Solar Physics 2010: Draft Notes on Polarization

Christoph U. Keller

May 15, 2010

Chapter 1

Basic Concepts of Optics

The polarization properties of electromagnetic waves is an integral part of optics. As such, it is useful to review the basic physics upon which polarimetry is based. Electromagnetic waves are a direct consequence of Maxwell's equations. The interaction of electromagnetic waves with matter as described by the material equations is the basis of optics. While it is possible to derive the properties of waves in anisotropic media of arbitrary electrical conductivity, it is more instructive to study the special cases that have a direct relation to most applications in polarimetry. In this chapter, we concentrate on the propagation of electromagnetic waves in isotropic, homogeneous media.

1.1 Electromagnetic waves in isotropic, homogeneous media

1.1.1 Maxwell and material equations

The basic physical equations on which polarimetry is based are the Maxwell equations ([**?**]). Here we look at the macroscopic electrical and magnetic fields, i.e. the fields are averaged over a volume that is large compared to the size of individual atoms and molecules. However, the volume may still be small with respect to the wavelength of light. The Maxwell equations are:

$$\nabla \cdot \boldsymbol{D} = 4\pi\rho, \tag{1.1}$$

$$\nabla \times \boldsymbol{H} - \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} = \frac{4\pi}{c}\boldsymbol{j}, \tag{1.2}$$

$$\nabla \times \boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = 0, \tag{1.3}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{1.4}$$

In the order of occurence, $\boldsymbol{D}$ is the *electric displacement*, ρ is the *electric charge density*, $\boldsymbol{H}$ is the *magnetic field vector*, c is the *speed of light in vacuum*, $\boldsymbol{j}$ is the *electric current density*, $\boldsymbol{E}$ is the *electric field vector*, and $\boldsymbol{B}$ is the *magnetic induction*. As usual, t is the time. The first equation describes Coulomb's law, the second describes Ampère's law, the third describes Faraday's law, and the fourth equation supresses magnetic monopoles.

The first two equations, and therefore $\boldsymbol{D}$ and $\boldsymbol{H}$ describe the interaction of electromagnetic fields with matter via electrical charges and currents. The Maxwell equations are, however, insufficient to describe electromagnetic fields in matter, and we need to add equations that relate $\boldsymbol{D}$, $\boldsymbol{H}$, and $\boldsymbol{j}$ to $\boldsymbol{E}$ and $\boldsymbol{B}$. In general, these *material equations* (or *constitutive equations*) are non-linear and time-dependent. Any anisotropy of the medium is reflected in these relations. It is often useful to write the relations in the following form:

$$\begin{aligned} \boldsymbol{D} &= \boldsymbol{E} + 4\pi\boldsymbol{P}\ , \\ \boldsymbol{B} &= \boldsymbol{H} + 4\pi\boldsymbol{M}\ , \\ \boldsymbol{j} &= \sigma\boldsymbol{E}\ . \end{aligned} \tag{1.5}$$

$\boldsymbol{P}$ is the *electric polarization*, $\boldsymbol{M}$ is the *magnetic polarization*, and *sigma* is the *eletric conductivity*, i.e. Ohm's law. Media for which $\sigma \neq 0$ are called *electric conductors*. For metals, σ decreases with temperature, while for semiconductors, σ increases with temperature. Media that have sufficiently small σ are called *insulators* or *dielectrica*.

Of course, the matieral equations 1.5 do not make the relation any simpler, but puts all the complications into $\boldsymbol{P}$ and $\boldsymbol{M}$. However, writing the relation in this way has some advantages: ferroelectric and ferromagnetic materials have non-vanishing $\boldsymbol{P}$ and $\boldsymbol{M}$, respectively, even in the absence of external fields.

External electric and magnetic fields can introduce internal electric and magnetic dipole (and higher order multipole) fields in matter, which are, to first order, linear in the electric and magnetic fields, respectively. The material equations 1.5 can then also be written as

$$\begin{aligned} \boldsymbol{D} &= \epsilon \boldsymbol{E} \, , \\ \boldsymbol{B} &= \mu \boldsymbol{H} \, , \\ \boldsymbol{j} &= \sigma \boldsymbol{E} \, . \end{aligned} \tag{1.6}$$

ϵ is the *dielectric constant* and μ is the *magnetic permeability*. For anisotropic media, these two scalars need to be replaced with the corresponding tensors of rank 2 (see Chapter **??**). Note that the isotropy of a medium can be broken by the anisotropy of the material itself (e.g. crystals, see Chapter **??**) or by external fields (e.g. Kerr effect, see Chapter **??**).

For sufficiently large field strengths, the relations between $\boldsymbol{D}, \boldsymbol{H}$ and $\boldsymbol{E}, \boldsymbol{B}$ become non-linear, e.g. $\boldsymbol{D}$ also depends on the product of components of $\boldsymbol{E}$. Such strong fields can be generated with focusing lasers or strong external fields. This is the area of *non-linear optics*, which is outside the scope of this book.

For vacuum, both scalars ϵ and μ are unity. For air, they are almost unity. For most materials, μ is almost unity, but for magnetic materials, it significantly deviates from unity. If $\mu < 1$, we call a medium *diamagnetic*, and for $\mu > 1$ it is called *paramagnetic*.

1.1.2 Wave equation

In a static, homogeneous medium (vanishing spatial and temporal derivatives of ϵ, μ, and σ) that has no (net) charge density ($\rho = 0$), Maxwell's equations 1.4 can be combined with the latter form of the material equations 1.6 (see e.g.[**?**]) to

$$\nabla^2 \boldsymbol{E} - \frac{\mu\epsilon}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \frac{4\pi\mu\sigma}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = 0, \tag{1.7}$$

which is the classical differential equation for a damped wave. The assumption of vanishing charge density ($\rho = 0$) is justified for any good conductor by the fact that the relaxation time for the charge density is much shorter than the inverse frequency of the electromagnetic wave (see e.g.[**?**]).

The magnetic field vector $\boldsymbol{H}$ obeys the equivalent damped wave equation

$$\nabla^2 \boldsymbol{H} - \frac{\mu\epsilon}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2} - \frac{4\pi\mu\sigma}{c^2}\frac{\partial \boldsymbol{H}}{\partial t} = 0, \tag{1.8}$$

The electric and magnetic field vector are therefore completely equivalent. However, in almost all materials relevant for polarimetry, the fundamental interaction between light and matter occurs through the magnetic field, which is the reason why we will often only consider the electric field vector, $\boldsymbol{E}$. Nevertheless, the magnetic field vector $\boldsymbol{H}$ is also important, in particular when considering interfaces between different media. The boundary conditions that apply to the magnetic field vector at these interfaces has considerable consequences for the transmission and reflection at such interfaces.

The magnitude of the damping term $\frac{\partial \boldsymbol{E}}{\partial t}$ in Eq. 1.8 is controlled by the finite conductivity σ. For non-conducting media ($\sigma = 0$), the wave is not attenuated by the medium. Finite conductivity ($\sigma > 0$) implies conversion of energy in the electromagnetic wave into thermal energy via Joule heating (e.g.[**?**]). The latter is proportional to the conductivity and the electrical field squared.

Good conductors such as metals therefore are extremely good absorbers. As we shall see, along with the absorption goes a high reflectivity, which makes metals good mirros and therefore useful in optics.

1.1.3 Plane-wave solutions to the wave equation

The wave equation can be solved by making the following damped plane-wave ansatz

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} \tag{1.9}$$

where the spatially and temporally constant *wave vector* $\boldsymbol{k}$ is normal to the surfaces of constant phase and its magnitude is the *wave number*. $\boldsymbol{x}$ is the spatial location, ω is the *angular frequency*, and t is the time. $\boldsymbol{E}_0$ is a (generally complex) vector independent of time and space. The damping is made possible by allowing the wave vector $\boldsymbol{k}$ to be complex. A complex $\boldsymbol{E}$ is, of course, not realistic, which implies that the real field vector is given by the real part of the right hand side of Eq. 1.9.

We have made a crucial choice in this plane wave ansatz, which will carry on throughout the rest of the book. An equally valid ansatz would be the one with the opposite sign of the phase term, i.e. $\boldsymbol{E}' = \boldsymbol{E}_0 e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$. The only important part is the fact that the spatial term $\boldsymbol{k}\cdot\boldsymbol{x}$ and the temporal term ωt have opposite signs. While one is free to choose one of the two forms, it is absolutely crucial to consistently use the same definition since this influences the signs of many equations to come, and in particular the sign of phase differences. Too many books on optics do not consistenly use the same definition of a plane wave, which has led to considerable confusion. In the following, I will always use the form shown in Eq. 1.9 and indicate when this choice has an influence on other equations, which should facilitate the comparison with equations in books that use the alternate plane wave ansatz. The sign convention used here is in agreement with the usual definition of plane waves in quantum mechanics as well as the books by Jackson and Born and Wolf.

When using the plane-wave ansatz in Eq. 1.9, it is useful to list the influence of the various vector operators and derivatives on these plane waves. They are summarized in the following relations

$$\nabla\cdot\boldsymbol{E} = i\boldsymbol{k}\cdot\boldsymbol{E} \tag{1.10}$$

$$\nabla\times\boldsymbol{E} = i\boldsymbol{k}\times\boldsymbol{E} \tag{1.11}$$

$$\frac{\partial}{\partial t}\boldsymbol{E} = -i\omega\boldsymbol{E} \tag{1.12}$$

$$\frac{\partial^2}{\partial t^2}\boldsymbol{E} = -\omega^2\boldsymbol{E} \tag{1.13}$$

By carrying out the temporal derivatives in 1.8 using this plane-wave ansatz, one obtains

$$\nabla^2\boldsymbol{E} + \frac{\omega^2\mu}{c^2}\left(\epsilon + i\frac{4\pi\sigma}{\omega}\right)\boldsymbol{E} = 0, \tag{1.14}$$

which is the Helmholtz wave equation. Of course, our ansatz is also a solution of this equation. In general, ϵ, μ, and even σ depend on the angular frequency ω.

In order for our ansatz to solve Eq. 1.14, the *dispersion relation* relating k^2 to the angular frequency ω by

$$\boldsymbol{k}\cdot\boldsymbol{k} = \frac{\omega^2\mu}{c^2}\left(\epsilon + i\frac{4\pi\sigma}{\omega}\right) \tag{1.15}$$

must hold.

By defining the *complex index of refraction* $\tilde{n}$ by

$$\tilde{n}^2 = \mu\left(\epsilon + i\frac{4\pi\sigma}{\omega}\right), \tag{1.16}$$

one obtains

$$\boldsymbol{k} \cdot \boldsymbol{k} = \frac{\omega^2}{c^2} \tilde{n}^2 \ . \tag{1.17}$$

It is customary to split the complex index of refraction into purely real and imaginary parts, i.e.

$$\tilde{n} = n \left(1 + i\kappa\right) = n + ik, \tag{1.18}$$

where n is the (real) index of refraction. κ and k have many names, and the same names have been used for either quanitities by different authors. In the following, we will call κ the *attenuation index* and k the *extinction coefficient*. To avoid confusion, the length of the wave vector will always be written in the form $|\boldsymbol{k}|$, and k is reserved for the imaginary part of the complex index of refraction. Either form of writing the complex index of refraction simplifies some of the equations in the following. For consistency, however, we will always use the form $\tilde{n} = n + ik$. When using the alternate form of the plane-wave ansatz, the sign of the imaginary part of the complex index of refraction changes, i.e. $\tilde{n}' = n - ik$.

The plane-wave ansatz must not only fulfill the wave equation, but it must also fulfill all of Maxwell's equations. For $\nabla \cdot \boldsymbol{E} = 0$ and $\nabla \cdot \boldsymbol{B} = 0$ to hold requires that

$$\boldsymbol{E}_0 \cdot \boldsymbol{k} = 0, \tag{1.19}$$

$$\boldsymbol{H}_0 \cdot \boldsymbol{k} = 0. \tag{1.20}$$

The electric and magnetic field vectors of electromagnetic waves in *isotropic* media are therefore perpendicular to the wave vector, i.e. we deal with purely transverse waves.

The other two Maxwell equations imply

$$\boldsymbol{H}_0 = \frac{\tilde{n}}{\mu} \boldsymbol{s} \times \boldsymbol{E}_0 \ , \tag{1.21}$$

where $\boldsymbol{s}$ is a unit vector in the direction of $\boldsymbol{k}$, i.e.

$$\boldsymbol{k} = |\boldsymbol{k}| \, \boldsymbol{s} \ . \tag{1.22}$$

$\boldsymbol{E}_0$, $\boldsymbol{H}_0$, and $\boldsymbol{k}$ are therefore orthogonal to each other and form a right-handed vector-triple. $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ only have the same phase if the the index of refraction of the medium, $\tilde{n}$, is a purely real quantity, i.e. the medium is not conductive. In a conductive medium with a complex $\tilde{n}$, $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are out of phase. Finally, $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ have a constant relationship, which makes it possible to only consider one of the two fields.

1.1.4 Energy propagation and the Poynting vector

The flow of energy density associated with an electromagnetic field is given by the *Poynting vector* $\boldsymbol{S}$,

$$\boldsymbol{S} = \frac{c}{4\pi} \left(\boldsymbol{E} \times \boldsymbol{H}\right) \ . \tag{1.23}$$

The magnitude of the Poynting vector is the amount of energy that flows through a unit area perpendicular to $\boldsymbol{S}$ within one time unit. Its direction is the direction of the energy flow.

For a plane-wave with complex amplitudes, the *time-averaged* Poynting vector is given by

$$\langle \boldsymbol{S} \rangle = \frac{c}{8\pi} \mathrm{Re} \left(\boldsymbol{E}_0 \times \boldsymbol{H}_0^*\right) \ , \tag{1.24}$$

where Re indicates the real part of a complex expression, * indicates the complex conjugate, and $\langle . \rangle$ indicates the time average. For a monochromatic wave as considered here, it is sufficient to average over one period of the wave to obtain the time average. The additional factor $\frac{1}{2}$ comes from the time average of the harmonic wave amplitude ($\langle \sin^2 \rangle = \frac{1}{2}$. Using the relation between $\boldsymbol{H}_0$ and $\boldsymbol{E}_0$ in Eq. 1.21, we can write the time-averaged Poynting vector as

$$\langle \boldsymbol{S} \rangle = \frac{c}{8\pi} \frac{|\tilde{n}|}{\mu} \left|E_0\right|^2 \boldsymbol{s} \ . \tag{1.25}$$

The Poynting vector is therefore parallel to the wave vector. However, this is only true for homogeneous, isotropic media and does not hold for arbitrary media.

The time-averaged energy density in a unit volume for a plane wave is given by

$$\langle u \rangle = \frac{|\tilde{n}|^2}{8\pi\mu} |E_0|^2 \ . \tag{1.26}$$

Dividing the length of the Poynting vector by the energy density provides the velocity of the energy density flow, which is given by

$$v = \frac{c}{|\tilde{n}|} \ , \tag{1.27}$$

which is the well-known result that electromagnetic waves propagate at the speed of light, which is reduced by a factor $|\tilde{n}|$ in a medium as compared to the speed of light in vacuum.

1.2 Polarization

As we have seen above, the spatially and temporally constant vector $\boldsymbol{E}_0$ lays in a plane perpendicular to the propagation direction $\boldsymbol{s}$. It therefore makes sene to represent $\boldsymbol{E}_0$ in a two-dimensional basis with unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, both of which are perpendicular to the propagation direction $\boldsymbol{s}$

$$\boldsymbol{E}_0 = E_1 \boldsymbol{e}_1 + E_2 \boldsymbol{e}_2. \tag{1.28}$$

E_1 and E_2 are arbitrary complex scalars.

For a damped plane-wave solution of the wave equation 1.8 with a given angular frequency ω and a given direction of the wave vector$\boldsymbol{s}$, we have four degrees of freedom (two complex scalars) in terms of the exact form of the wave. This additional property is called *polarization.* There are many ways to represent these four quantities, which we will discuss in detail in Chapter **??**. For the rest of this chapter it is sufficient to realize that if the phases of E_1 and E_2 are identical, the electric field vector will oscillate in a fixed plane, the orientation of which is such that its normal is perpendicular to $\boldsymbol{k}$ and determined by the ratio of the amplitudes of E_1 and E_2.

One could equally well define the polarization of light by using an equivalent decomposition of the magnetic field vector as compared to the electric field vector used above. The reason to prefer the electric field vector is due to the fact that the interaction of light with electrons is dominated by the electric field of the electromagnetic wave as compared to the magnetic field.

1.3 Quasi-monochromatic light

Monochromatic light as considered in the previous sections is a theoretical concept that has no equivalent in nature. This is true even for lasers that have clearly defined wavelengths. Laser lines have a very narrow distribution of wavelengths, but they are not monochromatic. By definition, monochromatic light is always fully polarized. Due to Heisenberg's uncertainty principle, monochromatic light implies infinite measuring times, which is unrealistic.

Light found in real life therefore needs to be expressed in terms of light that includes a range of wavelengths, even though this range can be very narrow, as is the case with lasers. This is called *quasi-monochromatic light*, since its properties are very similar to monochromatic light.

Quasi-monochromatic light can be described as a superposition of mutually incoherent monochromatic light beams whose wavelengths vary in a narrow range $\delta\lambda$ around a central wavelength λ_0. *Narrow* in the present context is defined as

$$\frac{\delta\lambda}{\lambda} << 1 \ . \tag{1.29}$$

The *coherence time* Δt is then given by

$$\Delta t = \frac{\lambda_0^2}{c\delta\lambda} \ . \tag{1.30}$$

For all practical purposes discussed in this book, the coherence time is much shorter than the typical speed with which common detectors work.

A measurement involving quasi-monochromatic light can then be written as the integral over the measurement time t_m. Since by definition, $t_m >> \Delta t$, it is customary to determine the limit of the integral when t_m goes to infinity. One can show that this limit exists for all practical purposes (see Born and Wolf for some mathematical details).

A quasi-monochromatic plane wave can be written in the same way as a monochromatic plane wave with the difference that the amplitude and phase are (slow) functions of the time for a given spatial location. *Slow* in this context means that variations occur on time scales much longer than the mean period of the wave, $\frac{2\pi}{\omega}$. The electric field vector for a quasi-monochromatic plane wave is the sum of electric field vectors of all monochromatic beams and is therefore given by

$$\boldsymbol{E}\left(t\right) = \boldsymbol{E}_0\left(t\right) e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} \ . \tag{1.31}$$

The reason for being able to write this is in the form is due to the fact that the range of wavelengths is small compared to the wavelength itself. This notation is, of course, only valid within the coherence length $\frac{\Delta t}{c}$.

The measured intensity of a quasi-monochromatic beam can then be expressed as

$$\langle \boldsymbol{E}_x \boldsymbol{E}_x^* \rangle + \left\langle \boldsymbol{E}_y \boldsymbol{E}_y^* \right\rangle = \lim_{t_m -> \infty} \frac{1}{t_m} \int_{-t_m/2}^{t_m/2} \boldsymbol{E}_x(t) \boldsymbol{E}_x^*(t) + \boldsymbol{E}_y(t) \boldsymbol{E}_y^*(t) dt \ , \tag{1.32}$$

where $\langle \cdots \rangle$ indicates the averaging over the measurement time t_m. Obviously, the measured intensity is independent of time.

Quasi-monochromatic also implies that frequency-dependent material properties such as the index of refraction can be assumed to be constant within the wavelength range $\Delta\lambda$.

1.4 Polychromatic light or white light

When the wavelength range of light is comparable to its wavelength, i.e. $\frac{\delta\lambda}{\lambda} \sim 1$, then we call this *polychromatic light*. Polychromatic light can be thought of as the (incoherent) sum of quasi-monochromatic beams that have large variations in wavelength. For such a broad wavelength range, we cannot write the electric field vector in a plane-wave form, and we have to explicitely take into account frequency-dependent meterial characteristics. Nevertheless, the intensity of polychromatic light is given by the sum of intensities of the constituting quasi-monochromatic beams since the (time-averaged) cross-products vanish because we assume incoherent superpositions.

Chapter 2

Description of Polarized Light

As we have seen in Chapter 1, the polarization of an electromagnetic wave can be expressed with two complex scalars. While this provides a very general description of polarized light, it is not always the most useful notation. Over the years, various other descriptions of the polarization have been developed, each having its particular advantages and disadvantages. This chapter presents the most common descriptions and discuss their range of applications.

2.1 Polarization Ellipse

We can write the plane-wave equation 1.9 in the form

$$\boldsymbol{E}(t) = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} \tag{2.1}$$

with

$$\boldsymbol{E}_0 = E_1 e^{i\delta_1}\boldsymbol{e}_x + E_2 e^{i\delta_2}\boldsymbol{e}_y. \tag{2.2}$$

$\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are unit vectors in the x and y directions, respectively. The beam propagates along the z-axis. The coefficients E_1 and E_2 are the (real) amplitudes and $\delta_{1,2}$ are the phases. At a given point $\boldsymbol{x}$, the time evolution of the electric field vector is described by an ellipse. Equation **??** describes two equations, one for each of the x and y components of the electrical field vector,$E_x(t)$ and $E_y(t)$, respectively. These are the real parts of the x and y components of Eq. **??**. The two equations can be combined to obtain

$$\left(\frac{E_x(t)}{E_1}\right)^2 + \left(\frac{E_y(t)}{E_2}\right)^2 - 2\left(\frac{E_x(t)}{E_1}\right)\left(\frac{E_y(t)}{E_2}\right)\cos\delta = \sin^2\delta\ , \tag{2.3}$$

where $\delta = \delta_1 - \delta_2$ (see e.g. Born and Wolf for the derivation). Figure 2.1 shows the ellipse with the various parameters used here.

The three parameters E_1, E_2, and δ are related to the major axis $2a$, the minor axis $2b$, and the orinentation ψ of the major axis with respect to the positive x-axis by the relations

$$a^2 + b^2 = E_1^2 + E_2^2 \tag{2.4}$$

$$\tan 2\psi = \tan 2\alpha \cos\delta \tag{2.5}$$

$$\sin 2\chi = \sin 2\alpha \sin\delta\ , \tag{2.6}$$

where the ellipticity is given by the ratio of the axes of the ellipse,

$$\tan\chi = \pm\frac{b}{a}\ . \tag{2.7}$$

The two signs indicate the direction in which the electric field vector describes the ellipse, either clockwise or counter-clockwise. α is given by

$$\tan\alpha = \frac{E_2}{E_1}\ . \tag{2.8}$$

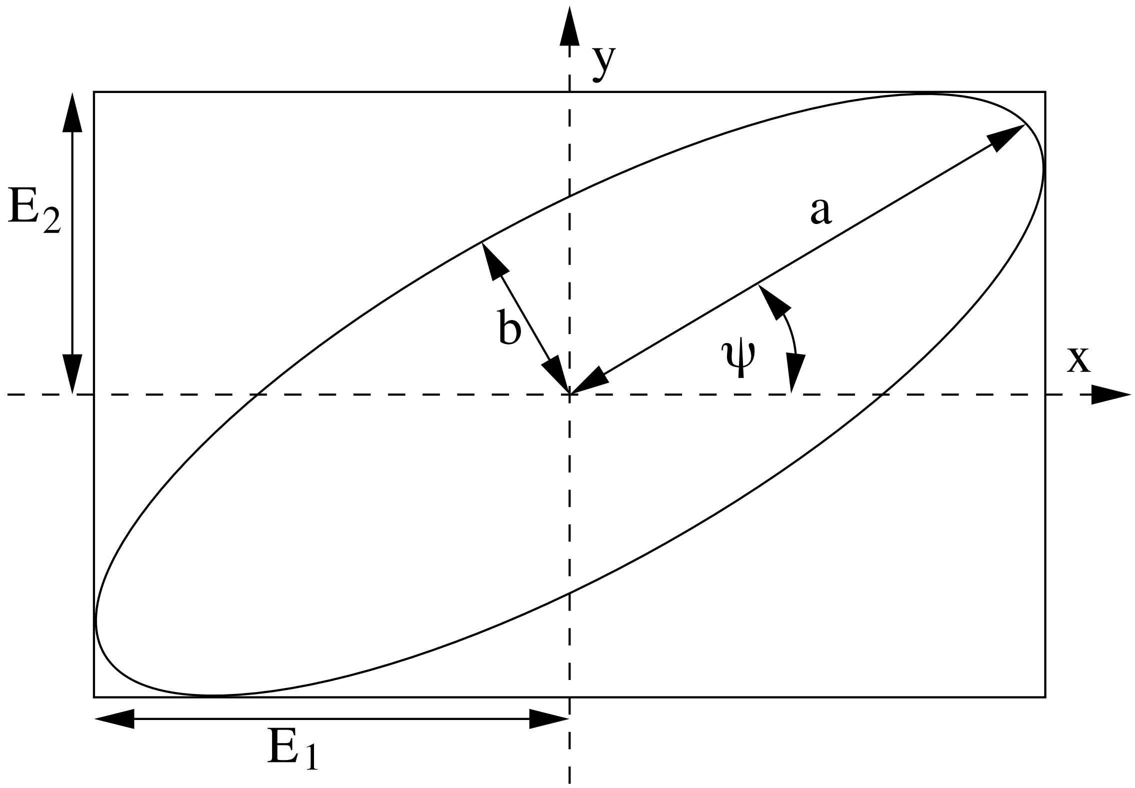

Figure 2.1: The polarization ellipse that describes the time evolution of the electric field vector at a fixed location in space. a and b are the major and minor semi-axes, ψ is the orientation of the ellipse, and $E_1, 2$ are the amplitudes in the x and y directions respectively.

For an ellipticity of 0, we have linear polarization with an orientation of ψ with respect to the positive x-axis, while for $\frac{b}{a} = 1$ we have circularly polarized light. For this case it becomes evident that we need both signs of the ratio. Otherwise it would be impossible to distinguish between the two senses of circular polarization since all the other parameters are identical for left- and right-circular polarization.

2.2 Jones formalism

2.2.1 Jones vector

The Jones formalism is most closely related to the description given in Chapter 1. There we had the following equation for the spatially and temporally invariant complex amplitude of the electric field vector:

$$\boldsymbol{E}_0 = E_1 \boldsymbol{e}_1 + E_2 \boldsymbol{e}_2. \tag{2.9}$$

At that point we only specified that $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are basis vectors in a plane perpendicular to the direction of propagation. In principle, the vectors could be complex. For the Jones formalism, we use unit vectors along the x and y axes as basis vectors. We then combine the two associated complex scalars E_x and E_y into a complex vector of length 2. If the electromagnetic wave travels along the z-axis, the *Jones vector* describing the polarization of this wave contains the complex amplitude electrical field in the form

$$\boldsymbol{e} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} . \tag{2.10}$$

The most intuitive way to understand the Jones vector is obtained by separating each vector component into an amplitude and a phase factor. The amplitudes can then be interpreted as the amplitude of the electric field vector projected onto the x and y directions, respectively. The phase difference between the x and y components represents the (constant) phase difference between the two electric field components. This phase difference is a crucial property that directly influences the type of polarization. If the two components have a phase difference that is a multiple of π, the electric field vector oscillates in a fixed plane. If the phase difference is $\pm\frac{\pi}{2}$, we have circularly polarized light. We are left with one more phase factor, which describes the *absolute* phase of the wave with respect to a given coordinate system. At the wavelengths discussed in this book, we only measure intensities, and the absolute phase is therefore not important. However, when superposing (adding) Jones vectors, one needs to keep track of the relative phases of the various vectors. In almost all cases, adding a constant phase to all vectors involved does not change observable effects. In many cases, it is therefore practical to set one of the phases or the sum of the phases to zero.

Since the wave equation derived from Maxwell's equations is linear, the sum of two solutions is again a solution. When considering two waves of the same angular frequency ω and the direction of propagation, the Jones vector of the sum is the sum of Jones vectors of the individual waves. The addition of Jones vectors therefore describes a *coherent* superposition of waves, and it inherently assumes that the waves have the same frequency and direction of propagation. it can be shown that is still is a good approximation as long as the directions of propagations are almost but not exactly the same.

Elements of Jones vectors, i.e. the complex electric field amplitudes are not observed directly by detectors in the wavelength range considered here. Therefore, observables always depend on products of the elements of Jones vectors such as the intensity I of the wave,

$$I = \boldsymbol{e} \cdot \boldsymbol{e}^* = e_x e_x^* + e_y e_y^*, \tag{2.11}$$

where * indicates the complex transpose operation. Similar to the phase issue, it is rare that we deal with absolute intensities, which is why the Jones vectors are often normalized to unit intensity. However, when adding Jones vectors, it is crucial to keep their relative amplitudes.

Let us look at some examples of normalized Jones vectors whose absolute phase is neglected. Light that is linearly polarized in the x-direction $\boldsymbol{e}_0$ (subscript 0 for horizontal) is represented by

$$\boldsymbol{e}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.12}$$

and similarly, light polarized in the y-direction $\boldsymbol{e}_90$ (subscript 90 for 90° or vertical) is represented by

$$\boldsymbol{e}_{90} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.13}$$

and light polarized at +45° is represented by

$$\boldsymbol{e}_{+45} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{2.14}$$

Finally, left and right circularly polarized light are represented by

$$\boldsymbol{e}_l = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{2.15}$$

and

$$\boldsymbol{e}_r = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ +i \end{pmatrix}. \tag{2.16}$$

Using the Jones vectors, we can create circularly polarized light of unit intensity by adding a horizontally and a vertically linearly polarized wave, each having an intensity of $\frac{1}{2}$ (corresponding to an amplitude of $\frac{1}{\sqrt{2}}$:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ e^{i\frac{\pi}{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ +i \end{pmatrix} = \boldsymbol{e}_r. \tag{2.17}$$

Here we have made use of the fact that $e^{i\phi} = \cos\phi + i\sin\phi$.

2.2.2 Jones matrix

The influence of a medium on the polarization property of an electromagnetic wave can be described by a 2 by 2 complex matrix J, the *Jones matrix*. If the original Jones vector is $\boldsymbol{e}$, the Jones vector after passing the medium is given by

$$\boldsymbol{e}' = \mathsf{J}\boldsymbol{e} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \boldsymbol{e} \ . \tag{2.18}$$

It is important to realize that this makes the assumption that the propoeries of the medium are not affected by the polarization state of the wave passing through it, i.e. this is valid in the area of linear optics.

The simplest form for a Jones matrix is the case of vacuum, i.e.

$$\mathsf{J}_{\text{vacuum}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ . \tag{2.19}$$

If a medium only changes the overall phase of the wave (e.g. a phase aberration in the Earth's atmosphere) by ϕ, the Jones matrix is described as

$$\mathsf{J} = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \ . \tag{2.20}$$

Such a matrix does not change the phase difference between the two Jones vector components, and therefore does not influence the polarization of the passing wave. Another simple Jones matrix is the one for a horizontal linear polarizer:

$$\mathsf{J} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ . \tag{2.21}$$

It is easy to verify that an arbitrarily polarized wave will be transformed into a horizontally linearly polarized wave. Of course, a vertically polarized wave will not be transmitted at all. We will introduce more complicated Jones matrices in Part II where we discuss various optical elements that affect the state of polarization.

When a wave successively passes through N different media (numbered 1 to N in order of the wave passing through them), the combined influence of all media on the polarization of the wave as indicated by the Jones matrix J is described by the product of the individual Jones matrices J_i, i.e.

$$\mathsf{J} = \mathsf{J}_N \mathsf{J}_{N-1} \cdots \mathsf{J}_2 \mathsf{J}_1 \tag{2.22}$$

The order of the matrices in the product is crucial because Jones matrices do not commute in general ($\mathsf{J}_1\mathsf{J}_2 \neq \mathsf{J}_2\mathsf{J}_1$).

Often we know the Jones matrix of a medium acting on a polarized wave. If we want to know the Jones matrix of the medium at a different angle, we can make use of the following relation:

$$\mathsf{J}' = \mathsf{R}(-\alpha)\mathsf{J}\mathsf{R}(\alpha) \ , \tag{2.23}$$

where the rotation matrix R is given by

$$\mathsf{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \ . \tag{2.24}$$

This is easy to derive when considering the transformation of a Jones vector under rotation around the z-axis (recall that the electric field vector only has x and y components).

The Jones calculus is the adequate way to describe the coherent superposition of polarized light because it operates on amplitudes rather than on intensities. Coherent superposition is important when considereing coherent light such as produced by lasers, interference effects, and the influence of optical aberrations on polarization. However, Jones vectors and matrices can only describe 100% polarized light because a monochromatic wave is always 100% polarized.

2.3 Coherency, Density, or Polarization Matrix

As we have seen above, observables need to be expressed as products of Jones vectors or their elements, which is somewhat unsatisfactory. It would be nice if we had a way to have a formalism that directly involves measurable quantities. Another limitation is the restriction to strictly monochromatic waves, which are always fully polarized. In nature, most of the light is not polarized because it is an *incoherent* superposition of many waves. The Jones formalism can only deal with coherent superpositions.

To deal with incoherent superpositions, i.e. adding the intensities and not the amplitudes of the waves, we need to find a description that involves suitably chosen products of the Jones vector components. The *coherency matrix* (also callsed *density matrix* or polarization matrix) formalism provides such an approach. It was introduced by Wiener (19??) and Wolf (????).

Going back to our representation of the spatially and temporally invariant part of the complex electric field vector in the plane perpendicular to the propagation direction,

$$\boldsymbol{E}_0 = E_x \boldsymbol{e}_x + E_y \boldsymbol{e}_y, \tag{2.25}$$

we can write the coherency matrix C as

$$\mathsf{C} = \boldsymbol{E} \otimes \boldsymbol{E}^*, \tag{2.26}$$

where $\otimes$ is the tensor product (also called outer product) and * indicates the complex conjugate transposed. Writing out the tensor product, we obtain

$$\mathsf{C} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} E_x E_x^* & E_x E_y^* \\ E_y E_x^* & E_y E_y^* \end{pmatrix} . \tag{2.27}$$

It is evident that the sum of the diagonal elements (also called the trace of a matrix and written as Tr) is equal to the intensity of the wave,

$$I = C_{11} + C_{22} = Tr\mathsf{C} = E_x E_x^* + E_y E_y^*. \tag{2.28}$$

It is clearly a Hermitian matrix since $\mathsf{C} = \mathsf{C}^*$. Furthermore, it can be shown that its determinant is positive:

$$det\mathsf{C} = C_{11}C_{22} - C_{12}C_{21} \geq 0. \tag{2.29}$$

A density matrix with purely diagonal elements of the same magnitude describes unpolarized light.

For a rotation of the coordinate system, the coherency matrix transforms as follows:

The coherency or density matrix approach is closely related to the quantum-mechanical density matrix approach. It is therefore particularly useful in the quantum-mechanical study of radiative transfer. However, for purely technical calculations, on which we focus here, it does not have any substantial advantages over the other approaches. Nevertheless, it is an important approach to describing polarized light.

2.4 Stokes parameters, vectors, and Mueller matrices

2.4.1 Stokes vector

"... the Stokes parameters are simple linear combinations of correlations that may exist between two mutually orthogonal components of the electric vector perpendicular to the direction of propagation of a fluctuating electromagnetic plane wave." (Wolf 2003).

The Stokes vector is similar to the density matrix formalism discussed above in that it describes averages of quantities related to the electric field vector. The components of the Stokes vector are measurable quantities, and Stokes vectors transform linearly when passing through a medium.

The four components of a Stokes vector $\boldsymbol{I}$ have been indicated with various characters:

$$\boldsymbol{I} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I \\ M \\ C \\ S \end{pmatrix} = \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} , \tag{2.30}$$

where the first form is used in many modern texts dealing with Stokes parameters, the second form can be found in some of the older texts (e.g. Perrin 1942), while the last two forms are useful when applying methods of linear algebra to Stokes vectors.

In terms of our plane wave descriptions using real or complex amplitudes, the Stokes vector has the following forms:

$$\boldsymbol{I} = \begin{pmatrix} E_x E_x^* + E_y E_y^* \\ E_x E_x^* - E_y E_y^* \\ E_x E_y^* + E_y E_x^* \\ i\left(E_x E_y^* - E_y E_x^*\right) \end{pmatrix} = \begin{pmatrix} E_1^2 + E_2^2 \\ E_1^2 - E_2^2 \\ 2E_1 E_2 \cos\delta \\ 2E_1 E_2 \sin\delta \end{pmatrix} \tag{2.31}$$

where $E_{x,y}$ are the complex amplitudes of the electric field vector, $E_{1,2}$ are the (real) amplitudes of the electric field vector in the x and y axes, and δ is the phase difference between the two components. The components of the Stokes vector are therefore real and obey the following inequality:

$$I^2 \geq Q^2 + U^2 + V^2 \; . \tag{2.32}$$

For completely polarized light, the equality holds. Otherwise, polarized light obeys the inequality. It makes therefore sense to define the *degree of polarization* as

$$P = \frac{\sqrt{Q^2 + U^2 + V^2}}{I} , \tag{2.33}$$

which is 1 for fully polarized light, and 0 for unpolarized light.

Addition of Stokes vectors corresponds to *incoherently* adding two beams, i.e. the two beams are uncorrelated in terms of amplitudes and phases.

2.4.2 Mueller matrices

Mueller matrices describe the (linear) transformation between Stokes vectors associated with optical elements and surfaces, i.e.

$$\boldsymbol{I}' = \mathsf{M}\boldsymbol{I} , \tag{2.34}$$

Mueller matrices have the following form:

$$\mathsf{M} = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} . \tag{2.35}$$

When a beam of light passes through N optical elements, each described by a Mueller matrix M_i, the combined Mueller matrix M' of the whole assembly is given by

$$\mathsf{M}' = \mathsf{M}_N \mathsf{M}_{N-1} \cdots \mathsf{M}_2 \mathsf{M}_1 \tag{2.36}$$

Note the reversed order of the Mueller matrices as compared to the order in which the light passes through the optical elements. This order is important since Mueller matrices do not commute in general.

Rotation of elements described by Mueller matrices are given by

$$\mathsf{M}' = \mathsf{R}(-\alpha)\mathsf{M}\mathsf{R}(\alpha) , \tag{2.37}$$

where α is the rotation angle and the rotation matrix R is given by

$$\mathsf{R}(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha & 0 \\ 0 & -\sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.38}$$

Since Stokes vectors and Mueller matrices operate on intensities and their differences, i.e. incoherent superpositions of light, they are not adequate to describe interference nor diffraction effects. However, they are ideally suited to describe partially polarized and unpolarized light.

2.5 Poincaré Sphere

Since a Stokes vector for a fully polarized beam obeys the following relationsship

$$I^2 = Q^2 + U^2 + V^2 , \tag{2.39}$$

we can think of this as the equation describing a sphere in cartesian coordinates labeled Q, U, and V. Without loss of generality, one can assume unit intensity, which makes the sphere have a radius of 1. A Stokes vector for a fully polarized beam then corresponds to a point on the sphere's surface. Points within the surface can be thought of as representing partially polarized light. This is the *Poincaré sphere* as first described by Henri Poincaré in 1892 in his book on the mathematical theory of light (Poincaré 1892).

The influence of optical elements on polarized light that do not change the degree of polarization can then be described by transformations on the sphere's surface, i.e. rotations around an axis that goes through the center of the sphere, which are largely exercies in spherical geometry. The Poincaré sphere has been used extensively in the past, in particular before the Mueller matrix approach was established. However, the advent of computers that can easily deal with large sets of Mueller matrices has rendered the Poincaré sphere formalism rather unimportant. However, it sometimes offers unique ways to understand certain issues such as the operation of achromatic retarders.

The plane defined by the Q and U axes defines the equator. Purely linearly polarized light is therefore represented by a point on the equator. Since Q and U are 90° apart on the Poincaré sphere, but in real space Q becomes U by a 45° rotation, the location of a linearly polarized beam with an orientation of its plane of polarization at angle θ is then represented by a point on the equator at 'longitude' 2θ, where the zero or Greenwich longitude is defined by a purely linearly polarized beam with only Stokes Q. Circularly polarized light is located at the poles of the sphere. Apart from this special locations, other locations on the sphere represent elliptically polarized light.

2.6 Important polarization states in various formalisms

Table 2.1: Normalized polarization states in various formulations.

description	Jones	coherency	Stokes	ellipse
unpolarized	—		(1,0,0,0)	—
linear at 0°			(1,1,0,0)	
linear at 90°			(1,-1,0,0)	
linear at 45°			(1,0,1,0)	
linear at -45°			(1,0,-1,0)	
linear at ϕ			(1,q,u,0)	
right circular			(1,0,0,1)	
left circular			(1,0,0,-1)	
elliptical			(1,q,u,v)	