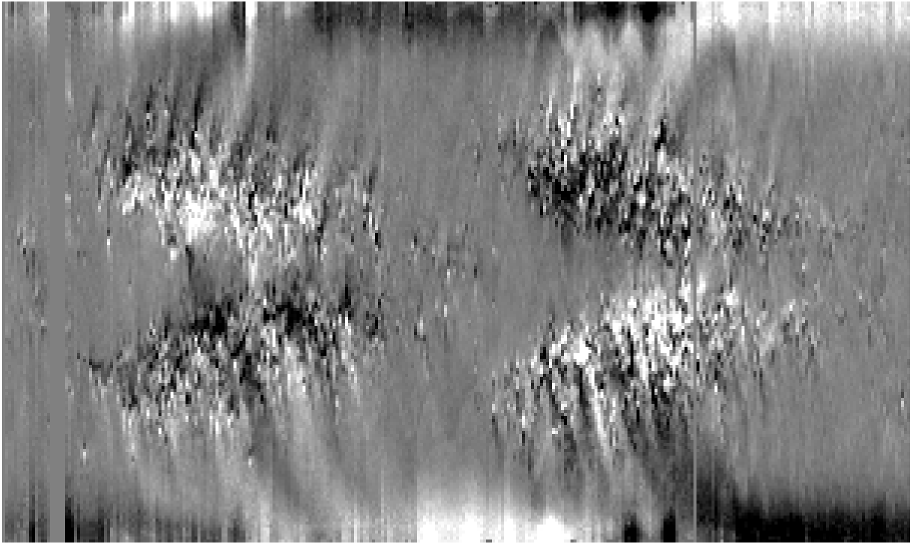
Outline

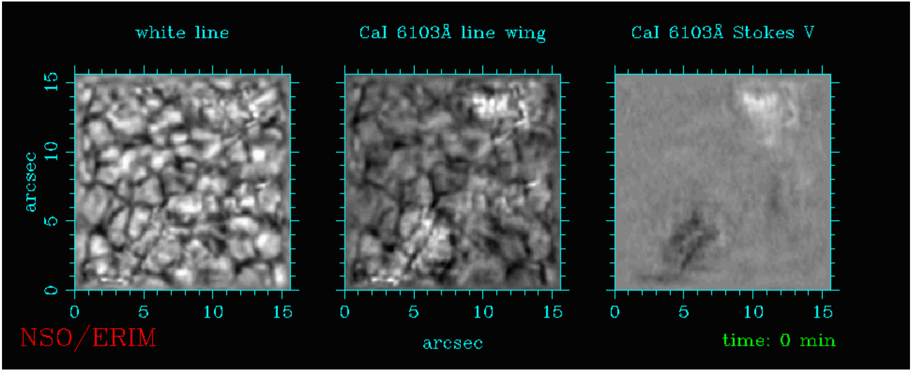
1. Motivation
2. Electromagnetic Equations
3. Plasma Equations
4. Frozen Fields
5. Cowling's Antidynamo Theorem

Why MHD in Solar Physics

Synoptic Kitt Peak Magnetogram over 2 Solar Cycles



Evolution of Small-Scale Fields in the Quiet Sun



Electromagnetic Equations (SI units)

Maxwell's and Matter Equations

$$
\begin{aligned}
\nabla \cdot \vec{D} &= \rho_c \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\
\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} &= \vec{j} \\
\vec{D} &= \epsilon \vec{E} \\
\vec{B} &= \mu \vec{H}
\end{aligned}
$$

Symbols

- $\vec{D}$ *electric displacement*
- ρ_c *electric charge density*
- $\vec{H}$ *magnetic field vector*
- c *speed of light in vacuum*
- $\vec{j}$ *electric current density*
- $\vec{E}$ *electric field vector*
- $\vec{B}$ *magnetic induction*
- t time
- ϵ *dielectric constant*
- μ *magnetic permeability*

Simplifications

- use vacuum values: $\epsilon = \epsilon_0$, $\mu = \mu_0$
- by definition: $(\epsilon_0 \mu_0)^{-\frac{1}{2}} = c$
- eliminate $\vec{D}$ and $\vec{H}$ and rearrange

Equations from before

$$
\begin{aligned}
\nabla \cdot \vec{D} &= \rho_c \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\
\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} &= \vec{j} \\
\vec{D} &= \epsilon \vec{E} \\
\vec{B} &= \mu \vec{H}
\end{aligned}
$$

Simplified Equations

$$\nabla \cdot \vec{E} = \frac{\rho_c}{\epsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

Further Simplifications

- magnetic field generation by currents and changing electrical fields (displacement current)

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

- Maxwell's equations are relativistic
- non-relativistic MHD, i.e. $v \ll c$ where v typical velocity
- neglect displacement current (see exercises)

$$\nabla \times \vec{B} = \mu_0 \vec{j}$$

- $\nabla \cdot \left(\nabla \times \vec{B}\right) = 0 \Rightarrow \nabla \cdot \vec{j} = 0$, no local charge accumulation, currents flow in closed circuits
- magnetic dominates over electrical energy density
- plasma is neutral, i.e. $\rho_c = 0$

Charge Neutrality

- electrically neutral plasma: $n_+ - n_- \ll n$
- charge imbalance $\rho_c = (n_+ - n_-)e$
- from $\nabla \cdot \vec{E} = \frac{\rho_c}{\epsilon_0}$ we get

$$\rho_c \approx \frac{\epsilon_0 E}{l}$$

- using $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\frac{E}{l} \approx \frac{B}{t}$$

- with $t = l/v$

$$\rho_c \approx \frac{\epsilon_0 v B}{l}$$

- charge neutrality condition becomes

$$\frac{\epsilon_0 v B}{e l} \ll n$$

- condition is well satisfied in solar photosphere

Generalized Ohm's Law

- normally $\vec{j} = \sigma \vec{E}$, σ is *electrical conductivity*
- plasma moving at non-relativistic speed with respect to electrical and magnetic fields
- $\vec{j}_1 = \sigma \vec{E}$ due to electrical field
- $\vec{j}_2 = \sigma \left(\vec{v} \times \vec{B} \right)$ due to transformation to rest frame
- Ohm's law for neutral plasma

$$\vec{j} = \sigma \left(\vec{E} + \vec{v} \times \vec{B} \right)$$

Induction Equation

$$\nabla \times \vec{B} = \mu_0 \vec{j}, \;\; \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \;\; \vec{j} = \sigma \left(\vec{E} + \vec{v} \times \vec{B} \right)$$

- eliminate $\vec{E}$ and $\vec{j}$

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \left(-\vec{v} \times \vec{B} + \frac{1}{\sigma}\vec{j} \right) = \nabla \times \left(\vec{v} \times \vec{B} \right) - \nabla \times \left(\eta \nabla \times \vec{B} \right)$$

  $\eta = 1/\left(\mu_0 \sigma\right)$: *magnetic diffusivity*

- using $\nabla \times \left(\nabla \times \vec{B} \right) = \nabla \left(\nabla \cdot \vec{B} \right) - \left(\nabla \cdot \nabla \right) \vec{B}$ we obtain the *induction equation*

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{v} \times \vec{B} \right) + \eta \nabla^2 \vec{B}$$

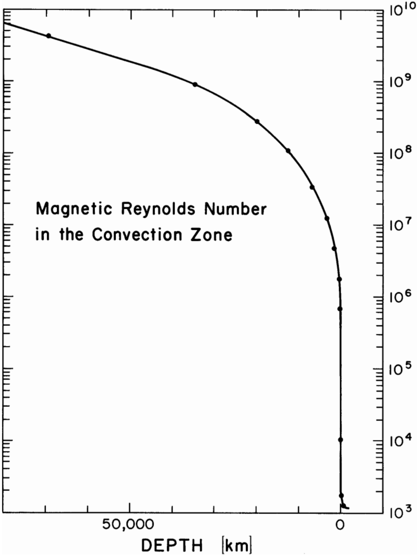
Interpretation of Induction Equation

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{v} \times \vec{B} \right) + \eta \nabla^2 \vec{B}$$

- for given $\vec{v}$, $\vec{B}$ can be determined with induction equation and $\nabla \cdot B = 0$
- first term describes generation of magnetic fields by plasma motions and magnetic field
- field cannot be created, only amplified
- second term describes Ohmic diffusion
- second term can mostly be neglected because of large length scales (often (wrongly) called *infinite conductivity limit*)
- ratio of magnitudes of the two terms with typical length, velocity scales l, v is *magnetic Reynolds number*

$$R_m = \frac{lv}{\eta}$$

Magnetic Reynolds Number in the Sun



Electric Field Interpretation

- electrical current is determined by $\vec{j} = \nabla \times \frac{\vec{B}}{\mu_0}$
- electrical field, but not current is determined by

$$\vec{E} = -\vec{v} \times \vec{B} + \frac{\vec{j}}{\sigma}$$

- $\vec{v} \times \vec{B}$ produces electric field of order

$$E_{\vec{v}\times\vec{B}} \sim vB \sim 100\mathrm{Vm}^{-1}$$

  with v=1000 ms^{-1} and B=1000 G
- $\frac{1}{\sigma}\vec{j}$ produces electric field of order

$$E_{\frac{1}{\sigma}\vec{j}} \sim \frac{1}{\sigma\mu_0}\frac{B}{l} \sim 10^{-5}\mathrm{Vm}^{-1}$$

  assuming a typical length scale of $l = 10^7$ m and a conductivity of $\sigma = 10^3$ mho m^{-1}

Electric Field and Electric Current

- generalized Ohm's law:

$$\vec{j} = \sigma\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

- electric current determined by

$$\vec{j} = \frac{1}{\mu_0}\left(\nabla \times \vec{B}\right)$$

- electric field almost always determined by

$$\vec{E} = -\vec{v} \times \vec{B}$$

- not infinite conductivity, but large length scale, because

$$E \approx vB$$

$$\frac{1}{\sigma} j \approx \frac{B}{\mu\sigma l}$$

Electrical Conductivity

- Spitzer conductivity provides easy way to calculate the conductivity of plasma
- in temperature minimum region, number of electrons to neutral atoms is $\frac{n_e}{n_n} = 0.001$
- since less than 10^{-6} of hydrogen is ionized, most electrons must come from metals
- collision frequency is high enough so that charged particles transfer momentum to neutrals
- despite small relative electron numbers, plasma can be considered as a single medium

Plasma Equations

Mass Conservation and Equation of Motion

- magnetic field and mass flows coupled by induction equation
- plasma motion must also obey other laws
- mass convservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

  where ρ is mass density

- equation of motion (force balance)

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = -\nabla p + \vec{j} \times \vec{B} + \vec{F}_{\mathrm{gravity}} + \vec{F}_{\mathrm{viscosity}}$$

- perfect gas law with gas constant R and mean atomic weight μ:

$$p = \frac{R}{\mu} \rho T$$

Lorentz Force

- Lorentz force $\vec{j} \times \vec{B}$ perpendicular to field lines
- motion and density variations along field lines must be produced by other forces
- rewrite Lorentz force in terms of $\vec{B}$ alone

$$\vec{j} \times \vec{B} = \left(\nabla \times \vec{B}\right) \times \frac{\vec{B}}{\mu_0}$$

- use vector identity for triple vector product

$$\vec{j} \times \vec{B} = \left(\vec{B} \cdot \nabla\right) \frac{\vec{B}}{\mu_0} - \nabla \left(\frac{B^2}{2\mu_0}\right)$$

- first term: *magnetic tension*, i.e. variations of $\vec{B}$ along $\vec{B}$, effect when field lines are curved
- second term: *magnetic pressure*
- along magnetic field lines, the two components cancel

Magnetic Tension Force

- magnetic tension force $\left(\vec{B}\cdot\nabla\right)\frac{\vec{B}}{\mu_0}$
- write magnetic field as $\vec{B} = B\vec{s}$ to obtain

$$
\begin{aligned}
\frac{B}{\mu_0}\frac{d}{ds}\left(B\vec{s}\right) &= \\
\frac{B}{\mu_0}\frac{dB}{ds}\vec{s} + \frac{B^2}{\mu_0}\frac{d\vec{s}}{ds} &= \\
\frac{d}{ds}\left(\frac{B^2}{2\mu_0}\right)\vec{s} + \frac{B^2}{\mu_0}\frac{\vec{n}}{R_c}
\end{aligned}
$$

where $\vec{n}$ is the principle normal to the field line and R_c is the radius of curvature of the field line

Frozen Fields

The Theorem

- for $R_m \gg 1$, typical for the Sun, induction equation becomes

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{v} \times \vec{B}\right)$$

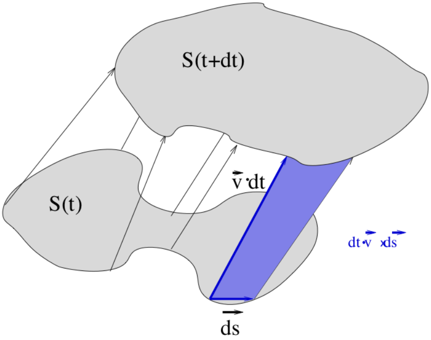
  and Ohm's law becomes

$$\vec{E} + \vec{v} \times \vec{B} = 0$$

- Frozen flux theorem by Alvén:

  In a perfectly conducting plasma, magnetic field lines behave as if they move with the plasma.

The Proof



- consider closed curve c enclosing surface S moving with plasma
- in time δt, a piece $\delta\vec{s}$ of curve c sweeps an element of area $\vec{v}\delta t \times \delta\vec{s}$
- magnetic flux of $\vec{B} \cdot \left(\vec{v}\delta t \times \delta\vec{s}\right)$ passes through this area
- magnetic flux through S is given by $\iint_S \vec{B} \cdot d\vec{S}$

The Proof

- rewrite flux through the sides $\vec{B} \cdot \left(\vec{v}\delta t \times \delta\vec{s}\right)$ as $-\delta t \vec{v} \times \vec{B} \cdot \delta\vec{s}$
- rate of change of magnetic flux through S is then given by

$$\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} - \oint_c \vec{v} \times \vec{B} \cdot d\vec{s}$$

  first term due to change of magnetic field in time, second due to motion of boundary

- with Stokes' theorem, second term becomes

$$-\iint_S \nabla \times \left(\vec{v} \times \vec{B}\right) \cdot d\vec{S}$$

- rate of change of magnetic flux through S

$$\iint_S \left(\frac{\partial \vec{B}}{\partial t} - \nabla \times \left(\vec{v} \times \vec{B}\right)\right) \cdot d\vec{S} = 0$$

Cowling's Antidynamo Theorem

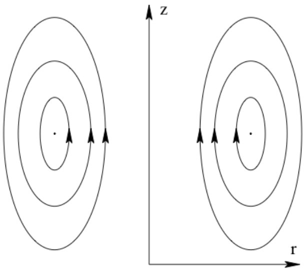
Why generating magnetic fields is not easy

T.G.Cowling (1934):

A steady axisymmetric magnetic field cannot be maintained.

- steady process $\Rightarrow \frac{\partial}{\partial t} = 0$
- axial symmetry $\Rightarrow \frac{\partial}{\partial \phi} = 0$ in cylindrical coordinate system (r, ϕ, z)

Toroidal and Poloidal Components



- separate magnetic field into azimuthal (toroidal) and poloidal (radial and axial) components

$$\vec{B}(r,\phi,z) = B_\phi(r,z)\vec{e}_\phi + \vec{B}_p(r,z)$$

- consider only $\vec{B}_p$ in meridional planes through axis

The Proof

- magnetic configuration must be the same in all meridional planes
- $\vec{B}_p$ field lines closed because $\frac{\partial}{\partial\phi} = 0$ and therefore $\nabla \cdot \vec{B}_p = 0$
- at least one *neutral point* where $\vec{B}_p(r,z) = 0$

The Proof

- in points where $\vec{B}_p = 0$: $\vec{B} = B_\phi \vec{e}_\phi$
- $j_\phi \neq 0$ because $\nabla \times \vec{B} = \mu_o \vec{j}$
- integrate Ohm's law $\frac{1}{\sigma}\vec{j} = \vec{E} + \vec{v} \times \vec{B}$ through curve c of all neutral points

$$\oint_c \frac{1}{\sigma}\vec{j} \cdot d\vec{s} = \oint_c \vec{E} \cdot d\vec{s} + \oint_c \vec{v} \times \vec{B} \cdot d\vec{s}$$

- since $d\vec{s}$ has only azimuthal component and using Stokes' theorem

$$\oint_c \frac{1}{\sigma} j_\phi ds = \int_S \left(\nabla \times \vec{E}\right) \cdot d\vec{S} + \oint_c \vec{v} \times \vec{B} \cdot d\vec{s}$$

- but $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0$ and $\vec{v} \times \vec{B} \cdot d\vec{s} = 0$
- therefore $\oint_c \frac{1}{\sigma} j_\phi ds = 0$, which cannot be