Problem 1. Molecular excitation: CO as an example

Consider the CO molecule. In order to determine the strength of a rotational transition, we need to know the fractional population of the molecule in rotational level J. In LTE, this is given by

$$p(J) = (2J+1)\exp(-E_J/kT)/Q(T),$$

where Q is the so–called partition function. At low temperatures,

$$Q(T) = \sum_{J=0}^{\infty}(2J+1)\exp(-E_J/kT),$$

where

$$E_J = BJ(J+1) \quad \mathrm{cm}^{-1}.$$

a. Derive an alternative, approximate form of this partition function by replacing the summation with an integral and by taking J to be a continuous variable. This formula should depend only on the temperature T and the rotation constant B.

b. Derive a formula for the value of $J{=}J_{max}$ at which the maximum population occurs for any specified temperature, based upon your integral form of the partition function.

c. Find J_{max} and the corresponding population for $T{=}10$ and $T{=}100$ K. The rotation constant for CO is $B{=}1.922529$ cm^{-1}, and $k{=}0.695$ cm^{-1} K^{-1}.

d. Which CO lines will be strongest (and most suitable to observe) in a (i) cold dark core; (ii) diffuse cloud; (iii) photon–dominated region; (iv) C–shock? Please motivate your answers.