Stellar Structure and Evolution 2017: Exercise list.

April 24, 2017

Directions

This list of exercises was made to help the master students to prepare the final examination for the Stellar Structure Lecture 2017 at Leiden Observatory. By studying and carefully solving this exercise list, you should be prepared for the exam. The exercise list was made using the lecture notes of the class (Prof. Pols: `https://www.astro.ru.nl/~onnop/education/stev_utrecht_notes/`). Please check the lecture notes and discuss with your fellow students in order to get a better understanding of the different topics. Please try to be precise and concise but also try to maintain the solutions as quantitative as possible by using equations, relations, back up statements, etc. If you have specific questions please send an email or make an appointment with the TAs: Ann-Sofie Bak Nielsen (nielsen@strw.leidenuniv.nl) and Luis Henry Quiroga Nuñez (quiroganunez@strw.leidenuniv.nl).

2.3 The virial theorem

An important consequence of hydrostatic equilibrium is that it links the gravitational potential energy E_{gr} and the internal thermal energy E_{int}.

(a) Estimate the gravitational energy Egr for a star with mass M and radius R, assuming (1) a constant density distribution and (2) the density distribution:

$$\rho = \rho_c \left(1 - \left(\frac{r}{R}\right)^2\right). \tag{1}$$

(b) Assume that a star is made of an ideal gas. What is the kinetic internal energy per particle for an ideal gas? Show that the total internal energy, E_{int} is given by:

$$E_{init} = \int_0^r \left(\frac{3}{2}\frac{k}{\mu m_u}\rho(r)T(r)\right) 4\pi r dr \tag{2}$$

(c) Estimate the internal energy of the Sun by assuming constant density and $T(r) \approx < T > \approx 1/2 T_C \approx 5 \times 10^6 K$ and compare your answer to your answer for a). What is the total energy of the Sun? Is the Sun bound according to your estimates?

It is not a coincidence that the order of magnitude for E_{gr} and E_{int} are the same[1]. This follows from hydrostatic equilibrium and the relation is known as the virial theorem. In the next steps we will derive the virial theorem starting from the pressure gradient in the form of:

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho. \tag{3}$$

[1] In reality E_{gr} is larger than estimated above because the mass distribution is more concentrated to the centre.

(d) Multiply by both sides by $4\pi r^3$ and integrate over the whole star. Use integration by parts to show that

$$\int_0^R 3P4\pi r^2 dr = \int_0^R \frac{Gm(r)}{r}\rho 4\pi r^2 dr \tag{4}$$

(e) Now derive a relation between E_{gr} and E_{int}, the virial theorem for an ideal gas.
(f) Also show that for the average pressure of the star

$$\langle P \rangle = \frac{1}{V}\int_0^{R_*} P4\pi r^2 dr = -\frac{1}{3}\frac{E_{gr}}{V} \tag{5}$$

where V is the volume of the star.

As the Sun evolved towards the main sequence, it contracted under gravity while remaining close to hydrostatic equilibrium. Its internal temperature changed from about 30 000 K to about 6×10^6K.

(g) Find the total energy radiated during away this contraction. Assume that the luminosity during this contraction is comparable to $L_\odot$ and estimate the time taken to reach the main sequence.

2.4 Conceptual questions

(a) Use the virial theorem to explain why stars are hot, i.e. have a high internal temperature and therefore radiate energy.

(b) What are the consequences of energy loss for the star, especially for its temperature?

(c) Most stars are in thermal equilibrium. What is compensating for the energy loss?

(d) What happens to a star in thermal equilibrium (and in hydrostatic equilibrium) if the energy production by nuclear reactions in a star drops (slowly enough to maintain hydrostatic equilibrium)?

(e) Why does this have a stabilizing effect? On what time scale does the change take place?

(f) What happens if hydrostatic equilibrium is violated, e.g. by a sudden increase of the pressure.

(g) On which timescale does the change take place? Can you give examples of processes in stars that take place on this timescale?

2.5 Three important timescales in stellar evolution

(a) The nuclear timescale τ_{nuc}
i. Calculate the total mass of hydrogen available for fusion over the lifetime of the Sun, if 70% of its mass was hydrogen when the Sun was formed, and only 13% of all hydrogen is in the layers where the temperature is high enough for fusion.

ii. Calculate the fractional amount of mass converted into energy by hydrogen fusion. (Refer to Table 1 for the mass of a proton and of a helium nucleus.)

iii. Derive an expression for the nuclear timescale in solar units, i.e. expressed in terms of $R/R_\odot$, $M/M_\odot$ and $L/L_\odot$.

iv. Use the mass-radius and mass-luminosity relations for main-sequence stars to express the nuclear timescale of main-sequence stars as a function of the mass of the star only.

v. Describe in your own words the meaning of the nuclear timescale.

(b) The thermal timescale τ_{KH}.

i-iii. Answer question (a) iii, iv and v for the thermal timescale and calculate the age of the Sun according to Kelvin.

iv. Why are most stars observed to be main-sequence stars and why is the Hertzsprung-gap called a gap?

(c) The dynamical timescale τ_{dyn}.

i-iii. Answer question (a) iii, iv and v for the dynamical timescale.

iv. In stellar evolution models one often assumes that stars evolve quasi-statically, i.e. that the star remains in hydrostatic equilibrium throughout. Why can we make this assumption?

v. Rapid changes that are sometimes observed in stars may indicate that dynamical processes are taking place. From the timescales of such changes - usually oscillations with a characteristic period - we may roughly estimate the average density of the Star. The sun has been observed to oscillate with a period of minutes, white dwarfs with periods of a few tens of seconds. Estimate the average density for the Sun and for white dwarfs.

(d) Comparison.

i. Summarize your results for the questions above by computing the nuclear, thermal and dynamical timescales for a 1, 10 and 25 $M_\odot$ main-sequence star. Put your answers in tabular form.

ii. For each of the following evolutionary stages indicate on which timescale they occur: premain sequence contraction, supernova explosion, core hydrogen burning, core helium burning.

iii. When the Sun becomes a red giant (RG), its radius will increase to $200R_\odot$ and its luminosity to $3000L_\odot$. Estimate τ_{dyn} and τ_{KH} for such a RG.

iv. How large would such a RG have to become for $\tau_{dyn} > \tau_{KH}$? Assume both R and L increase at constant effective temperature.

3.3 The $\rho - T$ plane

Consider a gas of ionized hydrogen. In the ρ–T plane compute the approximate boundary lines between the regions where:

(a) radiation pressure dominates,

(b) the electrons behave like a classical ideal gas,

(c) the electrons behave like a degenerate gas,

(d) the electrons are relativistically degenerate.

4.3 White dwarfs

To understand some of the properties of white dwarfs (WDs) we start by considering the equation of state for a degenerate, non-relativistic electron gas.

(a) What is the value of K for such a star? Remember to consider an appropriate value of the mean molecular weight per free electron μ_e.

(b) Derive how the central density ρ_c depends on the mass of a non-relativistic WD. Using this with:

$$m(z) = \int_0^{\alpha z} 4\pi r^2 \rho dr = -4\pi\alpha^3 \rho_c z^2 \frac{dw}{dz} \tag{6}$$

and

$$M = 4\pi\alpha^3 \rho_c \Theta_n = 4\pi \left(\frac{(n+1)K}{4\pi G} \right)^{3/2} \rho_c^{(3-n)/2n} \Theta_n, \tag{7}$$

derive a radius-mass relation $R = R(M)$, where $\Theta_n = (-z^2 dw/dz)_{z=z_n}$ (see section 4.1.1). Interpret this physically.

(c) Use the result of (b) to estimate for which WD masses the relativistic effects would become important.

(d) Show that the derivation of a $R = R(M)$ relation for the extreme relativistic case leads to a unique mass, the so-called "Chandrasekhar mass". Calculate its value, i.e. derive:

$$M_{Ch} = 5.836 \mu_e^{-2} M_\odot \tag{8}$$

4.4 Eddington's standard model (a) Show that for constant β the virial theorem leads to

$$E_{int} = \frac{\beta}{2} E_{gr} = -\frac{\beta}{2-\beta} E_{int} \tag{9}$$

for a classical, non-relativistic gas. What happens in the limits $\beta \to 1$ and $\beta \to 0$?

(b) Verify eq. (4.25), and show that the corresponding constant K depends on β and the mean molecular weight μ as

$$K = \frac{2.67 \times 10^{15}}{\mu^{4/3}} \left(\frac{1-\beta}{\beta^4} \right)^{1/3} \tag{10}$$

(c) Use the results from above and the fact that the mass of an n = 3 polytrope is uniquely determined by K, to derive the relation M = M(β, mu). This is useful for numerically solving the amount of radiation pressure for a star with a given mass.

(d) what does the relation M versus beta teaches us ?

5.1 Radiation transport

The most important way to transport energy form the interior of the star to the surface is by radiation, i.e. photons traveling from the center to the surface.

(a) How long does it typically take for a photon to travel from the center of the Sun to the surface? [Hint: estimate the mean free path of a photon in the central regions of the Sun.] How does this relate to the thermal timescale of the Sun?

(b) Estimate a typical value for the temperature gradient dT/dr. Use it to show that the difference in temperature ΔT between two surfaces in the solar interior one photon mean free path ℓ_{ph} apart is

$$\Delta T = \ell_{ph} \frac{dT}{dr} \approx 2 \times 10^{-4} K \tag{11}$$

In other words the anisotropy of radiation in the stellar interior is very small. This is why radiation in the solar interior is close to that of a black body.

(c) Verify that a gas element in the solar interior, which radiates as a black body, emits $\approx 6{\times}10^{23}$ erg $\mathrm{cm}^{-2}\mathrm{s}^{-1}$.

If the radiation field would be exactly isotropic, then the same amount of energy would radiated into this gas element by the surroundings and so there would be no net flux.

(d) Show that the minute deviation from isotropy between two surfaces in the solar interior one photon mean free path apart at r $\sim$ $R_\odot/10$ and T $\sim$ 107 K, is sufficient for the transfer of energy that results in the luminosity of the Sun.

(e) Why does the diffusion approximation for radiation transport break down when the surface (photosphere) of a star is approached?

5.4 Conceptual questions: convection

(a) Why does convection lead to a net heat flux upwards, even though there is no net mass flux (upwards and downwards bubbles carry equal amounts of mass)?

(b) Explain the Schwarzschild criterion

$$\left(\frac{dlnT}{dlnP}\right)_{rad} > \left(\frac{dlnT}{dlnP}\right)_{ad} \tag{12}$$

in simple physical terms (using Archimedes law) by drawing a schematic picture . Consider both cases $\nabla_{rad} > \nabla_{ad}$ and $\nabla_{rad} < \nabla_{ad}$. Which case leads to convection?

(c) What is meant by the superadiabaticity of a convective region? How is it related to the convective energy flux (qualitatively)? Why is it very small in the interior of a star, but can be large near the surface?

5.5 Applying Schwarzschild's criterion

(a) Low-mass stars, like the Sun, have convective envelopes. The fraction of the mass that is convective increases with decreasing mass. A 0.1 $M_\odot$ star is completely convective. Can you qualitatively explain why?

(b) In contrast higher-mass stars have radiative envelopes and convective cores. Determine if the energy transport is convective or radiative at two different locations (r = $0.242R_\odot$ and r = $0.670R_\odot$) in a $5M_\odot$ main sequence star. Use the data of a 5 $M_\odot$ model in the table below. You may neglect the radiation pressure and assume that the mean molecular weight $\mu = 0.7$.

$r/R_\odot$	$m/M_\odot$	$L_r/L_\odot$	T[K]	ρ [g cm^{-3}]	κ [g−1cm^2]
0.242	0.199	3.40×10^2	2.52×10^7	18.77	0.435
0.670	2.487	5.28×10^2	1.45×10^7	6.91	0.585

6.2 Hydrogen burning

(a) Calculate the energy released per reaction in MeV (the Q-value) for the three reactions in the pp1 chain. (Hint: first calculate the equivalent of $m_u c^2$ in MeV.)

(b) What is the total effective Q-value for the conversion of four 1H nuclei into 4He by the pp1 chain?
Note that in the first reaction (1H + 1H → 2H + e^+ + ν) a neutrino is released with (on average) an energy of 0.263 MeV.

(c) Calculate the energy released by the pp1 chain in erg/g

6.4 Helium burning

(a) Calculate the energy released per gram for He burning by the 3α reaction and the 12C+α reaction, if the final result is a mixture of 50% carbon and 50% oxygen (by mass fraction).

(b) Compare the answer to that for H-burning. How is this related to the duration of the He-burning phase, compared to the main-sequence phase?

7.2 Dynamical Stability

(a) Show that for a star in hydrostatic equilibrium (dP/dm = -Gm/(4πr^4)) the pressure scales with density as P $\propto \rho^{4/3}$.

(b) If $\gamma_{ad} < 4/3$ a star becomes dynamically unstable. Explain why.

(c) In what type of stars $\gamma_{ad} \approx 4/3$?

(d) What is the effect of partial ionization (for example H $\leftrightarrows$ H$^+$ + e$^-$) on γ_{ad}? So what is the effect of ionization on the stability of a star?

(e) “Pair creation” and “photo-disintegration” of Fe have a similar effect on γ_{ad}. In what type of stars (and in what phase of their evolution) do these processes play a role?

8.1 Homologous contraction (1)

(a) Explain in your own words what homologous contraction means.

(b) A real star does not evolve homologous. Can you give a specific example? [Think of core versus envelope]

(c) Fig. 8.3 shows the central temperature versus the central density for schematic evolution tracks assuming homologous contraction. Explain qualitatively what we can learn form this figure (nuclear burning cycles, difference between a 1 M$_\odot$ and a 10 M$_\odot$ star, ...)

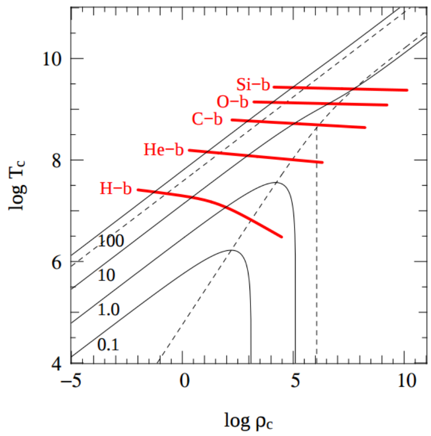

Figure 8.3. The same schematic evolution tracks as in Fig. 8.2, together with the approximate regions in the $\log T_c - \log \rho_c$ plane where nuclear burning stages occur.

(d) Fig. 8.4 shows the same diagram with evolution tracks from detailed (i.e. more realistic) models. Which aspects were already present in the schematic evolution tracks? When and where do they differ?

8.2 Homologous contraction (2)

In this question you will derive the equations that are plotted in Figure 8.2b.

(a) Use the homology relations for P and ρ to derive:

$$P_c = CGM^{2/3}\rho_c^{4/3} \tag{13}$$

To see what happens qualitatively to a contracting star of given mass M, the total gas pressure can be approximated roughly by:

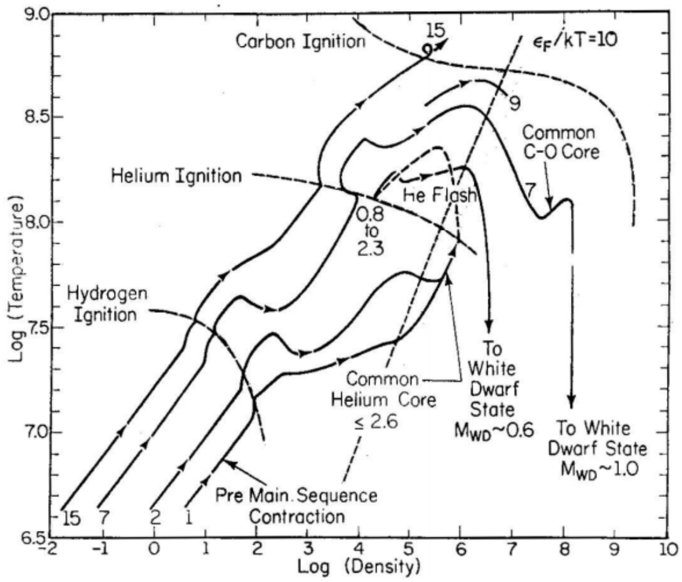

Figure 8.4. Detailed evolution tracks in the $\log \rho_c$ - $\log T_c$ plane for masses between 1 and 15 $M_\odot$. The initial slope of each track (labelled pre-main sequence contraction) is equal to $\frac{1}{3}$ as expected from our simple analysis. When the H-ignition line is reached wiggles appear in the tracks, because the contraction is then no longer strictly homologous. A stronger deviation from homologous contraction occurs at the end of H-burning, because only the core contracts while the outer layers expand. Accordingly, the tracks shift to higher density appropriate for their smaller (core) mass. These deviations from homology occur at each nuclear burning stage. Consistent with our expectations, the most massive star (15 $M_\odot$) reaches C-ignition and keeps evolving to higher T and ρ. The core of the 7 $M_\odot$ star crosses the electron degeneracy border (indicated by $\epsilon_F/kT = 10$) before the C-ignition temperature is reached and becomes a C-O white dwarf. The lowest-mass tracks (1 and 2 $M_\odot$) cross the degeneracy border before He-ignition because their cores are less massive than 0.3 $M_\odot$. Based on our simple analysis we would expect them to cool and become He white dwarfs; however, their degenerate He cores keep getting more massive and hotter due to H-shell burning. They finally do ignite helium in an unstable manner, the so-called He flash.

$$P \approx P_{id} + P_{deg} = \frac{\Re}{\mu}\rho T + K\left(\frac{\rho}{\mu_e}\right)^\gamma \tag{14}$$

where γ varies between 5/3 (non-relativistic) and 4/3 (extremely relativistic).

(b) Combine this equation, for the case of NR degeneracy, with the central pressure of a contracting star in hydrostatic equilibrium (eq. 13, assuming C $\approx$ 0.5) in order to find how T_c depends on ρ_c.

(c) Derive an expression for the maximum central temperature reached by a star of mass M.

8.3 Application: minimum core mass for helium burning

Consider a star that consists completely of helium. Compute an estimate for the minimum mass for which such a star can ignite helium, as follows.

- Assume that helium ignites at $T_c = 10^8$ K.

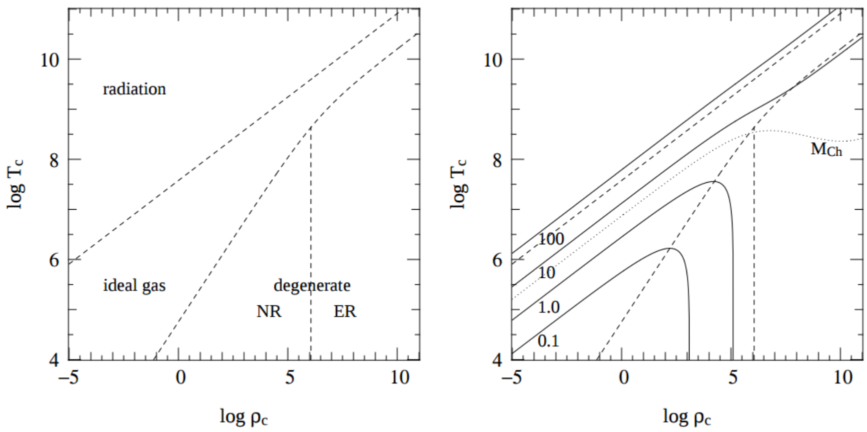

Figure 8.2. The equation of state in the $\log T_c - \log \rho_c$ plane (left panel), with approximate boundaries between regions where radiation pressure, ideal gas pressure, non-relativistic electron degeneracy and extremely relativistic electron degeneracy dominate, for a composition of $X = 0.7$ and $Z = 0.02$. In the right panel, schematic evolution tracks for contracting stars of $0.1 - 100\,M_\odot$ have been added.

- Assume that the critical mass can be determined by the condition that the ideal gas pressure and the electron degeneracy pressure are equally important in the star at the moment of ignition.

- Use the homology relations for the pressure and the density. Assume that $P_{c,\odot} = 10^{17}$g cm^{-1}s $^{-2}$ and $\rho_{c,\odot} = 60$ g cm^{-3}

9.1 Kippenhahn diagram of the ZAMS

Figure 9.8 indicates which regions in zero-age main sequence stars are convective as a function of the mass of the star.

(a) Why are the lowest-mass stars fully convective? Why does the mass of the convective envelope decrease with M and disappear for $M \sim > 1.3\ M_\odot$?

(b) What changes occur in the central energy production around $M = 1.3\ M_\odot$, and why? How is this related to the convection criterion? So why do stars with $M \approx 1.3\ M_\odot$ have convective cores while lower-mass stars do not?

(c) Why is it plausible that the mass of the convective core increases with M?

9.2 Conceptual questions

(a) What is the Hayashi Line (HL)? Why is it a line, in other words: why is there a whole range of possible luminosities for a star of a certain mass on the HL?

(b) Why do no stars exist with a temperature cooler than that of the HL? What happens if a star would cross over to the cool side of the HL?

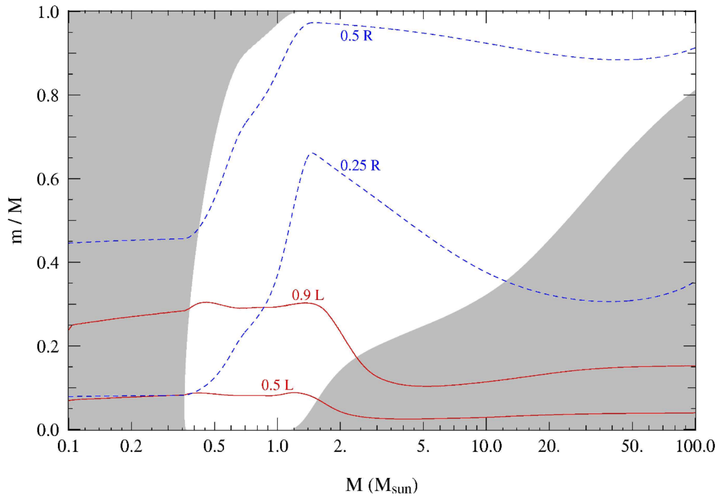

Figure 9.8. Occurrence of convective regions (gray shading) on the ZAMS in terms of fractional mass coordinate m/M as a function of stellar mass, for detailed stellar models with a composition $X = 0.70$, $Z = 0.02$. The solid (red) lines show the mass shells inside which 50% and 90% of the total luminosity are produced. The dashed (blue) lines show the mass coordinate where the radius r is 25% and 50% of the stellar radius R. (After Kippenhahn & Weigert.)

(c) Why is there a mass-luminosity relation for ZAMS stars? (In other words, why is there a unique luminosity for a star of a certain mass?)

(d) What determines the shape of the ZAMS is the HR diagram?

9.3 Central temperature versus mass

Use the homology relations for the luminosity and temperature of a star to derive how the central temperature in a star scales with mass, and find the dependence of T_c on M for the pp-chain and for the CNO-cycle. To make the result quantitative, use the fact that in the Sun with Tc $\approx 1.3\times10^7$ K the pp-chain dominates, and that the CNO-cycle dominates for masses M $\approx >$ 1.3 $M_\odot$. (Why does the pp-chain dominate at low mass and the CNO-cycle at high mass?)

10.1 Conceptual questions

(d) Explain the existence of a *Hertzsprung gap* in the HRD for high-mass stars. Why is there no Hertzsprung gap for low-mass stars?

(e) What do we mean by the *mirror principle?*

(f) Why does the envelope become convective on the red giant branch? What is the link with the *Hayashi line*?

10.3 Red giant branch stars

(a) Calculate the total energy of the Sun assuming that the density is constant, i.e. using the equation for potential energy E_{gr} = -3/5 GM^2/R. In later phases, stars like the Sun become red giants, with R $\approx$ $100R_\odot$. What would be the total energy, if the giant had constant density. Assume that the mass did not change either. Is there something wrong? If so, why is it?

(b) What really happens is that red giants have a dense, degenerate, pure helium cores which grow to $\sim 0.45M_\odot$ at the end of the red giant branch (RGB). What is the maximum radius the core can have for the total energy to be smaller than the energy of the Sun? (N.B. Ignore the envelope – why are you allowed to do this?)

(c) For completely degenerate stars, one has

$$R = 2.6 \times 10^9 \mu_e^{-5/3} \left(\frac{M}{M_\odot}\right)^{-1/3} \tag{15}$$

where μ_e is the molecular weight per electron and $\mu_e = 2$ for pure helium. Is the radius one finds from this equation consistent with upper limit derived in (b)?

10.4 Core mass-luminosity relation for RGB stars

Low-mass stars on the RGB obey a core mass-luminosity relation, which is approximately given by:

$$L \approx 2.3 \times 10^5 L_\odot \left(\frac{M_c}{M_\odot}\right)^6 \tag{16}$$

The luminosity is provided by hydrogen shell burning.

(a) Derive relation between luminosity L and the rate at which the core grows dM_c/dt. Use the energy released per gram in hydrogen shell burning.

(b) Derive how the core mass evolves in time, i.e, $M_c = M_c(t)$.

(c) Assume that a star arrives to the RGB when its core mass is 15% of the total mass, and that it leaves the RGB when the core mass is 0.45 $M_\odot$. Calculate the total time a 1 $M_\odot$ star spends on the RGB and do the same for a 2 $M_\odot$ star. Compare these to the main sequence (MS) lifetimes of these stars.

(d) What happens when the core mass reaches 0.45 $M_\odot$? Describe the following evolution of the star (both its interior and the corresponding evolution in the HRD).

(e) What is the difference in evolution with stars more massive than 2 $M_\odot$?

Extra Exercise 1

a) Using the homology relation for ZAMS stars derive

$$L \propto 1/k\mu^4 M^3 \tag{17}$$

$$R \propto \mu^{(\nu-4)/(\nu+3)} M^{(\nu-1)/(\nu-3)} \tag{18}$$

Using the following assumptions:

- Radiative stars
- Thermal equilibrium
- Hydrostatic equilibrium
- Constant opacity
- Ideal gas EOS
- Hydrogen burning $\epsilon \propto \rho$ T$^\nu$

b) Compare the above relation with data in Figure 9.5 and comment on why there are differences and by which violation of assumptions are they caused.

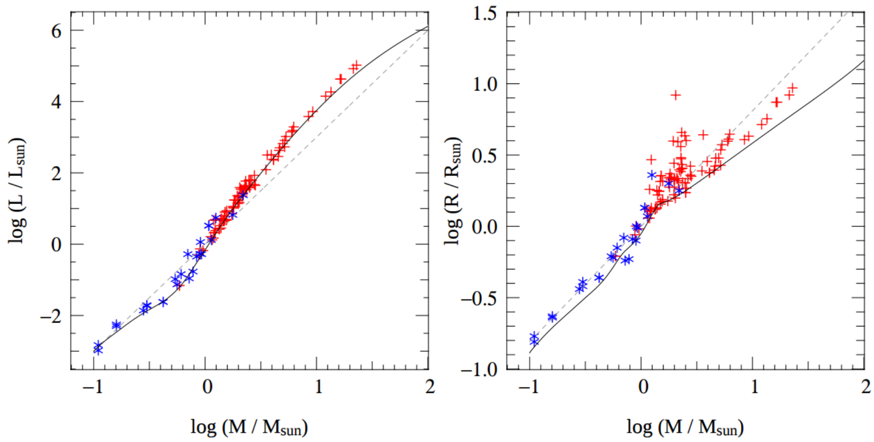

Figure 9.5. ZAMS mass-luminosity (left) and mass-radius (right) relations from detailed structure models with $X = 0.7, Z = 0.02$ (solid lines) and from homology relations scaled to solar values (dashed lines). For the radius homology relation, a value $\nu = 18$ appropriate for the CNO cycle was assumed (giving $R \propto M^{0.81}$); this does not apply to $M < 1\,M_\odot$ so the lower part should be disregarded. Symbols indicate components of double-lined eclipsing binaries with accurately measured M, R and L, most of which are MS stars.

Extra Exercise 2

Write and discuss any 3 outstanding features of massive stars' evolution.

Extra Exercise 3

Describe the unique features of nuclear burning in massive stars.

Extra Exercise 4

Derive the Mestel cooling law for white dwarfs (see equation below) and discuss it's dependences on M and μ_{ion}. The derivation should be accompanied by written comments that allow us to follow the mathematics.

$$\tau \approx \frac{1.05 \times 10^8 yr}{\mu_{ion}} \left(\frac{L/L_\odot}{M/M_\odot} \right)^{-5/7} \tag{19}$$

Extra Exercise 5

Describe mass loss in massive stars.

Extra Exercise 6

Describe in detail the properties of AGB stars.