Solutions 2: CO vs. CS excitation

a.

$$E_J = B[J(J+1)] \Rightarrow \nu = E_{J+1} - E_J = B[(J+1)(J+2) - J(J+1)] = 2B(J+1) \text{ cm}^{-1} \quad (1)$$

Convert cm^{-1} to GHz: $\nu[\text{GHz}] = \nu[\text{cm}^{-1}] \times c[\text{cm s}^{-1}] \times 10^{-9}$.

Molecule	$J+1 \rightarrow J$	ν (cm^{-1})	ν (GHz)
CO	$1 \rightarrow 0$	2B = 3.8	115
CO	$7 \rightarrow 6$	14B = 26.9	807
CS	$1 \rightarrow 0$	2B = 1.6	49
CS	$7 \rightarrow 6$	14B =11.4	343

b. In the two-level approximation, neglecting stimulated absorption and emission, we have:

$$\frac{n_u}{n_l} = \frac{n_c \, q_{lu}}{A_{ul}} \left[\frac{1}{1 + n_c \, q_{ul}/A_{ul}} \right] \quad (2)$$

n_c is the density of the collision partner.

q_{lu} is related to q_{ul} by: $q_{lu} = q_{ul} \dfrac{g_u}{g_l} \exp\left(-\dfrac{h\nu}{kT}\right)$. Hence:

$$\frac{n_u}{n_l} = \frac{n_c \, q_{ul}}{A_{ul}} \frac{g_u}{g_l} \exp\left(-\frac{h\nu}{kT}\right) \cdot \left[\frac{1}{1 + n_c \, q_{ul}/A_{ul}} \right] \quad (3)$$

$g = 2J+1 \Rightarrow \dfrac{g_u}{g_l} = \dfrac{2J_u + 1}{2J_l + 1} = 3$, for J_l=0 and J_u=1. Therefore:

$$\frac{n_1}{n_0} = 3\frac{n_c \, q_{10}}{A_{10}} \exp\left(-\frac{h\nu}{kT}\right) \cdot \left[\frac{1}{1 + n_c \, q_{10}/A_{10}} \right] \quad (4)$$

For CO:

- $\dfrac{n_c \, q_{10}}{A_{10}} = \dfrac{10^3 \, 2.6 \times 10^{-11}}{7.17 \times 10^{-8}} = 0.3626$
- $\dfrac{h\nu}{kT} = \dfrac{3.8}{0.695 \times 20} = 0.2766$
- $\dfrac{n_1}{n_0} = 3 \times 0.3626 \times \exp(-0.2766) \cdot \dfrac{1}{1 + 0.3626} = 0.6$

For CS:

- $\dfrac{n_c \, q_{10}}{A_{10}} = \dfrac{10^3 \, 2.2 \times 10^{-11}}{1.75 \times 10^{-6}} = 1.26 \times 10^{-2}$
- $\dfrac{h\nu}{kT} = \dfrac{1.6}{0.695 \times 20} = 0.1176$
- $\dfrac{n_1}{n_0} = 3 \times 1.26 \times 10^{-2} \times \exp(-0.1176) \cdot \dfrac{1}{1 + 1.26 \times 10^{-2}} = 0.03$

c. The outputs from the RADEX program can be found in "Radex_CO.pdf" and "Radex_CS.pdf" for CO and CS, respectively. These outputs were obtained using the line-command version of RADEX and include additional output (compared to the online version): wavelength, fluxes in $\mathrm{K\,km\,s^{-1}}$ and in $\mathrm{erg\,cm^{-2}\,s^{-1}}$, and populations of the upper and lower levels of the transitions.

Table 1: Some numbers and information

	CO(1–0)	CO(7–6)	CS(1–0)	CS(7–6)
ν (GHz)	115	807	49	392
A_{ul} (s^{-1})	$7.17{\times}10^{-8}$	$3.42{\times}10^{-5}$	$1.75{\times}10^{-6}$	$8.39{\times}10^{-4}$
q_{ul} (cm^3 s^{-1})	$2.6{\times}10^{-11}$	$7.7{\times}10^{-11}$	$2.2{\times}10^{-11}$	$4.2{\times}10^{-11}$
n_{crit} (cm^{-3})	$2.8{\times}10^3$	$4.4{\times}10^5$	$7.95{\times}10^4$	$2.0{\times}10^6$
Wavelength regime, 20 K	RJ	–	RJ	–
Wavelength regime, 60 K	RJ	–	RJ	RJ

First note/recall that:

- Critical density: $n_{\mathrm{crit}} = A_{ul}/q_{ul}$
  Optical depth: $\tau \propto N_u\, A_{ul}$
  Einstein-A coefficient: $A_{ul} \propto \mu^2$, where μ = dipole moment; since $\mu(\mathrm{CS}) > \mu(\mathrm{CO})$, CS has larger A_{ul} and a larger n$_{\mathrm{crit}}$ (see table).
  $\Rightarrow n_{\mathrm{crit}} \propto \mu^2$
  and $\tau \propto N_u\, \mu^2$

- $T_{\mathrm{R}} \propto T_{\mathrm{A}} \propto I_\nu = I_\nu(0)e^{-\tau} + B_\nu(T)(1 - e^{-\tau})$
  - In the optically thin case ($\tau \ll 1$) and weak background ($I_\nu(0) \ll 1$), we have $I_\nu \sim \tau\, B_\nu(T)$.
    $\rightarrow$ Additionally, in RJ regime[1], $I_\nu \propto \tau\, T \propto N_u\, \mu^2\, T \Rightarrow T_{\mathrm{R}} \propto N_u\, \mu^2\, T$
  - In the optically thick case ($\tau \gg 1$), we have $I_\nu \sim B_\nu(T)$.
    $\rightarrow$ Additionally, in RJ regime, $I_\nu \propto T \Rightarrow T_{\mathrm{R}} \propto T$

Taking the above into consideration, we see that:

- For a given transition, since $\tau \propto N$, an increase in N leads to an increase in τ.

- As $n(\mathrm{H_2})$ increases, T_{ex} increases towards T_{kin} and when $n(\mathrm{H_2}) > n_{\mathrm{crit}}$, $T_{\mathrm{ex}} \sim T_{\mathrm{kin}}$ (LTE).

- If $n(\mathrm{H_2}) > $ n$_{\mathrm{crit}}$ and $\tau \gg 1$, then $T_{\mathrm{R}} \sim T_{\mathrm{kin}} \sim T_{\mathrm{ex}}$.

- Increase in T_{kin} will lead to increase in T_{R} and τ.
  $\rightarrow$ Additionally, as $n(\mathrm{H_2})$ tends to n_{crit}, increase in T_{kin} will also lead to increase in T_{ex}.

- CS vs CO: due to the higher dipole moment of CS, rotational excitation is more difficult to achieve for this molecule, hence $T_{\mathrm{ex}}(\mathrm{CS})$ is lower for low $n(\mathrm{H_2})$ and low T_{kin}. Also at low $n(\mathrm{H_2})$, N and T, population is dominated by radiation and CMB dominates the radiation field such that $T_{\mathrm{ex}} \sim T_{\mathrm{CMB}} \sim 2.8$ K for the low-J transitions of CS.

- High-J vs low-J lines: since $n_{\mathrm{crit}} \propto \nu^3$, higher densities are needed to reach LTE for high-J compared to low-J transitions (e.g. $J = 7 - 6$ vs $J = 1 - 0$). Hence T_{ex} is generally lower with increasing J.

---

[1]Rayleigh-Jeans regime: $h\nu \ll kT \Rightarrow B_\nu(T) \sim 2kT\nu^2/c^2 \propto T$

Also note that the following temperature and density conditions:

$$T = 60\,\text{K},\ n(\text{H}_2) = 10^5\,\text{cm}^{-3},$$

in combination with $N = 10^{12}$ or $10^{14}\,\text{cm}^{-2}$ yield a very interesting result for CS, namely that we get a negative T_{ex}! This in fact corresponds to a maser, i.e. a population inversion.

<u>Final note</u>: As you may have noticed, T_{ex} is very high for, e.g., CS(1–0) with $T = 60\,\text{K}$, $n(\text{H}_2) = 10^6\,\text{cm}^{-3}$ and $N = 10^{14}\,\text{cm}^{-2}$. This is the explanation from Floris van der Tak:

"Suprathermal excitation of the ground state lines is a well-known effect in linear molecules, first described by Köppen & Kegel 1980 (A&AS 42, 59). The effect occurs for CO at densities around 10^4 and for CS around 10^6 cm^{-3}, and modest column densities. The reason is that collision rates for $\Delta J = 2$ tend to be larger than for $\Delta J = 1$, leading to a "traffic jam" in the lowest levels. The effect is strongest at densities just below critical; at lower densities, the lines become weak masers whereas at higher densities, the excitation becomes thermalized. The main observable consequence is a large uncertainty on column densities derived from ground-state lines at densities just below critical. In this regime, the excitation temperature is very sensitive to the volume density."

d. This is because we assume a H$_2$ density below the critical density of each of these transitions. Hence we are not dominated by collisions and effects from background radiation are not negligible. For $n(\text{H}_2) = 10^5$ or 10^6 cm^{-3} the ratios given by RADEX will be very close to what you would calculate using the method in problem *(b)*.