical sampling given by Nyquist theorem

Given f(x), its Fourier Transform F(s) defined on $[-s_{max}, s_{max}]$.

Sampled distribution of the form

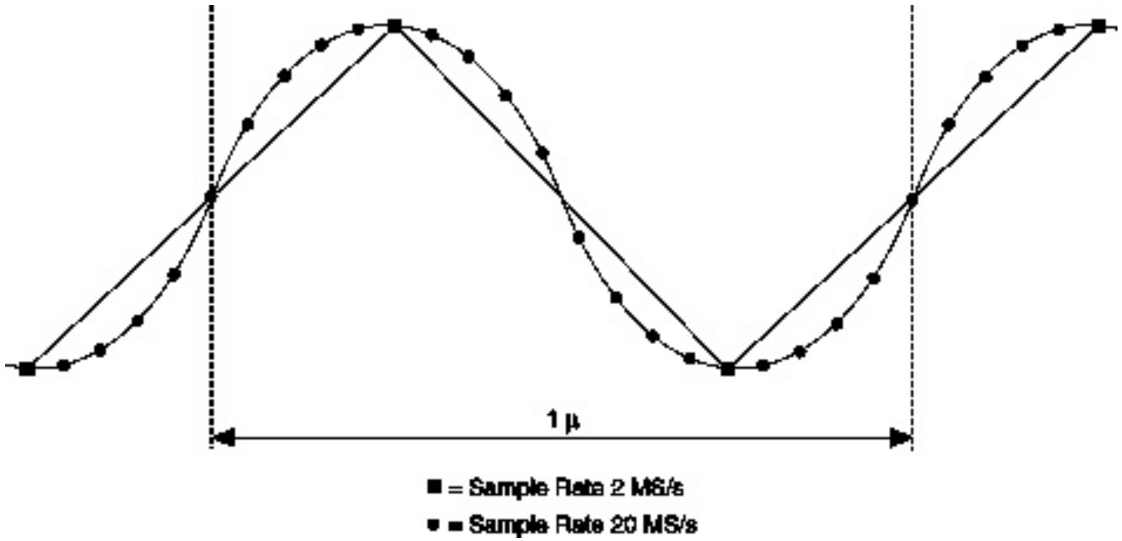
$$g(x) = f(x) \cdot \Xi\left(\frac{x}{\Delta x}\right)$$

with a sampling rate of $\Delta x = 1/(2s_{max})$ is enough to reconstruct *f(x)* for all *x*.

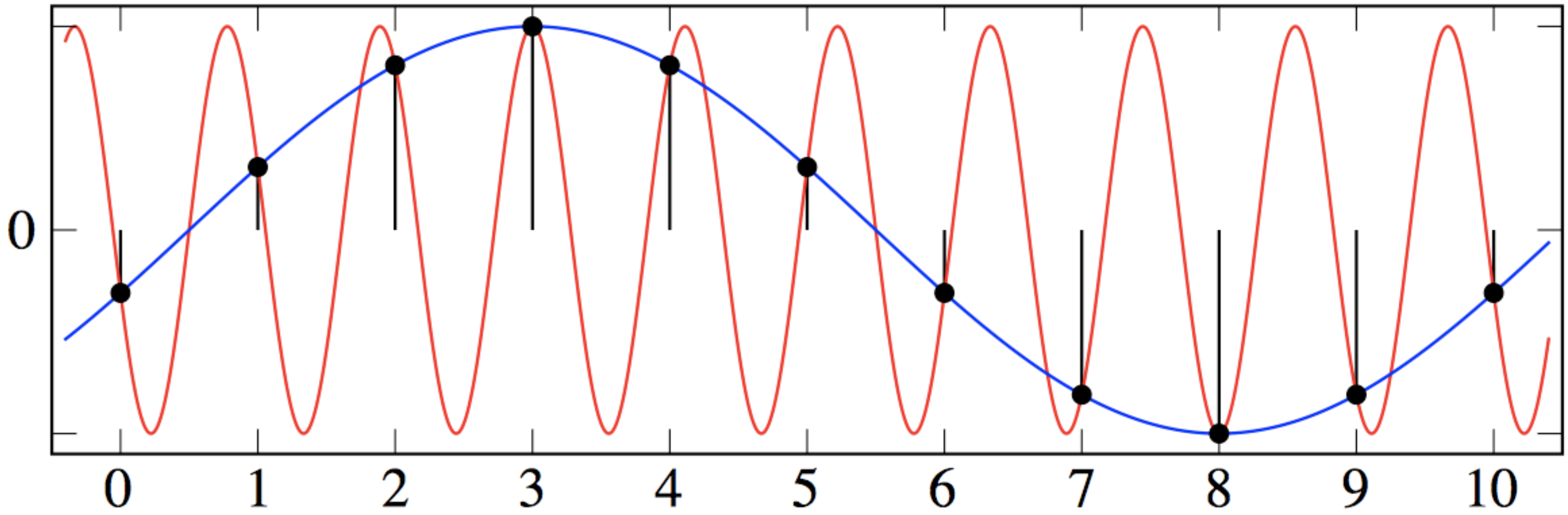


Sampling Rate

Oversampling	Sampling rate above critical sampling rate: - redundant measurements/too much data - often lowering the S/N
Undersampling	Sampling rate below critical sampling rate: - signal contains frequencies higher than $1/(2s_{max})$ - source signal cannot be determined after sampling - loss of fine details - must apply low-pass filter before sampling

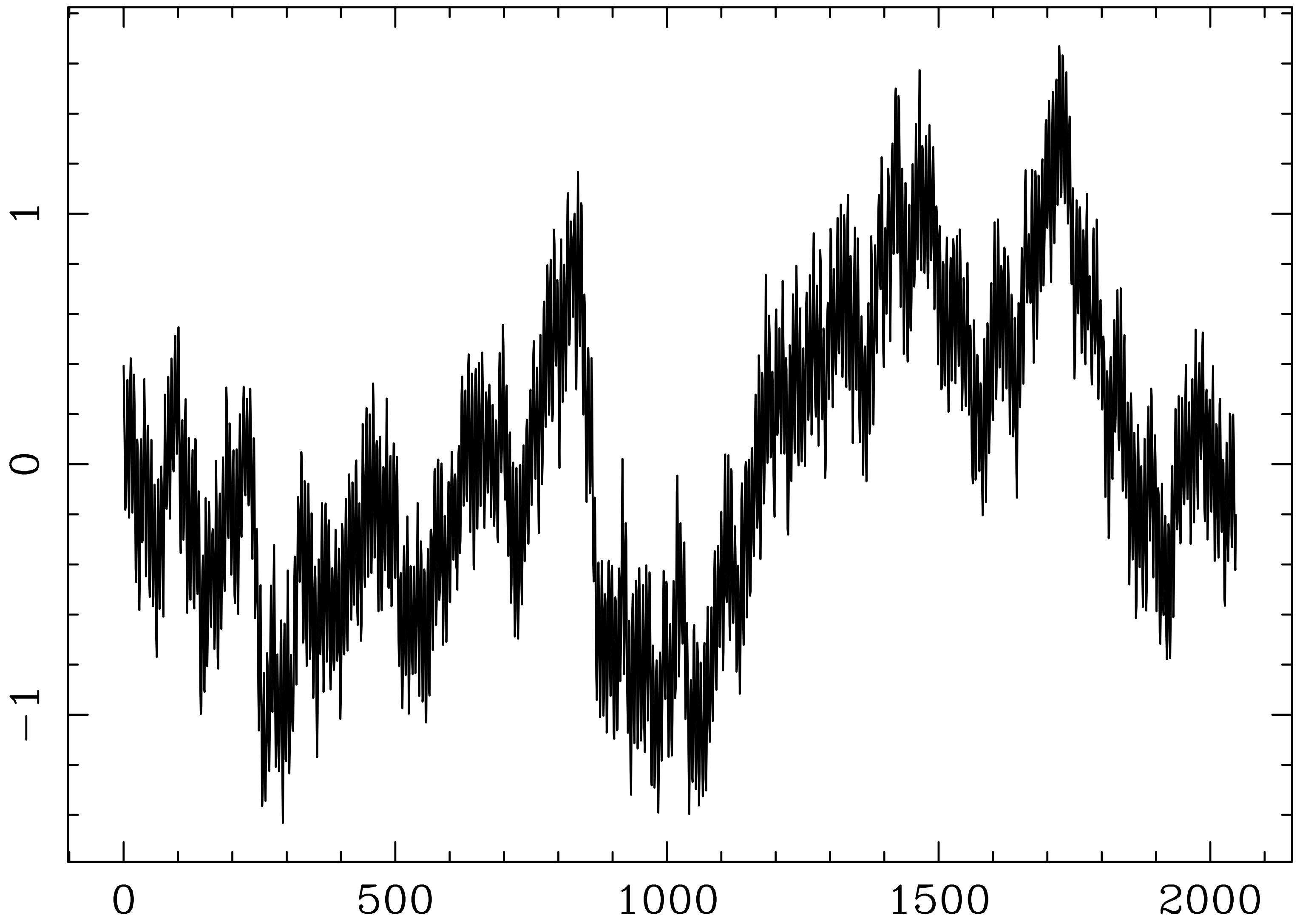


Aliasing

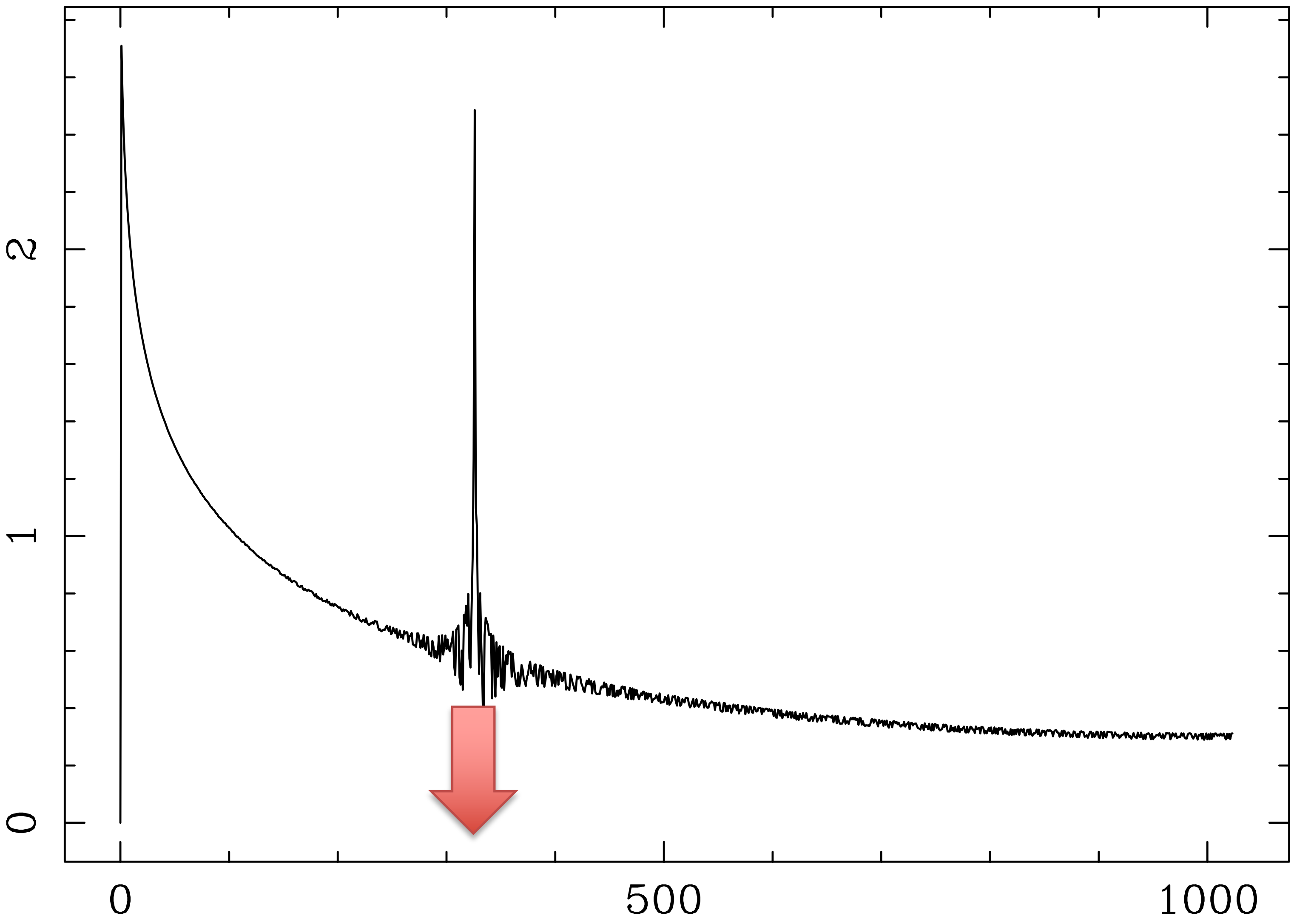


- undersampled, high frequencies look like well-sampled low frequencies
- create spurious components below Nyquist frequency
- may create major problems and uncertainties in determination of original signal

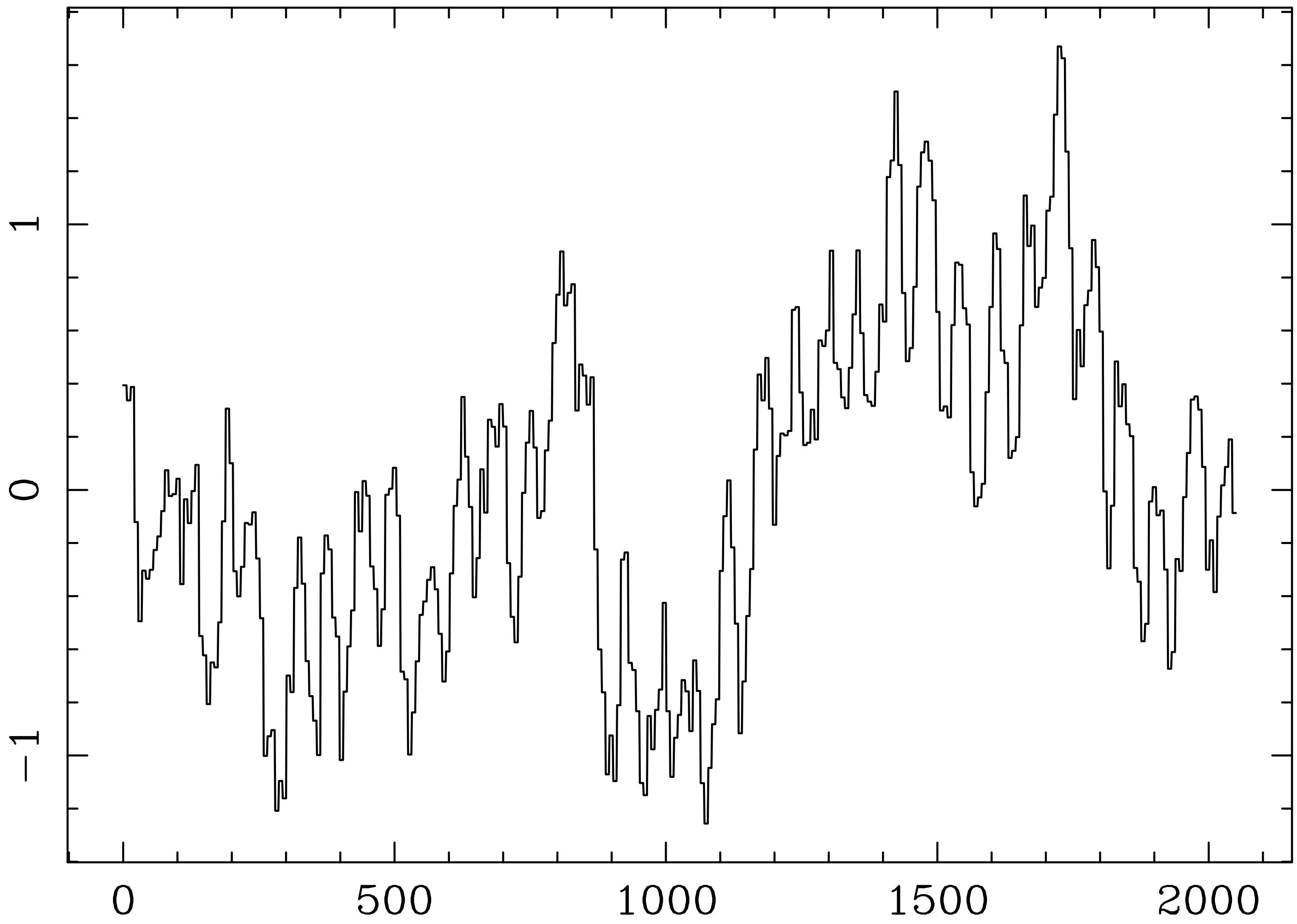
1/f noise with periodic signal



Fourier Transform of Well-Sampled Signal



Undersampled Signal



FT of Undersampled Signal