Assignments 4

March 2 2014

Exercise 1:

Consider an adiabatic process.

i. Use the first law of thermodynamics (dU+PdV=0) and the ideal gas law to write the pressure-temperature relation as P~ρ^{Γ_1} and give the value for Γ_1.

ii. Derive the expression for the adiabatic temperature gradient,

$$\nabla_a = \left(\frac{d\ln T}{d\ln P}\right)_{ad} = \left(\frac{\gamma - 1}{\gamma}\right)$$

iii. Show that for a mixture of an ideal gas plus radiation, the adiabatic exponent and the adiabatic temperature gradient are given by

$$\Gamma_1 = \frac{32 - 24\beta - 3\beta^2}{24 - 21\beta}$$

$$\nabla_{ad}^{-1} = \frac{32 - 24\beta - 3\beta^2}{2(4 - 3\beta)}$$

Hint: start with

$$\frac{dP}{P} = \chi_T \frac{dT}{T} + \chi_\rho \frac{d\rho}{\rho}$$

and use the first law of thermodynamics.

iv. What are the limits when gas pressure and radiation pressure dominate ?

Exercise 2:

In the temperature-density plane sketch the regimes where the ideal gas law applies, and where radiation pressure (P_{rad}), and non-relativistic ($P_{e,deg}$) and relativistic degenerate electron pressure ($P_{e,r\text{-}deg}$) dominate. As boundaries use, $P_{rad} = 10\,P_g$, $P_g = P_{e,deg}$, $P_{e,deg} = P_{e,r\text{-}deg}$.

Exercise 3:

In class, we have derived the Saha equation for a fiducial H-atom with one bound state and free electrons/protons starting with Fermi-Dirac statistics and the chemical potential. The partition functions are very simple then. Here, you will derive the Saha equation starting from the Boltzmann equation, taking the partition function into account.

$$n^0 \leftrightarrow n^+ + e^-$$

i. Write the Boltzmann equation relating the abundances of neutral H-atoms in the ground state and protons in the ground state. Give an expression for the energy E in this equation.

ii. The statistical weight on the right-hand-side includes ion and electron. Evaluate the statistical weight of the free electron. Use that

$$dxdydz = V_e = 1/n_e$$

Remember Pauli !

iii. Write an expression for the fraction of the protons in the ground-state in terms of all protons.

iv. Write an expression for the fraction of the H-atoms in the ground-state in terms of all H-atoms. Keep explicitly track of the nuclear spin state of the H atom.

v. What is the proton partition function ? And what is the H-atom partition function ? In these evaluations, allow for the nuclear spin.

vi. The proton partition function is 1. Often when the Saha equation is used to evaluate the ionization fraction around $T \approx 10^4$ K, the neutral hydrogen partition function is set equal to 2. Why is this a reasonable approximation ?