Planets and Exoplanets 2010:
Exercises to Planet Formation and Migration

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1 A collapsing molecular cloud

The gravitational potential energy $e_{\rm G}$ of a mass m at a distance r from a mass M equals

$$e_{\rm G} = -\frac{GmM}{r}, \tag{1}$$

with G the gravitational constant.

a. Derive the following formula that describes the gravitational potential energy, $E_{\rm G}$, of a uniform, spherical cloud of density ρ and radius R:

$$E_{\rm G} = -\frac{3}{5}\frac{GM^2}{R} \tag{2}$$

b. Derive the formula for the Jeans mass, $M_{\rm J}$, of this cloud:

$$M_{\rm J} = \left(\frac{5kT}{G\mu}\right)^{3/2}\left(\frac{3}{4\pi\rho}\right)^{1/2}, \tag{3}$$

with μ the average molecular mass in the cloud. Hints: if the mass of a cloud equals its Jeans mass, its gravitational potential energy $E_{\rm G}$ equals (minus) twice its kinetic energy, $E_{\rm K}$. Assume the cloud's kinetic energy equals its thermal energy, thus $E_{\rm K} = (3/2)NkT$, with N the total number of molecules in the cloud, k Boltzmann's constant, and T the temperature of the cloud.

c. Show that if the cloud collapses isothermally, it becomes more *unstable* as it shrinks.

2 Orbiting a planet

A satellite or ring particle that orbits a planet experiences gravitational attraction from both the planet and the star. The maximum orbital distance of such a satellite is given by the radius $R_{\rm H}$ of the so-called *Hill sphere*, which can be approximated as follows

$$R_{\rm H} \approx \left(\frac{m_{\rm p}}{m_{\rm s}}\right)^{1/3} a, \tag{4}$$

with $m_{\rm s}$ and $m_{\rm p}$ the masses of the star and the planet, and a the planet's orbital distance. Here, we assume circular orbits.

a. Derive Equation 4. Hint: assume that at the edge of the Hill sphere, the velocity of the satellite due to the attraction of the star equals its velocity due to the attraction of the planet.

b. Calculate the radius of the Earth's Hill sphere.

c. How does this compare to the orbital distance of the moon?

d. Due to tidal effects, the distance between the moon and the Earth is increasing. Calculate the orbital period of the moon just before it is lost to the Earth.

3 Bonus exercise: Building planetesimals

During the early stages of planet formation, planetesimals can grow through *runaway* or *oligarchic growth*. This rapid growth will cease when a planetary embryo has consumed most of the planetsimals within its gravitational reach. Planetsimals that come within about 4 times the planetary embyro's Hill sphere will eventually come close enough to the planetary embryo during one of their orbits that they may be accreted.

The mass of a planetary embryo orbiting its star at a distance r, which has accreted all of the planetsimals within an annulus of width $2\Delta r$ is (Eq. 12.27 from *Planetary Sciences*)

$$M_{\rm p} = \int_{r-\Delta r}^{r+\Delta r} 2\pi r' \sigma_\rho(r') dr' \approx 4\pi r \Delta r \sigma_\rho(r), \tag{5}$$

with $\sigma_\rho(r)$ the disk's surface density at distance r.

Setting $\Delta r = 4R_{\rm H}$, we obtain the *isolation mass*, $M_{\rm i}$, which is the largest mass to which a planetary embryo can grow by runaway accretion.

a. Using the expression for $R_{\rm H}$ from Eq. 4, derive an expression for $M_{\rm i}$ of a planetesimal orbiting a star with mass $m_{\rm s}$ at a distance r.

b. Assuming a solar mass star (2×10^{33} g), and a disk surface density that varies as $\sigma_\rho = 10\ \ r^{-1}$ g cm^{-2}, with r the distance to the star in AU, calculate $M_{\rm i}$ at distances of 1 and 5 AU, using the expression derived under a. How do these masses compare to the actual masses of the Earth and Jupiter?

c. Repeat the calculation at 5 AU with a surface density that is twice as large (to account for condensed ices on the dust particles). How does this mass compare to the actual mass of Jupiter?