NOTES CH 3

# EQUATION OF STATE FOR STARS

# LOCAL THERMODYNAMICAL EQUILIBRIUM=LTE

## 1.The interior of a star is a very optically thick medium:

$$\tau \simeq \frac{R}{l_{\rm ph}} \gg 1 \qquad \text{e.g.} \quad \tau_{\rm Sun} \simeq \frac{R_{\rm Sun}}{< l_{\rm ph} >} \approx 10^{11}$$

~1 cm

where $l_{\rm ph}$ is the mean free path of photons between scatterings and/or absorption

In a region of several mean free path but << R
matter and radiation are in *thermal equilibrium* and
can be described **by the same *local* temperature**

mixing locally is very effective: **well defined local composition**

we can calculate microscopic thermodynamical properties
as a function of **the *local* temperature, density and composition**

In each position within the star,
radiation has a **black body spectrum** with **that *local* temperature**

Note: thermal equilibrium $\neq$ LTE

# THE EQUATION OF STATE

## EOS

- *Def:* A relation between the pressure exerted by a system of particles of known composition, temperature and density:

$$P = P(T, \rho, X)$$

# IDEAL OR PERFECT GAS

## EOS

- **Def: an ensemble of free, non-interacting particle so that the internal energy is given by the sum of the particles' kinetic energy**

- **Good description for ionised gas in a star with T~10$^6$ K. One can show that the ratio of Coulomb interaction energy per particle and mean kinetic energy per particle is**

$$\frac{\epsilon_{\mathrm{C}}}{k_{\mathrm{b}} < T >} \approx 1\%$$

see 3.1 in Prialnik's book and section 3.6.1 notes

- **Note:** Cristallization can happen in cooling white dwarfs (3.6.1)

- **Note:** gas is completely ionised in the hot stellar interiors but in cooler (<10 6 K)  atmospheric layers there maybe partial ionization (3.5)

# THERMODYNAMICAL QUANTITIES

| | | |
|---|---|---|
| number density | $n = \int_0^\infty n(p)\,\mathrm{d}p$ | definition of n(p) =momenta distribution |
| internal energy density | $U = \int_0^\infty \epsilon_p n(p)\,\mathrm{d}p = n\langle\epsilon_p\rangle$ | definition of perfect gas |
| pressure | $P = \frac{1}{3}\int_0^\infty p v_p n(p)\,\mathrm{d}p = \frac{1}{3}n\langle p v_p\rangle$ | P= flux of momentum |

## Relation between U & P

We need the particle energy $\epsilon_\mathrm{p} = \epsilon - mc^2$ and velocity $v_\mathrm{p} = \frac{pc^2}{\epsilon}$

**NR limit:** $p \ll mc$

$$< p v_\mathrm{p} >= \left\langle \frac{p^2}{m} \right\rangle = 2 < \epsilon_\mathrm{p} > \;\Rightarrow\; \boxed{P = \frac{2}{3} n < \epsilon_\mathrm{p} >= \frac{2}{3} U}$$

**ER limit** $p \gg mc$

$$\left.\begin{array}{l} \epsilon_\mathrm{p} = pc \\ v_\mathrm{p} = c \end{array}\right\} < p v_\mathrm{p} >=< pc >=< \epsilon_\mathrm{p} > \;\Rightarrow\; \boxed{P = \frac{1}{3} U}$$

$$P = \frac{1}{3}\int_0^{\infty} p v_p n(p)\,\mathrm{d}p$$

“

**TO COMPUTE THE EOS WE NEED**

$$n(p)dp$$

**LET’S REFRESH OUR KNOWLEDGE OF STATISTICAL MECHANICS**

”

# CLASSICAL IDEAL GAS

Consider:

- Gas of identical, non-interacting particles (*perfect* gas), with mass "m", non relativistic (p << mc)
- At LTE, at temperature T

**the momentum distribution is a Maxwell-Boltzmann distribution:**

$$n(p)\,\mathrm{d}p = \frac{n}{(2\pi mkT)^{3/2}} e^{-p^2/2mkT}\, 4\pi p^2\, \mathrm{d}p.$$

$$P = \frac{1}{3}\int_0^\infty p v_p n(p)\,\mathrm{d}p$$ with $v_\mathrm{p} = \dfrac{p}{m}$

$$\rightarrow \boxed{P = n k_\mathrm{b} T}$$

NOTE: true also for a relativistic Maxwellian distribution

# MIXTURE OF *IDEAL NON DEGENERATE GASES*

$$P_{\text{gas}} = P_{\text{ion}} + P_{\text{e}} = \sum_i P_i + P_{\text{e}} = \left(\sum_i n_i + n_{\text{e}}\right)kT = nkT$$

$$n_{\text{ion}} = \Sigma_i n_i = \Sigma_i \frac{X_i}{A_i}\frac{\rho}{m_{\text{u}}} \equiv \frac{1}{\mu_{\text{ion}}}\frac{\rho}{m_{\text{u}}}$$

mean atomic mass per ion

$$m_{\text{u}} = \frac{1}{12} m^{12}\text{C}$$

$$n_{\text{e}} = \Sigma_i Z_i n_i = \Sigma_i \frac{Z_i X_i}{A_i}\frac{\rho}{m_{\text{u}}} \equiv \frac{1}{\mu_{\text{e}}}\frac{\rho}{m_{\text{u}}}$$

mean molecular weight per free electron

$$P_{\text{gas}} = P_{\text{ion}} + P_{\text{e}} = \left(\frac{1}{\mu_{\text{ion}}} + \frac{1}{\mu_{\text{e}}}\right) R\rho T = \frac{R}{\mu}\rho T$$

$R = k_b/m_u$ = the universal gas constant

# MEAN MOLECULAR WEIGHT

$$\frac{1}{\mu} = \frac{1}{\mu_{\rm ion}} + \frac{1}{\mu{\rm e}} = \Sigma_i \frac{(Z_i + 1)X_i}{A_i}$$

it is the inverse sum of the mean atomic mass per ion and the mean molecular weight per electron

Fully ionised hydrogen Z=A =1 gas while fully ionised He and metals Z/A ~1/2

$$\frac{1}{\mu} \approx \frac{1}{2X + \frac{3}{2}Y + \frac{1}{2}Z} \approx 0.6$$

X=0.7, Y=0.28, Z=0.02

# QUANTUM MECHANICAL STATISTICAL DESCRIPTION OF GAS

- P*erfect* gas, with mass "m" and spin "s"
- At, at LTE temperature T
- Fermions

More generally, then in the previous case, we state that the distribution of its numerical density as a function of its linear momentum is

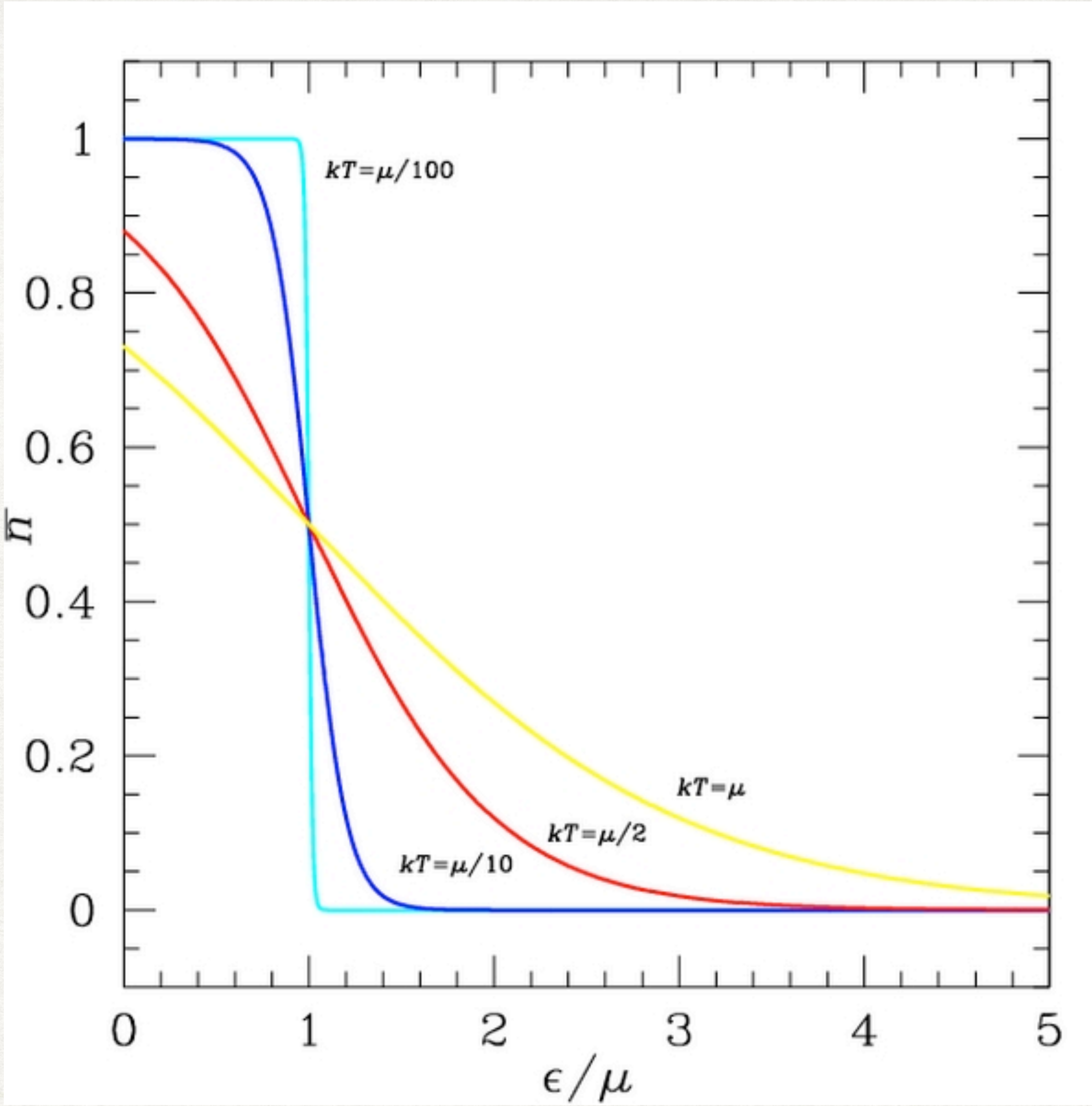
$$\frac{dn(p)}{dp} = \frac{4\pi g}{h^3}\frac{p^2}{e^{\frac{\epsilon-\mu}{k_\mathrm{b}T}} - 1}$$

**Fermi-Dirac distribution**

g = 2s+1 energy level degeneracy

μ chemical potential = is the work necessary to change the particle number by dN: dW/dN

$\epsilon = (pc)^2 + m^2c^4$ total energy




$kT \gg \mu$ ==> classical Maxwell-Boltzmann distribution

# WDS ARE MADE OF "COLD" GAS

Degenerate gas of fermions at T—>0

$$\frac{dn}{dp} = \frac{4\pi g}{h^3} p^2 \times \begin{cases} 1, & \epsilon \leq \mu \\ 0, & \epsilon > \mu \end{cases}$$

Here the chemical potential is called *Fermi energy* $E_{\mathrm{F}} = \mu$

The corresponding *Fermi momentum* $p_{\mathrm{F}} = mc\sqrt{\left(\frac{E_{\mathrm{F}}}{mc^2}\right)^2 - 1}$

*Fermi temperature* $k_{\mathrm{b}}T = E_{\mathrm{F}} - mc^2$

"zero" temperature distribution can be assumed if a gas has $T \ll T_F$ (i.e. $KT \ll \mu$)

# THERMODYNAMICAL PROPERTIES: NUMBER DENSITY

$$n = \int n(p)dp$$

$$n = \frac{g}{2}\frac{p_F^3}{3\pi^2\hbar^3}$$

$$\begin{aligned} p_F &= \left(\frac{2}{g}3\pi^2\hbar^3 n\right)^{1/3}, \\ x_F = \frac{p_F}{mc} &= \left(\frac{2}{g}3\pi^2\left(\frac{\hbar}{mc}\right)^3 n\right)^{1/3}. \end{aligned}$$

simply counting particles in a sphere with radius $p_F$

particle density determines *Fermi Energy*

# THERMODYNAMICAL PROPERTIES: PRESSURE

$$P = \tfrac{1}{3}\int_0^\infty p v_p n(p)\,\mathrm{d}p$$

$$\begin{aligned} P &= \frac{g}{2}\frac{m^4c^5}{24\pi^2\hbar^3}F_\mathrm{P}(x_\mathrm{F}), \\ F_\mathrm{P}(x) &= x\left(2x^2-3\right)\sqrt{x^2+1}+3\ln\left(x+\sqrt{x^2+1}\right) \end{aligned}$$

let's now specialise to WD typical conditions...

# DEGENERATE ELECTRON PRESSURE IN WDS

$$g = 2 \; \& \; m = m_e$$

$$\begin{aligned} P &= 6.002 \times 10^{21}\,\mathrm{Pa}\, F_P(x_F)\,, \\ x_F &= \frac{p_F}{m_e c} = 0.801 \left(\frac{Y_e}{0.5}\right)^{1/3} \left(\frac{\rho}{10^9\,\mathrm{kg.m^{-3}}}\right)^{1/3}, \\ F_P(x) &= x\left(2x^2 - 3\right)\sqrt{x^2+1} + 3\ln\left(x + \sqrt{x^2+1}\right) \end{aligned}$$

for typical density and number of free electron per nucleon ($Y_e$), electron are mildly relativistic $X_F \sim 1$.

# DEGENERACY TEMPERATURE FOR ELECTRONS

Degeneracy if T is << $T_F$

$$T_{\rm F} = \frac{m_{\rm e}c^2}{k}\left(\sqrt{1+x_{\rm F}^2}-1\right) \simeq 5.93\times 10^9\,{\rm K}\left(\sqrt{1+x_{\rm F}^2}-1\right)$$

Typical central temperature ~$10^7$ K

electrons are fully degenerate in a WD

# NUCLEONS ARE NON DEGENERATE IN WDS

inversely proportional to mass

$$T_\mathrm{F} \propto n^{2/3}/m$$

$$T_\mathrm{F,e} \sim 2000 \times T_\mathrm{F,p/n}$$

$$T_\mathrm{F,p/n} = 3 \times 10^6 K < T_\mathrm{WD} \ll T_\mathrm{F,e} \approx 6 \times 10^9 K$$

In a WD the condition $T_e << T_{F,e}$ is attained well before $T_{n/p} << T_{F,e/p}$.
In fact, in WD only electrons are degenerate

# DEGENERATE ELECTRON PRESSURE IN WD

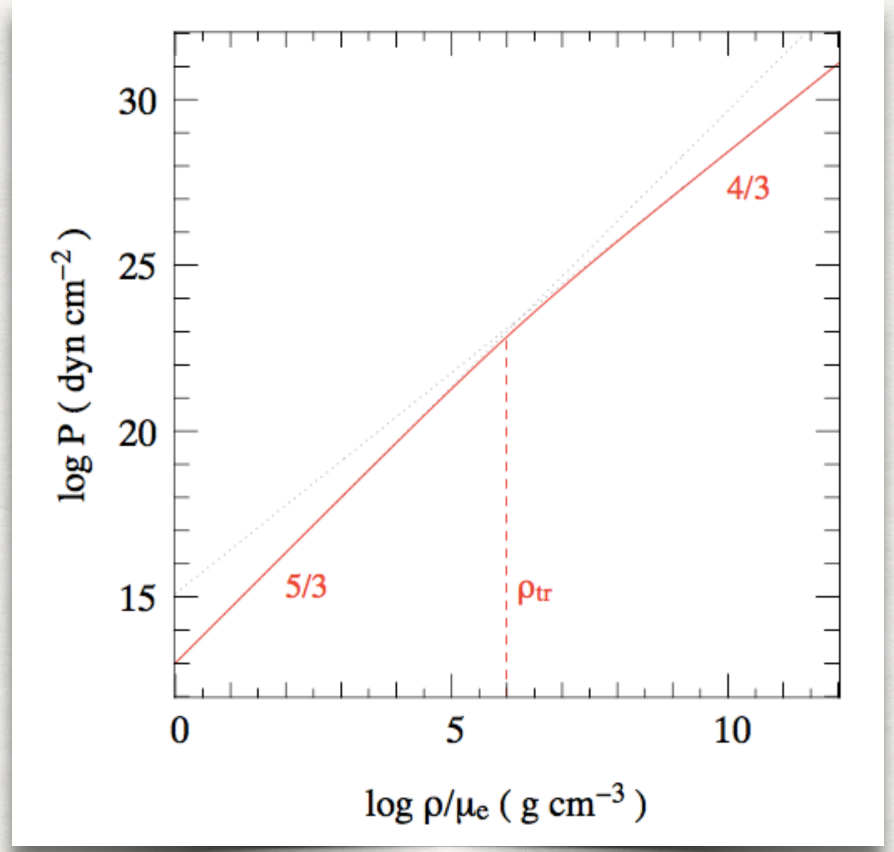
## two extreme regimes:

$$x_{\rm F} \ll 1 \qquad \rho \ll 10^6 g\ cm^{-3}$$

$$P \approx 3 \times 10^{20}\ \mathrm{Ba}\ \ (Y_{\rm e}/0.5)^{5/3} \left(\frac{\rho}{10^6 g/cm^3}\right)^{5/3}$$

$$x_{\rm F} \gg 1 \qquad \rho \gg 10^6 g\ cm^{-3}$$

$$P \approx 5 \times 10^{20}\ \mathrm{Ba}\ \ (Y_{\rm e}/0.5)^{4/3} \left(\frac{\rho}{10^6 g/cm^3}\right)^{4/3}$$




$$P \propto \rho^{\gamma}$$

WD's pressure does not depend on temperature =>
does have to be hot to be in hydrostatic equilibrium

# RADIATION

photon are bosons with $g_\mathrm{s} = 2$ and $\mu = 0$ and completely relativistic

$$n(p)\,\mathrm{d}p = \frac{2}{h^3}\,\frac{1}{e^{\epsilon_p/kT} - 1}\,4\pi p^2\,\mathrm{d}p$$

Bose-Einstein distribution

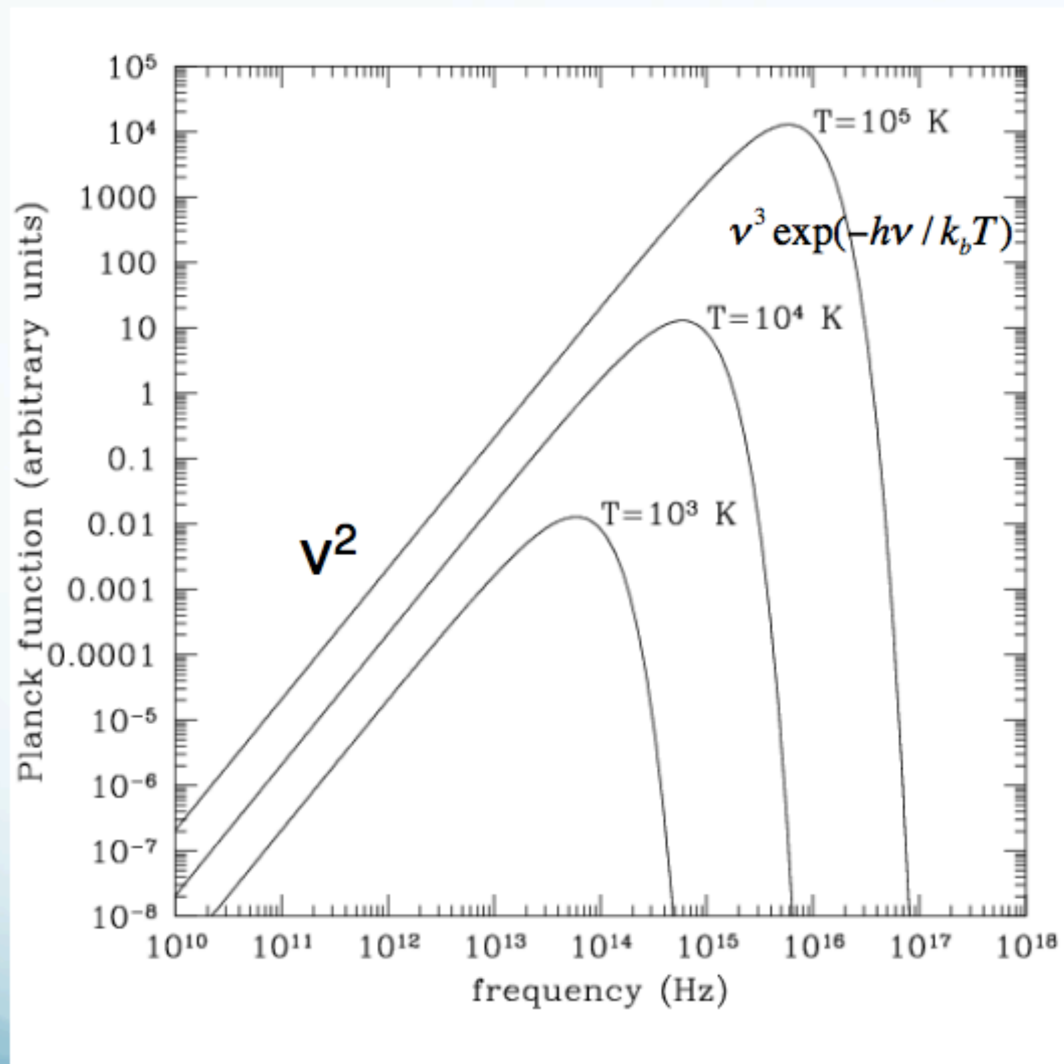
$$\epsilon_\mathrm{p} = pc = h\nu$$

$$n(\nu)\,\mathrm{d}\nu = \frac{8\pi}{c^3}\,\frac{\nu^2\,\mathrm{d}\nu}{e^{h\nu/kT} - 1}$$

**Planck function**

# BLACK BODY RADIATION

## Properties of blackbody radiation




- Isotropic
- Continuous spectrum
- Increasing T increases $B_\nu$ at **all** frequencies
- **peak** $h\nu_{\max} = 2.82kT$

$$u(T) = aT^4$$

$$B(T) = \frac{ac}{4\pi} T^4$$

$$F(T) = \pi B = \sigma T^4$$

Stefan-Boltzmann law

# RADIATION PRESSURE

$$U = aT^4$$ energy density

$$\boxed{P_{\rm rad} = \frac{1}{3}U = \frac{1}{3}aT^4}$$

## A MIXTURE OF GAS AND RADIATION

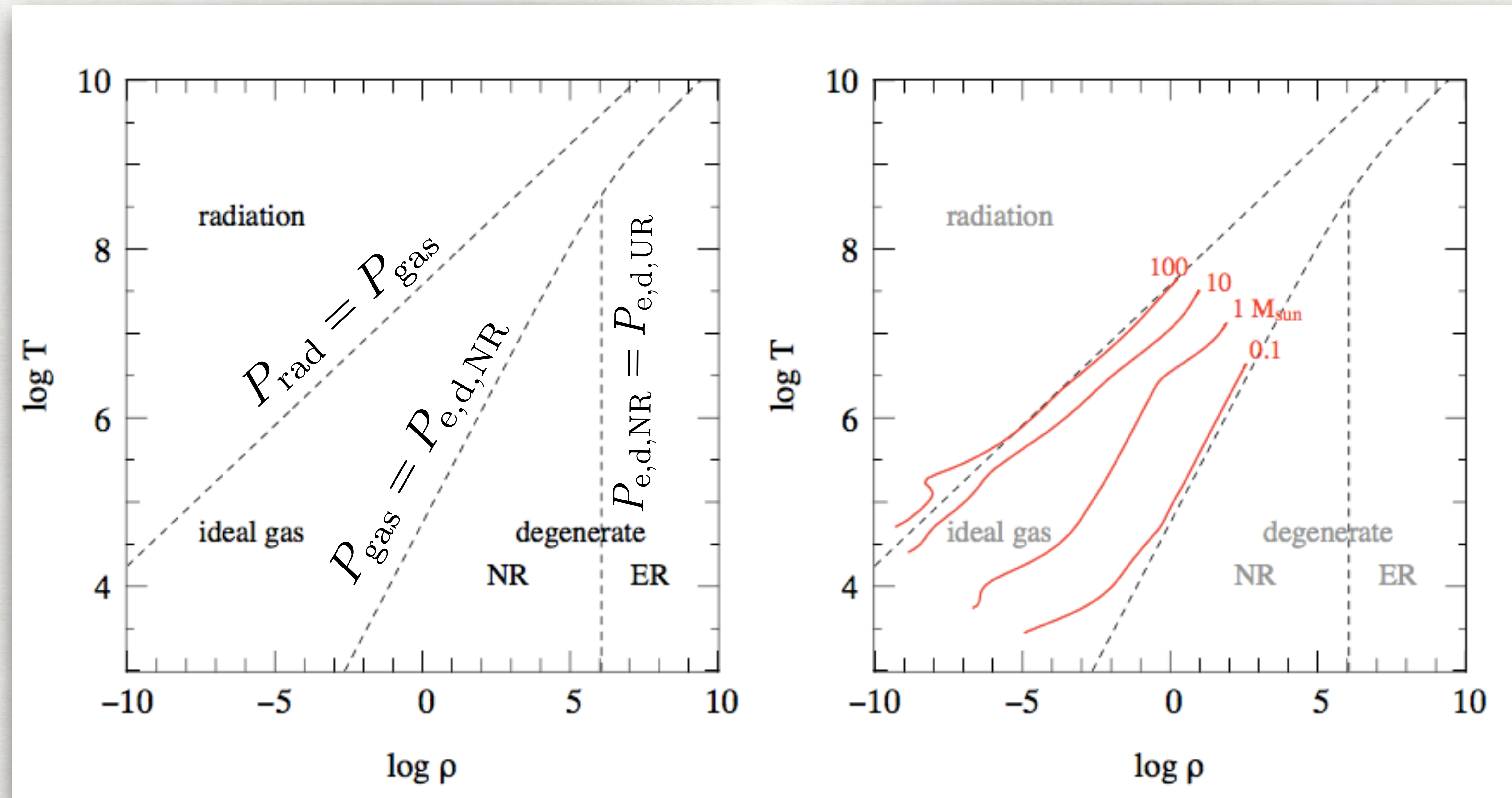
$$P = P_{\rm rad} + P_{\rm ion} + P_{\rm e-}$$

$$P_{\rm gas} = P_{\rm ion} + P_{\rm e-} = \beta P$$

$$P_{\rm rad} = (1-\beta)P$$

$$P_{\rm rad} = P_{\bf gas,\rm classical} \Rightarrow T/\rho^{1/3} \approx 10^7 \mu^{-1/3}$$

# SUMMARY




from O.R. Pols