CH 5.5 AND CH 6

CONVECTION& NUCLEAR ENERGY

SUMMARY PREVIOUS CLASS

POLYTROPIC STELLAR MODELS

$$\boxed{\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}} \quad + \quad \frac{dP}{dr} = -\frac{Gm}{r^2}\rho, \quad + \quad P = K\rho^\gamma =$$

$$n = \frac{1}{\gamma - 1} \qquad \text{or} \qquad \gamma = 1 + \frac{1}{n}$$

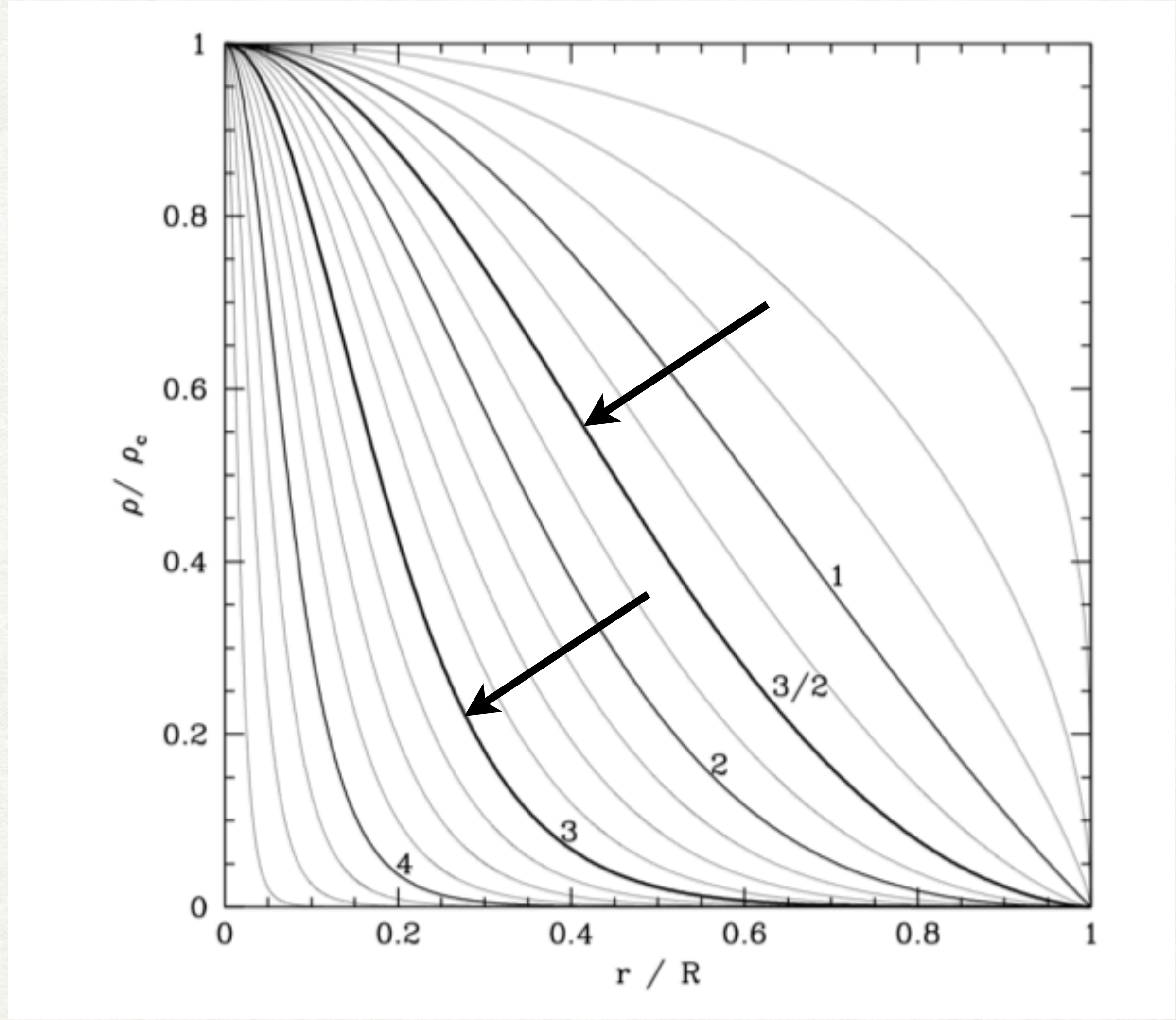
only models with n< 5 are stars

$$n = 3/2 \rightarrow \gamma = \frac{5}{3}$$

$$M \propto R^{-3}$$

$$n = 3 \rightarrow \gamma = \frac{4}{3}$$

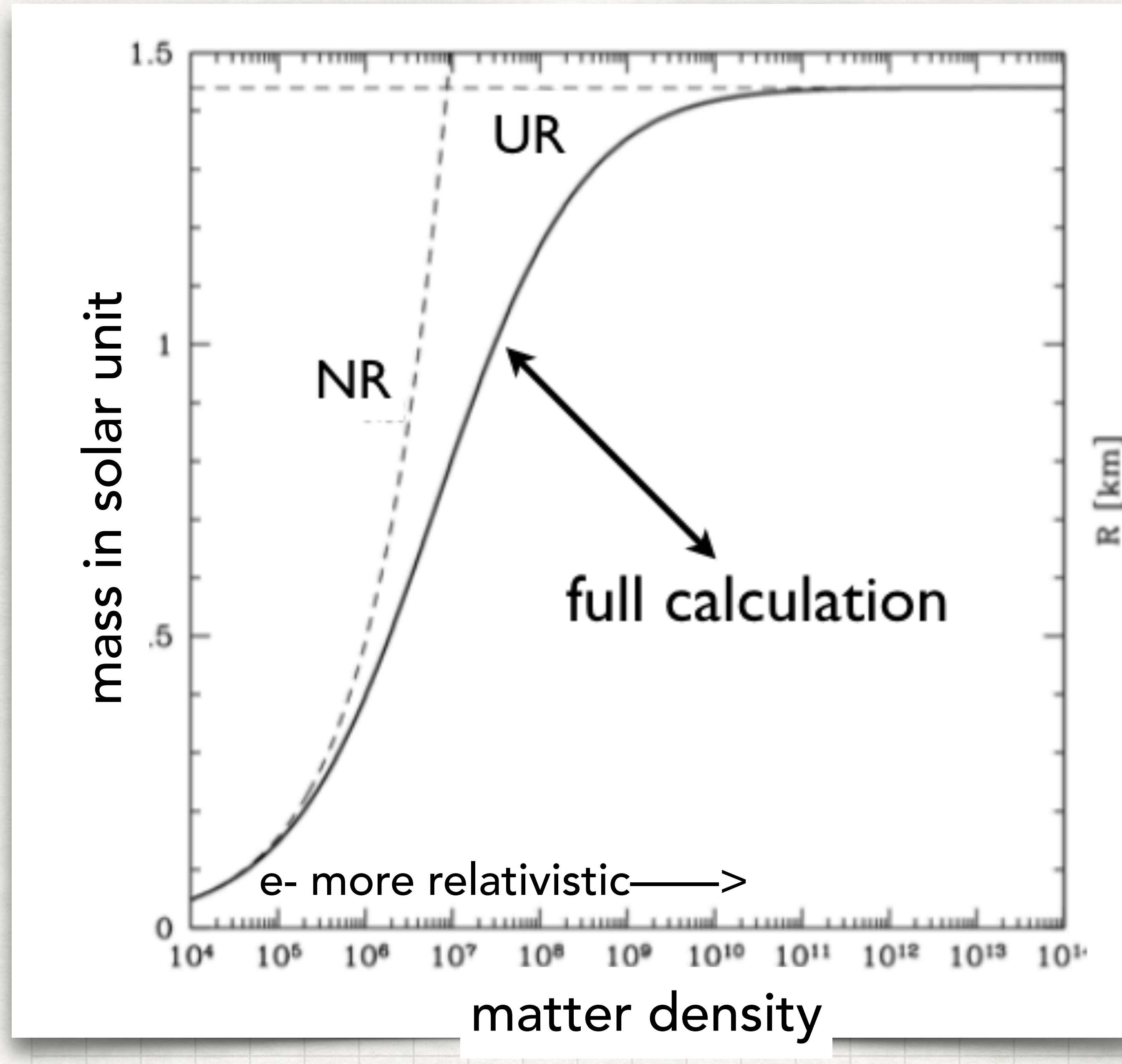
$$M = f(K) = \text{ independent of radius}$$



LANE-EMDEN SOLUTIONS

3 IMPORTANT INSIGHTS

- 1. White Dwarfs (n=3/2 NR): the more massive the smaller
- 2. There is a maximum predicted mass for White Dwarfs (Chandrasekhar's mass), when electrons become fully relativistic (n=3)
- 3. Main sequence stars: the more massive the more radiation pressure in important (Eddington's model)



$$P = \left(\frac{3\mathcal{R}^4}{a\mu^4} \frac{1-\beta}{\beta^4} \right)^{1/3} \rho^{4/3}$$

Polytropic relation with n=3

$$M = 4\pi\,\Theta_3 \left(\frac{K}{\pi G} \right)^{3/2}$$

SUMMARY CONT.

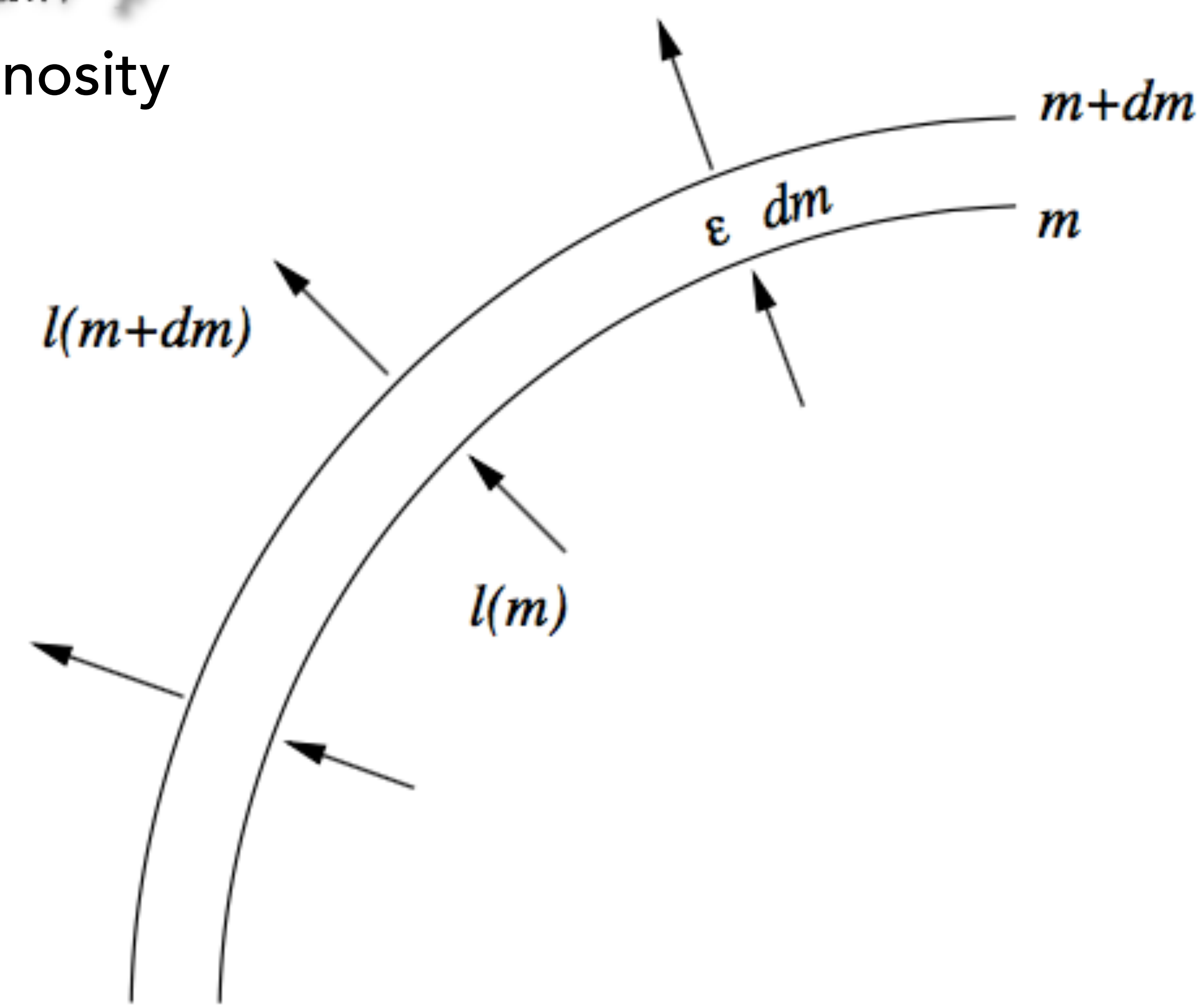
LOCAL ENERGY CONSERVATION EQUATION

$$\frac{\mathrm{d}l}{\mathrm{d}m} = \epsilon_{\mathrm{nuc}} - \epsilon_{\nu}.$$

equilibrium = no changes in time

$l = a\pi r^2 F$

luminosity



At equilibrium over whole star

nuclear reaction luminosity

$$L = \int_0^M \epsilon_{\mathrm{nuc}}\,\mathrm{d}m - \int_0^M \epsilon_{\nu}\,\mathrm{d}m \equiv L_{\mathrm{nuc}} - L_{\nu}$$

neutrino luminosity

SUMMARY CONT.

RADIATIVE/CONDUCTIVE ENERGY TRANSPORT

- **HEAT DIFFUSION** → **radiative diffusion**; → **heat conduction**

  random thermal motion of particles

$$\boldsymbol{F}_{\rm rad} = -K_{\rm rad}\,\boldsymbol{\nabla} T = -\frac{4}{3}\frac{acT^3}{\kappa\rho}\boldsymbol{\nabla} T.$$

$$\boxed{\nabla_{\rm rad} = \left(\frac{\mathrm{d}\log T}{\mathrm{d}\log P}\right)_{\rm rad} = \frac{3}{16\pi acG}\frac{\kappa l P}{m T^4}}$$

Dominant in main sequence stars

$$\boxed{F_{\rm cd} = -K_{\rm cd}\nabla T}$$

Dominant in WD

$$\boldsymbol{F} = -\frac{4acT^3}{3\kappa\rho}\boldsymbol{\nabla} T \qquad \text{with} \qquad \frac{1}{\kappa} = \frac{1}{\kappa_{\rm rad}} + \frac{1}{\kappa_{\rm cd}}$$

total

SUMMARY CONT.

EDDINGTON LUMINOSITY

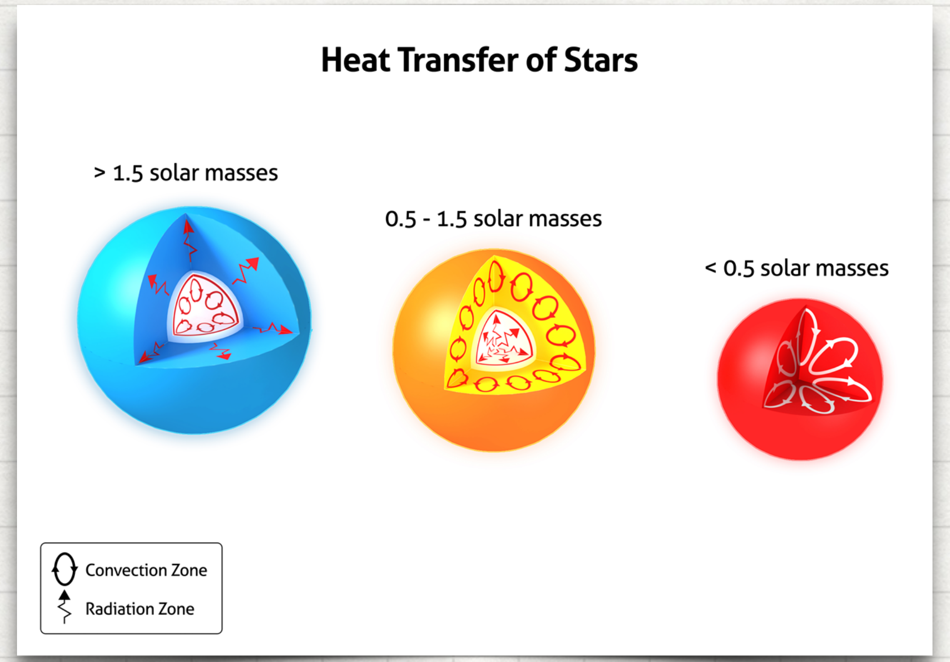
We require: $$\frac{1}{\rho}\frac{dP_{\mathrm{rad}}}{dr} \leq \frac{Gm}{r^2}$$

and using the radiative temperature gradient we get:

$$l \leq \frac{Gm4\pi c}{\kappa\rho} = 3.8 \times 10^4 \left(\frac{m}{M_{\mathrm{sun}}}\right)\left(\frac{0.34}{\kappa}\right) L_{\mathrm{sun}} \equiv L_{\mathrm{edd}}$$

- $l=L_{edd}$ when P ~ P_{rad}: i.e. for radiation dominated stars=> for massive stars
- Since L ~M^x x > 1 eventually L>=L_{edd} as M increases
- $l>L_{edd}$ in zones of large opacity (low T) like outer layers of Sun

When l > L_{edd} convection must take over to ensure hydrostatic equilibrium



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CONVECTION

Ch 5.5 notes

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CONVECTION

- Cyclic macroscopic motions of the gas causing a net heat flux against the direction of gravity without net mass displacement

- It can be described as an instability which actually leads to a stable stellar configuration: small density/temperature fluctuations/perturbations that grow (or not)

A (IN)STABILITY CRITERIUM

INVOLVING THE DENSITY GRADIENT

- eddy has same pressure as surrounding medium
- Eddy's expansion/contraction occurs adiabatically = $v<c_s$

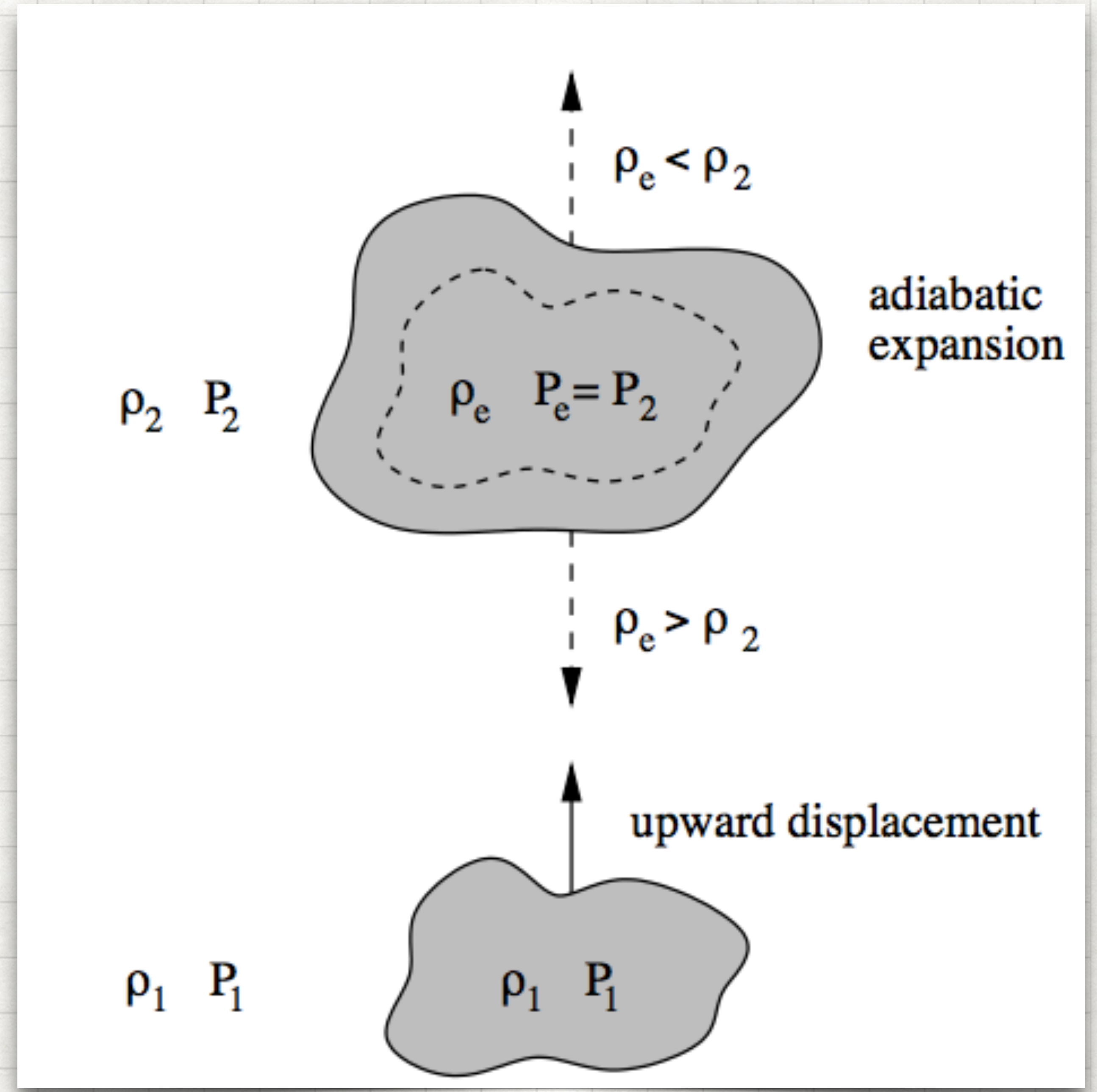
$$\frac{\delta P_e}{P_e} = \gamma_{\rm ad} \frac{\delta \rho_e}{\rho_e}$$

stability

$$\partial\rho_{\rm e} > \frac{d\rho}{dr}\Delta r$$

$$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} < \frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}r}\frac{1}{\gamma_{\rm ad}},$$

$$\frac{\mathrm{d}\log\rho}{\mathrm{d}\log P} > \frac{1}{\gamma_{\rm ad}}.$$

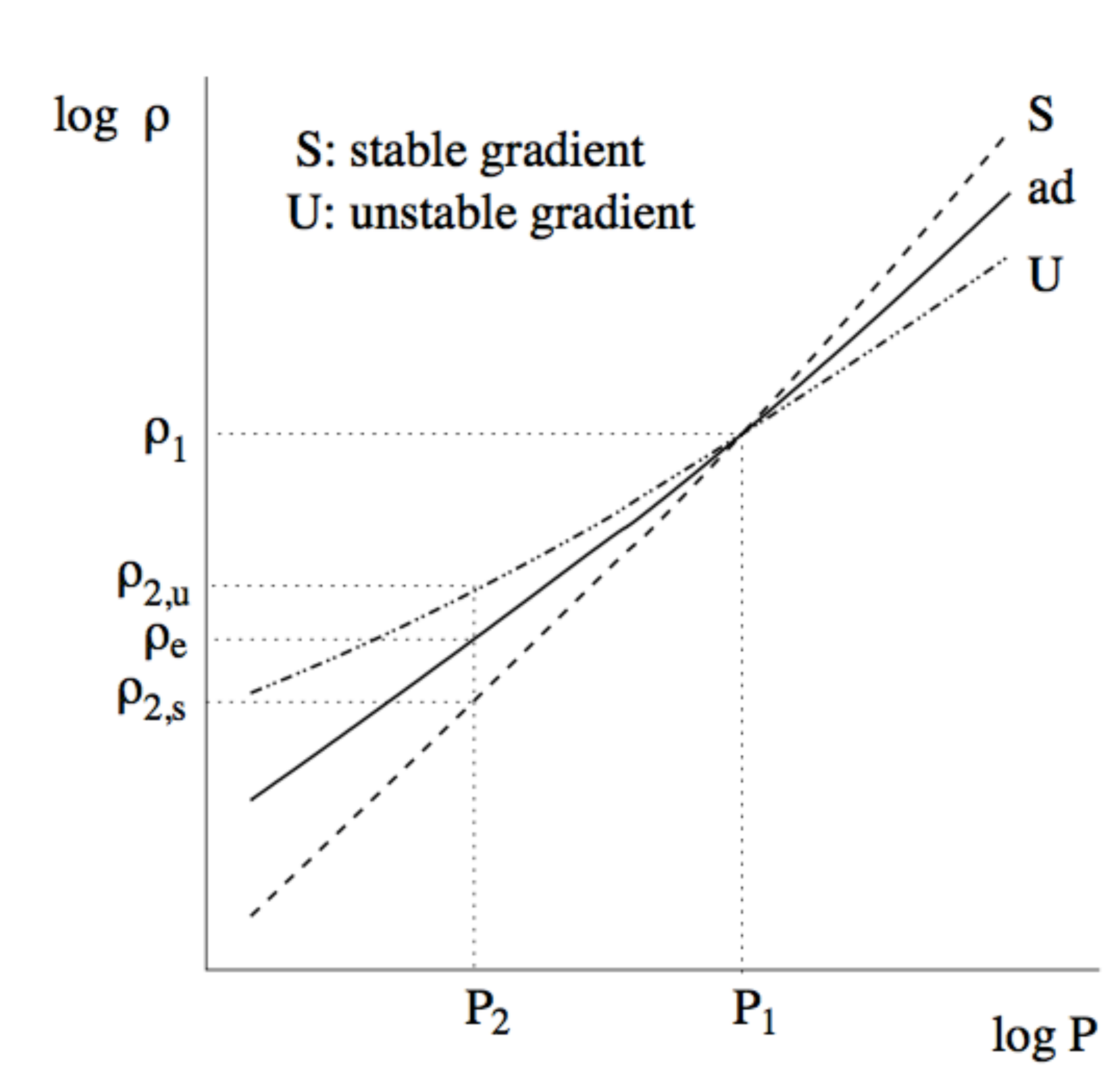


A (IN)STABILITY CRITERIUM

INVOLVING THE DENSITY GRADIENT

stability against convection if density profile steeper than adiabatic

$$\boxed{\frac{d \log \rho}{d \log P} > \frac{1}{\gamma_{\mathrm{ad}}}.}$$



THE SCHWARZSCHILD AND LEDOUX CRITERIA

A STABILITY CRITERIA INVOLVING <u>TEMPERATURE GRADIENTS</u>

in parallel with the eq. of radiative transfer

To go from a criterium in density grad to one in temperature grad we use the equation of state:

$$\frac{d\log\rho}{d\log P} > \frac{1}{\gamma_{ad}}.$$

use definition of adiabatic index and temperature gradient

$$P = P(\rho, T, \mu)$$

$$\gamma_{ad} = \left(\frac{\partial \log P}{\partial \log\rho}\right)_{ad}$$

$$\frac{d\log\rho}{d\log P} = \frac{1}{\chi_\rho}\left(1 - \chi_T \frac{d\log T}{d\log P} - \chi_\mu \frac{d\log\mu}{d\log P}\right) =$$

$$\nabla_{ad} = \left(\frac{\partial \log T}{\partial \log P}\right)_{ad}$$

$$= \frac{1}{\chi_\rho}\left(1 - \chi_T\nabla - \chi_\mu\nabla_\mu\right) = \frac{1}{\gamma_{ad}} = \frac{1}{\chi_\rho}\left(1 - \chi_T\nabla_{ad}\right)$$

$$\nabla < \nabla_{ad} - \frac{\chi_\mu}{\chi_T}\nabla_\mu$$

μ mean molecular weight: a proxy for composition

$$\frac{dx}{x} = d\log x$$

ADIABATIC GRADIENT FOR IDEAL GAS

CH 3.4.2

- The *adiabatic temperature gradient* is defined as

$$\nabla_{\text{ad}} = \left(\frac{\partial \log T}{\partial \log P}\right)_{\text{ad}}$$

$$\nabla_{\text{ad}} = \frac{\gamma_{\text{ad}} - \chi_\rho}{\gamma_{\text{ad}} \chi_T}, \tag{3.63}$$

This gives the following limiting cases:

- for an ideal gas without radiation ($\beta = 1$) we have $\chi_T = \chi_\rho = 1$, which together with $\gamma_{\text{ad}} = \frac{5}{3}$ gives $\nabla_{\text{ad}} = \frac{2}{5} = 0.4$.
- for a radiation-dominated gas ($\beta = 0$) $\chi_T = 4$ and $\chi_\rho = 0$ so that $\nabla_{\text{ad}} = \frac{1}{4} = 0.25$.

$$P_{\text{rad}} = \frac{1}{3} a T^4$$

$$P_{\text{gas}} = \beta P = \frac{R}{\mu} \rho T$$

where χ_T and χ_ρ are defined as

$$\chi_T = \left(\frac{\partial \log P}{\partial \log T}\right)_{\rho, X_i} = \frac{T}{P}\left(\frac{\partial P}{\partial T}\right)_{\rho, X_i} = 1$$

$$\chi_\rho = \left(\frac{\partial \log P}{\partial \log \rho}\right)_{T, X_i} = \frac{\rho}{P}\left(\frac{\partial P}{\partial \rho}\right)_{T, X_i} = -1$$

STABILITY CRITERIA FOR CONVECTION

A STABILITY CRITERIA INVOLVING TEMPERATURE GRADIENTS

$$\nabla < \nabla_{\mathrm{ad}} - \frac{\chi_\mu}{\chi_T}\nabla_\mu$$

ideal gas

$$\nabla < \nabla_{\mathrm{ad}} + \overset{>0}{\nabla_\mu}$$

composition makes it more stable

If energy transport is radiative

$$\nabla_{\mathrm{rad}} < \nabla_{\mathrm{ad}} - \frac{\chi_\mu}{\chi_T}\nabla_\mu$$

Ledoux

$$\nabla_{\mathrm{rad}} < \nabla_{\mathrm{ad}}$$

Schwarzschild $\nabla_\mu = 0$

Actual temperature gradient in a radiative zone:

$$\boxed{\nabla_{\mathrm{rad}} = \left(\frac{d\log T}{d\log P}\right)_{\mathrm{rad}} = \frac{3}{16\pi acG}\frac{\kappa l P}{mT^4}}$$

temperature variation in a specific gas element undergoing an adiabatic change in pressure

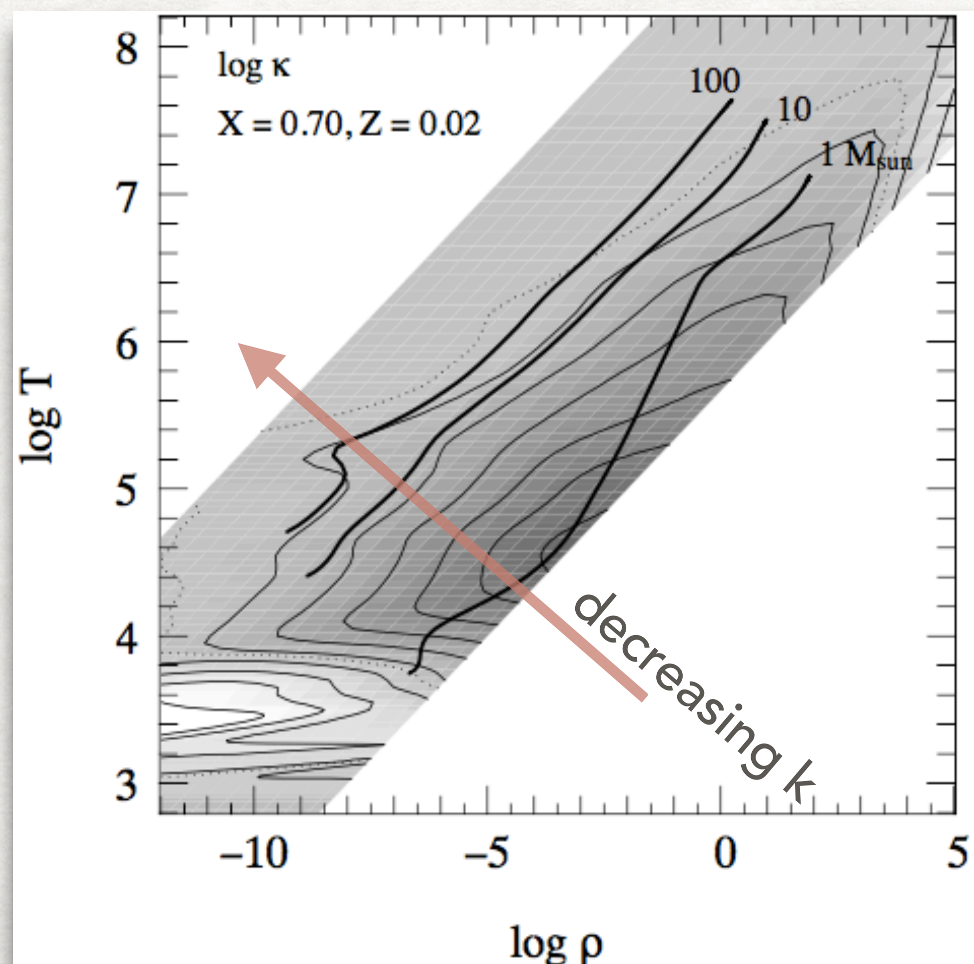
IMPLICATIONS

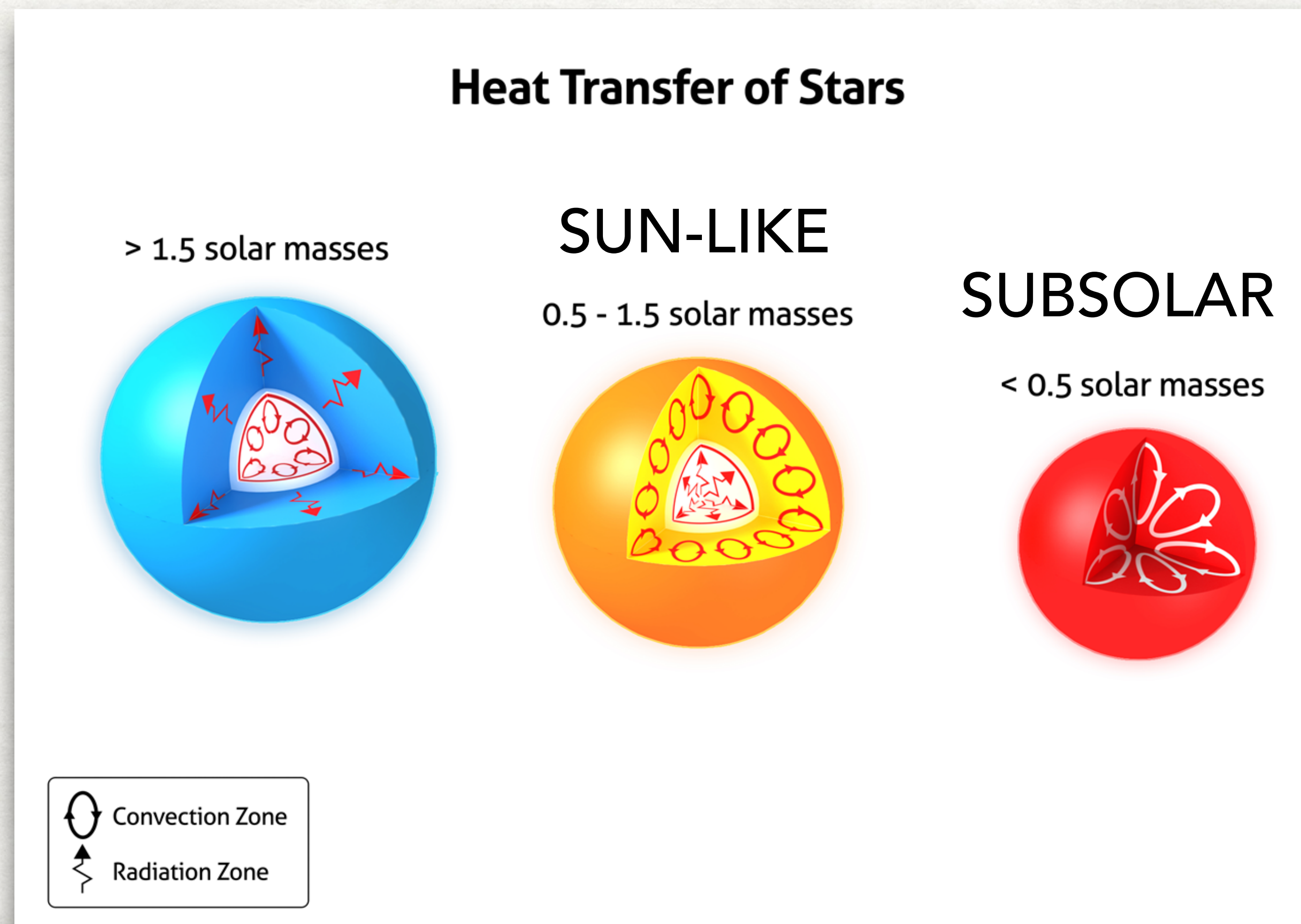
CONVECTION OCCURS IF

$$\nabla_{\rm rad} = \frac{3}{16\pi acG}\frac{P\kappa l}{T^4 m} > \nabla_{\rm ad}.$$

IT REQUIRES:

1. **A large value of the opacity : convection occurs in the more opaque regions of the stars.** Examples: outer envelope of the Sun and cool stars as opacity increases with lower T. Cool stars may be mostly convective





IMPLICATIONS

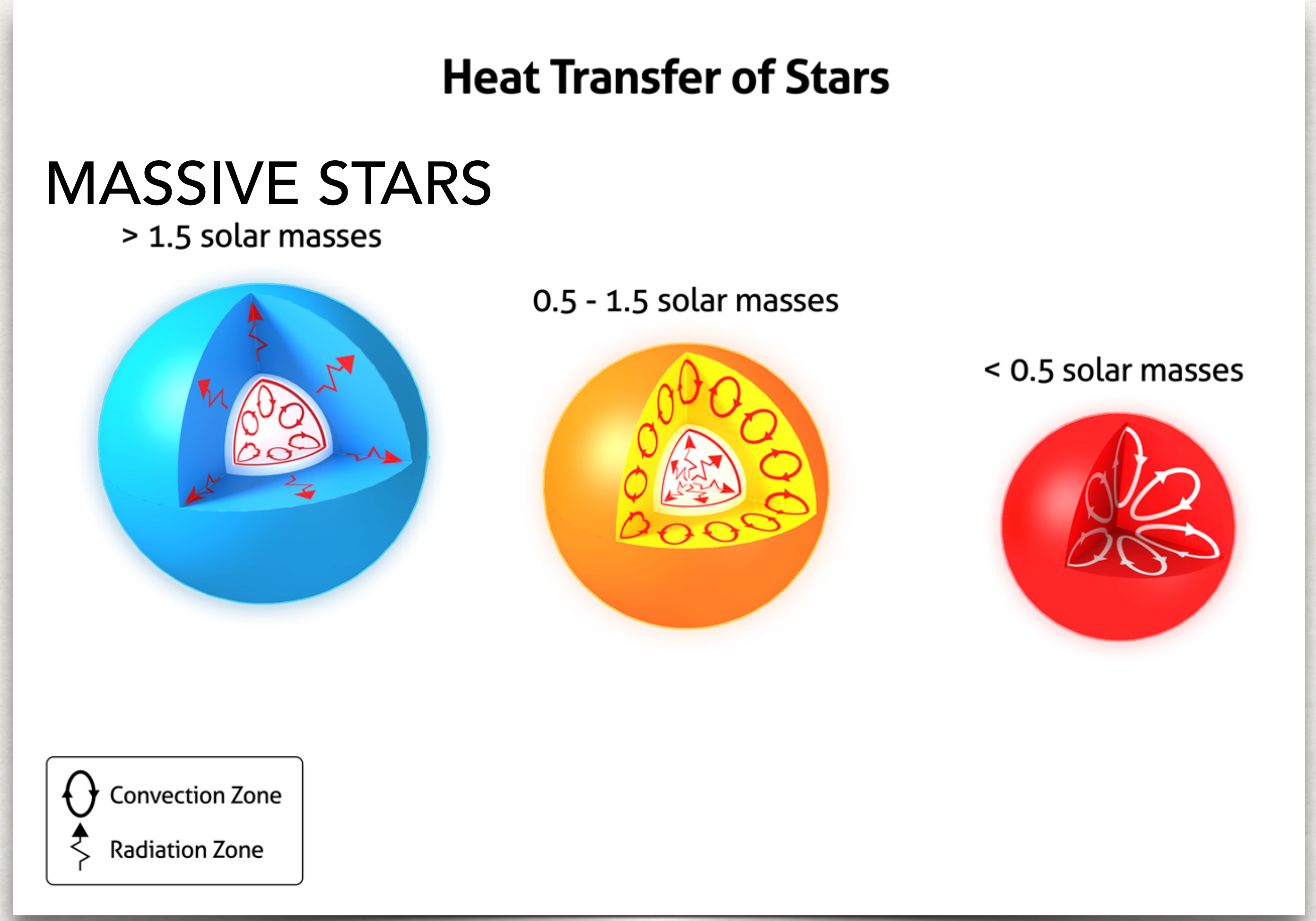
CONVECTION OCCURS IF

$$\nabla_{\mathrm{rad}} = \frac{3}{16\pi acG} \frac{P}{T^4} \frac{\kappa l}{m} > \nabla_{\mathrm{ad}}.$$

IT REQUIRES:

2.large value of l/m => large energy flux. In the nuclear region $l/m \approx \epsilon_{\mathrm{nuc}}$

relative massive stars have nuclear energy production concentrated in the relative small cores: they typically have convective inner regions



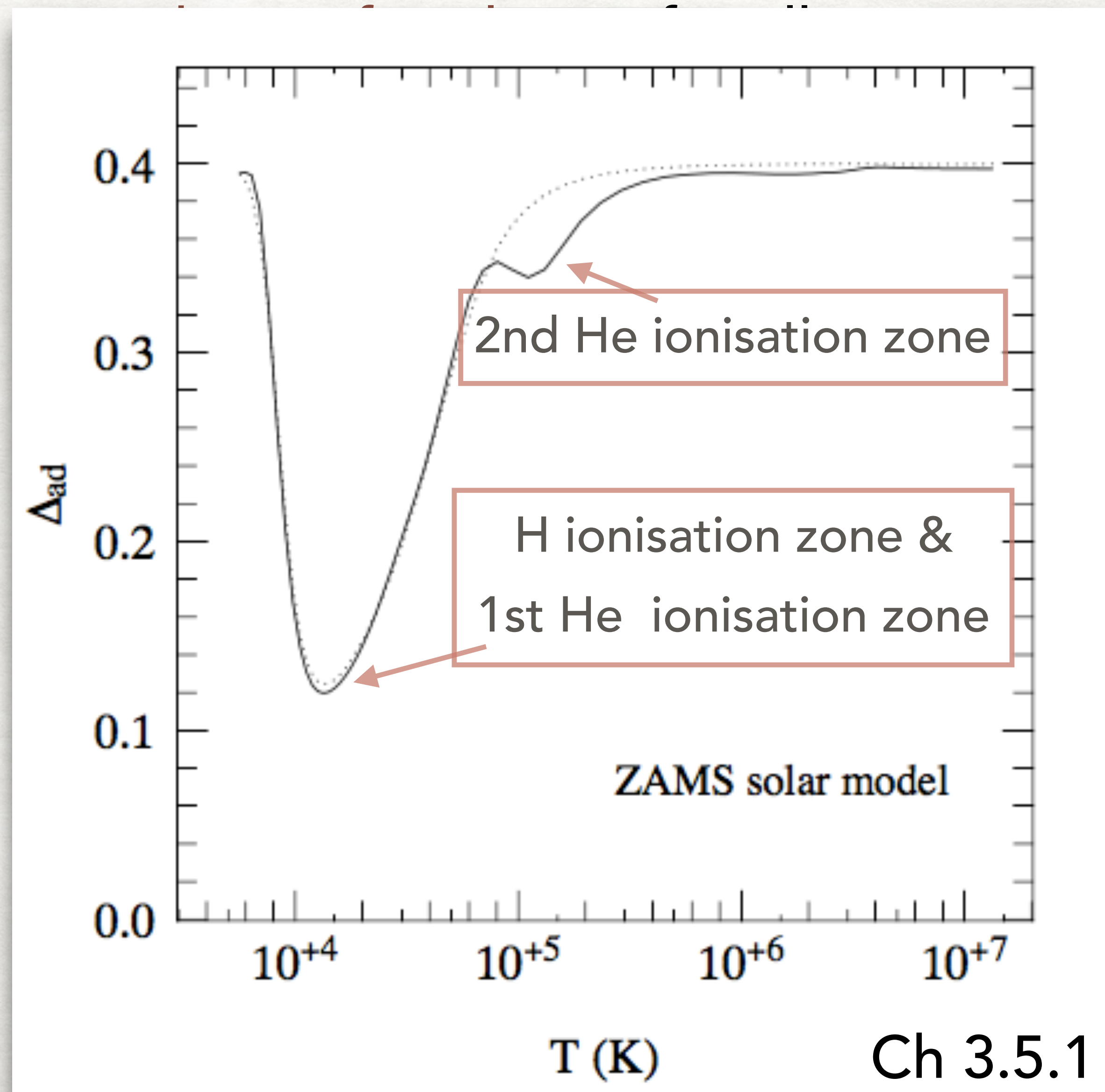
IMPLICATIONS

CONVECTION OCCURS IF

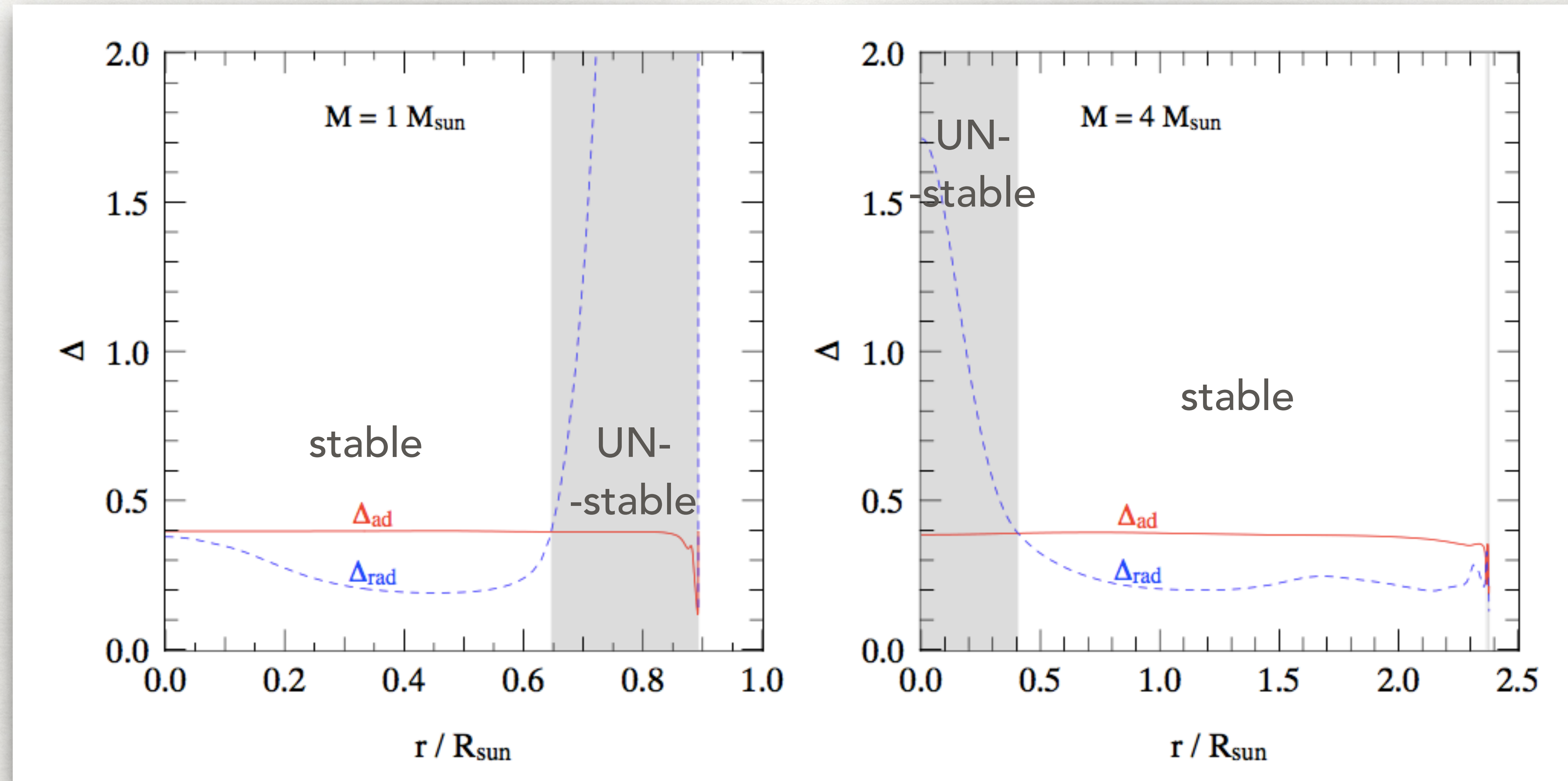
$$\nabla_{\rm rad} = \frac{3}{16\pi acG}\frac{P}{T^4}\frac{\kappa l}{m} > \nabla_{\rm ad}.$$

IT REQUIRES:

3. $\nabla_{\rm ad} < 0.4$ (the ideal gas value) in region where H & He are partially ionised



SUMMARY



M = 1 Msun
stable
UN-
-stable
Δad
Δrad
Δ
r / Rsun
M = 4 Msun
UN-
-stable
stable
Δad
Δrad
Δ
r / Rsun

CONVECTIVE ENERGY TRANSPORT

- detailed theory: difficult unsolved problem
- expensive simulations of star in a moment of their life (not applied in stellar evolution)
- To study evolution : approximate one dimensional theory: mixing length theory (MLT)
- mixing length is the average length travelled by a blob before dissolving

$$\ell_\mathrm{m} \sim H_P$$

$$\alpha_\mathrm{m} = l_\mathrm{m}/H_\mathrm{P}$$

unknown parameter, calibrated against observation~1.5-2

$$H_P = \left|\frac{\mathrm{d}r}{\mathrm{d}\ln P}\right| = \frac{P}{\rho g}.$$

Hp=Pressure scale hight

CONVECTIVE ENERGY FLUX

IN MLT

surrounding

$$F_{\text{conv}} = v_c \rho \Delta u = v_c \rho c_P \Delta T$$

T gradient:

$$\Delta T = T_e - T_s = \left[\left(\frac{dT}{dr}\right)_e - \frac{dT}{dr}\right] \ell_m$$

$$\Delta T = T \frac{\ell_m}{H_P} (\nabla - \nabla_{ad}).$$

convective average velocity:

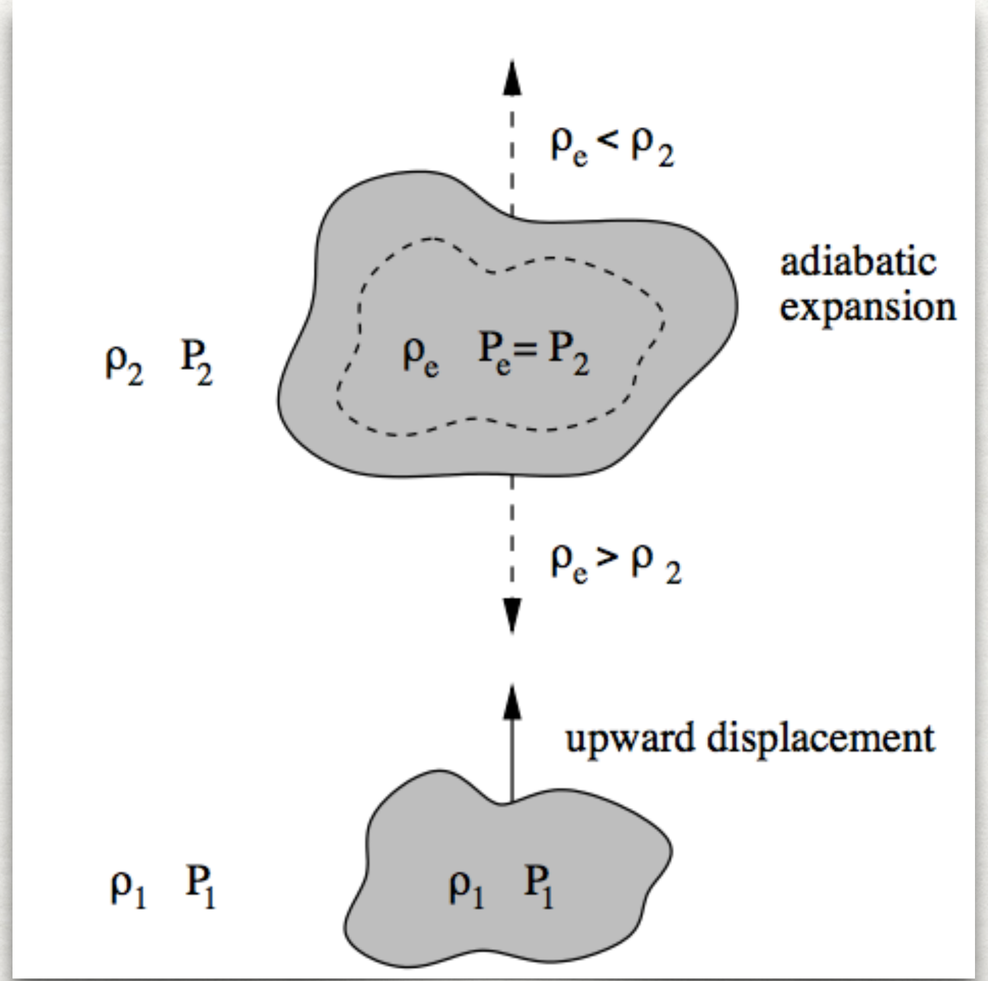
buoyancy acceleration:

$$a = -g\frac{\Delta\rho}{\rho} = g\frac{\Delta T}{T}$$

$$\Delta P = 0; P \propto T\rho$$

$$l_m = \frac{1}{2}at^2$$

$$v_c \approx l_m/t = \sqrt{\frac{1}{2} l_m g \frac{\Delta T}{T}}$$



CONVECTIVE ENERGY FLUX

IN MLT

$$F_{\rm conv} = v_{\rm c}\,\rho\Delta u = v_{\rm c}\,\rho c_P \Delta T$$

$$F_{\rm conv} = \rho c_P T \left(\frac{\ell_{\rm m}}{H_P}\right)^2 \sqrt{\tfrac{1}{2} g H_P}\,(\nabla - \nabla_{\rm ad})^{3/2}.$$

$$T \approx \bar{T} \sim \frac{\mu}{\mathcal{R}}\frac{GM}{R} \qquad c_P = \frac{5}{2}\frac{\mathcal{R}}{\mu} \qquad \sqrt{g H_P} = \sqrt{\frac{P}{\rho}} = \sqrt{\frac{\mathcal{R}}{\mu}T} \sim \sqrt{\frac{GM}{R}} \approx c_{\rm s}$$

rate of internal energy transport (VIRIAL TH>)

$$F_{\rm conv} \sim \rho c_{\rm s} \frac{GM}{R} (\nabla - \nabla_{\rm ad}) \sim \rho c_{\rm s} E_{\rm in}$$

numerical factor

Superadiabaticity

$$\nabla - \nabla_{\rm ad}$$

how much steeper is the convective temperature gradient from the adiabatic one?

SUPERADIABATICITY

convection requires SuperAdiabaticity : $\nabla > \nabla_{\rm ad}$ but by how much ?

$$\begin{cases} F_{\rm conv} \sim \rho c_{\rm s} \dfrac{GM}{R} (\nabla - \nabla_{\rm ad}) \sim \dfrac{M}{R^3} \left(\dfrac{GM}{R} \right)^{3/2} (\nabla - \nabla_{\rm ad}) \\ F_{\rm conv} = \dfrac{L}{4\pi r^2} \sim \dfrac{L}{R} \end{cases}$$

$$\rho \approx \frac{3M}{4\pi R^3}$$

$$\nabla - \nabla_{\rm ad} \sim \left(\frac{LR}{M} \right)^{2/3} \frac{R}{GM} \sim 10^{-8}$$

conclusion: in a convective zone:

$$\boxed{\nabla \approx \nabla_{\rm ad}}$$

Note: detailed calculations give 10^{-5} - 10^{-7}

CONVECTIVE TEMPERATURE GRADIENT

IN MLT

In deep stellar layers where $\nabla \approx \nabla_{\rm ad}$ and hydrostatic equilibrium:

$$\frac{dT}{dm} = \frac{dP}{dm}\frac{dT}{dP} = -\frac{Gm}{4\pi r^4}\frac{T}{P}\frac{dlogT}{dlogP} = -\frac{Gm}{4\pi r^4}\frac{T}{P}\nabla_{\rm ad}$$

$$\boxed{\frac{dT}{dm} = -\frac{Gm}{4\pi r^4}\frac{T}{P}\nabla_{\rm ad}}$$

CONVECTIVE TEMPERATURE GRADIENT

IN MLT

At outer layers density and temperature are very different from the mean and super adiabaticity becomes substantial :

Detailed calculations are needed for the temperature gradient

At the surface layers

convective flux << radiative flux even if convective unstable

$$\nabla \approx \nabla_{\mathrm{rad}}$$

CONVECTIVE MIXING

$$v_{\rm c} \approx c_{\rm s}\sqrt{\nabla - \nabla_{\rm ad}}$$

highly subsonic in the inner layers (so gentle and not disruptive): HE can be maintained

the mixing timescale for a region with size d = q R (q = a fraction of radius) is

$$\tau_{\rm mix} \approx v_{\rm c}/d \sim q\left(R^2 M/L\right)^{1/3} = q \times 10^7\ {\rm s}$$

weeks to months , very short!

$$\tau_{\rm mix} \ll \tau_{\rm KH} \ll \tau_{\rm nuc}$$

A couple conclusions:

1. a convective zone inside a star will be mixed homogeneously
2. convective mixing remains efficient in the outer layers even if convection is inefficient in transporting energy

CONSEQUENCE OF CONVECTIVE MIXING

- If nuclear burning occurs in a ***convective core,*** convection will keep transporting He (burning ashes) outward and H (fuel) inward: there is a larger fraction of the stellar mass involved in the nuclear reaction compared to a stable core: it's lifetime will be longer.

- *``Dgredge-up"* A star with a convective envelope extending into regions where nuclear burning occur will move the burning ashes outward towards the surface where may be directly observed: **red-giants**

OVERSHOOTING

CH 5.5.4: PLEASE, GO THROUGH YOURSELF

what happens at the boundary of a convective envelope