Planets and Exoplanets 2009: Exercises to Lecture 5
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1 Black Body Radiation

A blackbody is so named because it absorbs all electromagnetic energy incident upon it; it is completely black. To be in thermal equilibrium, however, such a body must radiate energy at the same rate as it absorbs energy; otherwise, the body will heat up or cool down.

The energy that a blackbody radiates per unit frequency interval is described by Planck's Radiation Law:

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}, \tag{1}$$

with B_ν the *specific intensity* or *brightness* of the radiation (in J m^{-2} Hz^{-1} s^{-1} ster^{-1} or W m^{-2} Hz^{-1} ster^{-1}), T the body's temperature, h Planck's constant, c the speed of light, and k Boltzmann's constant:
$h = 6.626 \cdot 10^{-34}$ m^2 kg s^{-1}
$c = 2.998 \cdot 10^8$ m s^{-1}
$k = 1.381 \cdot 10^{-23}$ m^2 kg s^{-2} K^{-1}
Because frequency ν and wavelength λ of electromagnetic radiation are related by $\lambda\nu = c$, Planck's Law may also be expressed in terms of the intensity emitted per unit wavelength interval:

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5} \left[\frac{1}{e^{hc/\lambda kT}} - 1 \right], \tag{2}$$

with B_λ the specific intensity or specific brightness of the radiation (in J m^{-2} m^{-1} s^{-1} ster^{-1} or W m^{-2} m^{-1} ster^{-1}).

a. Derive Eq. 2 from Eq. 1.

Hint: the intensity $B_\nu \Delta\nu$ that is emitted in frequency interval $\Delta\nu$ equals the intensity $B_\lambda \Delta\lambda$ that is emitted in wavelength interval $\Delta\lambda$, where $|\Delta\nu| = |c\Delta\lambda/\lambda^2|$.

b. A blackbody appears to become bluer as its temperature increases. The wavelength $\lambda_{\rm max}$ at which a Planck curve peaks is described by Wien's Displacement Law:

$$\lambda_{\rm max} = 2.898 \times 10^{-3}/T, \tag{3}$$

with $\lambda_{\rm max}$ in meters and T in kelvins.

Derive Wien's Displacement Law in units of wavelength (Eq. 3).

Hint: Calculate the wavelength at which the derivative $dB_\lambda/d\lambda = 0$.

c. Write down Wien's Displacement Law in units of frequency.

d. Calculate the wavelength at which the continuum spectrum of our Sun peaks, assuming that $T = 5700$ K.

e. Calculate the wavelength at which the emitted continuum spectrum of Jupiter peaks, assuming that $T = 125$ K.

f. When Jupiter was much younger, it was much hotter. Calculate the wavelength at which the emitted continuum spectrum of a younger Jupiter peaked, assuming that $T = 500$ K.

g. The specific intensity B_ν or B_λ of a body is the intensity at frequency ν or wavelength λ that is emitted in one direction. To calculate the *flux density* F_ν or F_λ above the surface of a body, this intensity has to be integrated over the directions into which the radiaton is emitted. It can be shown that

$$F_\nu(T) = \pi B_\nu(T) \quad \text{or} \quad F_\lambda(T) = \pi B_\lambda(T), \tag{4}$$

where F_ν has the dimension W m^{-2} Hz^{-1}, and F_λ W m^{-2} m^{-1}).

Derive an expression for the total emitted flux F_ν (in W Hz^{-1}) or F_λ (in W m^{-1}) of a body with radius R that arrives at a distance d from the body.

h. The continuum spectrum of a planet consists of emitted, thermal radiation at the longer wavelengths (lower frequencies) and reflected stellar radiation at the shorter wavelengths (higher frequencies). Derive an expression for the total continuum flux of a planet (with radius r and temperature T_p) that orbits its star (with radius R and temperature T_s) at a distance D, and that arrives at an observer located at distance d. The phase function of the planet, i.e. the function that describes which part of the incident stellar radiation the planet reflects in the direction of the observer, can be written as ϕ (this phase function depends on the reflection properties of the planet and the phase angle of the planet).

i. The contrast between a star and a planet is defined as the ratio of the observed planetary flux and the stellar flux.

Using the equation found under h, calculate the contrast between the Sun and Jupiter at the wavelength where the thermally emitted flux of Jupiter peaks and at the wavelength where the solar flux that is reflected by Jupiter peaks for the following 2 cases:
1. Jupiter's temperature T is 500 K
2. Jupiter's temperature T is 125 K

Use these values: the Solar System is at a distance of 10 pc from the alien observer, the distance D between the Sun and Jupiter is 5 AU, the radius R of the Sun is $7.0 \cdot 10^8$ m, the radius of Jupiter is $7.0 \cdot 10^7$ km, and $\phi/4 = 0.5$ (for simplicity, we assume that the planet's radius and phase function are independent of its temperature). Furthermore, if needed, 1 AU$= 1.5 \cdot 10^{11}$ m, and 1 pc$= 2.06 \cdot 10^8$ AU.

j. Comment on which planet you think will be easier to detect when observing thermally emitted fluxes, the hot or the cold one.

2 Martian thermal spectra

Figure 1 shows two brightness temperature spectra observed for two different locations on Mars, at mid-latitudes (upper curve, labelled Rev 92) and at the South pole (lower curve, labelled Rev 30) (note that the mid-latitude curve is the lowest curve around 700 cm^{-1}). The South polar spectrum shows emission features where the mid-latitude spectrum shows absorption features (apart from the deepest part of the 700 cm^{-1} CO_2 band).

a. Explain using the atmospheric temperatures why you see the CO_2 (around 700 cm^{-1}) aborption band in absorption above the mid-latitudes and mostly in emission above the South pole.

b. Explain why the deepest part of the CO_2 band is seen in absorption in both spectra.

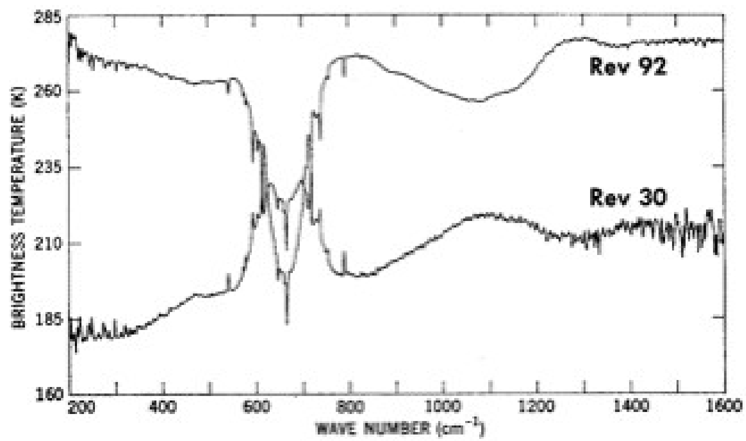

Fig. 4. Brightness temperatures calculated from spectra recorded on revolutions 30 and 92. The midlatitude spectrum shows a minimum due to silicate dust near 1085 cm^{-1}. The south polar spectrum shows a maximum at approximately the same wavenumber.

Figure 1: Brightness spectra on Mars

3 Bonus exercise: scattering and absorption

a. We have a planetary atmosphere that consists of a single, homogeneous layer of gas.

At 400 nm, the scattering optical thickness b_{sca} of the layer equals 10. Calculate the layer's scattering optical thickness at 600 and at 800 nm, respectively. You can assume that the refractive index and the depolarisation factor of the gas are independent of the wavelength.

b. A beam of stellar flux F_0 is incident on the top of the layer (the flux is measured perpendicular to the direction of incidence) under an angle θ_0 equal to 30°. Calculate the flux F_b of the beam (expressed as a fraction of F_0) that emerges from the bottom of the layer at 400, 600 and 800 nm, ignoring absorption.

c. The atmospheric layer is bounded below by an optically thick cloud. For simplicity, we will assume that all radiation that is incident on the cloud will be reflected upwards, independent of the wavelength. The cloud is thus approximated as a horizontal surface with a wavelength independent albedo that is equal to one. We furthermore assume that the reflected flux is independent of the angle under which it is reflected. The flux that is reflected by the cloud in a direction that makes an angle θ with the vertical can thus be written as

$$F_\text{c}(\theta, \lambda) = \cos\theta_0 F_\text{b}(\theta_0, \lambda), \tag{5}$$

where the $\cos\theta_0$ is included to obtain the incident flux per unit surface of the cloud. Flux F_b is the flux of the incident beam that emerges from the bottom of the atmosphere (see b). Note that F_c is the flux above the cloud, not the flux that emerges from the top of the atmosphere.

Calculate the flux F_t of radiation that is reflected by the cloud (expressed as a fraction of F_0) and that emerges in the vertical direction ($\theta = 0°$) from the top of the atmospheric layer at 800 nm. The angle of the incident radiation is 30° (see b) and absorption is ignored.

d. The gas in our atmospheric layer absorbs radiation at 810 nm. Because of this absorption, the flux that has been reflected by the cloud layer and that emerges in the vertical direction from the top of the atmosphere at 810 nm is as low as 0.1 F_0. Calculate the absorption optical thickness of the gas at 810 nm. You can assume that the atmospheric scattering optical thickness at 810 nm equals that at 800 nm (see a). Like before, the angle of incidence of the radiation at the top of the atmosphere is 30°.