Planets and Exoplanets 2009: Exercises to Lecture 3
Due: 6 October 2009 at 11:15

D. M. Stam

September 29, 2009

1 A collapsing molecular cloud

The gravitational potential energy $e_{\rm G}$ of a mass m at a distance r from a mass M equals

$$e_{\rm G} = -\frac{GmM}{r}, \tag{1}$$

with G the gravitational constant.

a. Derive the formula that describes the gravitational potential energy, $E_{\rm G}$, of a uniform, spherical cloud of density ρ and radius R:

$$E_{\rm G} = -\frac{3}{5}\frac{GM^2}{R} \tag{2}$$

b. Derive the formula for the Jeans mass, $M_{\rm J}$, of this cloud:

$$M_{\rm J} = \left(\frac{5kT}{Gm}\right)^{3/2} \left(\frac{3}{4\pi\rho}\right)^{1/2}, \tag{3}$$

with m the average molecular mass in the cloud. Hints: if the mass of a cloud equals its Jeans mass, its gravitational potential energy $E_{\rm G}$ equals twice its kinetic energy, $E_{\rm K}$. Assume the cloud's kinetic energy equals its thermal energy, thus $E_{\rm K} = 3/2NkT$, with N the total number of molecules in the cloud, k Boltzmann's constant, and T the temperature of the cloud.

c. Show that if the cloud collapses isothermally, it becomes more *unstable* as it shrinks.

d. Show that if the cloud retains the gravitational energy of its collapse as heat (it is thus not radiated away), it becomes more *stable* as it shrinks.

2 Orbiting a planet

A satellite or ring particle that orbits a planet experiences gravitational attraction from both the planet and the star. The maximum orbital distance of such a satellite is given by the radius $R_{\rm H}$ of the so-called *Hill sphere*, which can be approximated by

$$R_{\rm H} = \left(\frac{m_{\rm p}}{3(m_{\rm s}+m_{\rm p})}\right)^{1/3} a \approx \left(\frac{m_{\rm p}}{m_{\rm s}}\right)^{1/3} a, \tag{4}$$

with $m_{\rm s}$ and $m_{\rm p}$ the masses of the star and the planet, and a the planet's orbital distance. Here, we assume circular orbits.

a. Derive the right-hand side of Equation 4.

b. Calculate the radius of the Earth's Hill sphere.

c. How does this compare to the orbital distance of the moon?

d. Due to tidal effects, the distance between the moon and the Earth is increasing. Calculate the orbital period of the moon just before it is lost to the Earth.

e. Which planet in the Solar System has the largest Hill radius?

3 Building planetesimals

During the early stages of planet formation, planetesimals can grow through *runaway* or *oligarchic growth*. This rapid growth will cease when a planetary embryo has consumed most of the planetsimals within its gravitational reach. Planetsimals within 4 times the planetary embyro's Hill sphere eventually will come close enough to the planetary embryo during one of their orbits that they may be accreted.

The mass of a planetary embryo orbiting its star at a distance r, which has accreted all of the planetsimals within an annulus of width $2\Delta r$ is (Eq. 12.27 from *Planetary Sciences*)

$$M_{\mathrm{p}} = \int_{r-\Delta r}^{r+\Delta r} 2\pi r' \sigma_\rho(r') dr' \approx 4\pi r \Delta r \sigma_\rho(r), \tag{5}$$

with $\sigma_\rho(r)$ the disk's surface density at distance r.

Setting $\Delta r = 4R_{\mathrm{H}}$, we obtain the *isolation mass*, M_{i}, which is the largest mass to which a planetary embryo can grow by runaway accretion.

a. Using the expression for R_{H} from Eq. 4, derive an expression for M_{i} (in grams) of a planetesimal orbiting a star with mass m_{s}.

b. Assuming a solar mass star, and a disk surface density that varies as $\sigma_\rho = 10\ r^{-1}$ g cm^{-2}, with r the distance to the star in AU, calculate M_{i} at distances of 1 and 5 AU.

c. Repeat the calculations at 5 AU with a surface density that is twice as large (to account for condensed ices on the dust particles).

4 Bonus exercise: The evolving composition of the planets

The chemical composition of solar system planets is a strong function of their distance from the sun. This is largely because when a chemical is vaporized, it is not available for constructing the solid body of a planet. Solar system materials are often categorized as either 'gas', 'ice' or 'rock', based on their volatility. It is reasonable to assume that planets retain most of their 'rocky' materials, but lose some 'ices' and 'gases' during their evolution.

Table 1 shows the real or estimated rock masses of the solar system planets in Earth masses. Table 2 shows the cosmic abundances, by number, of the most common elements. We'll assume that they represent the original abundances in the solar system, and that they were well-mixed throughout the solar nebula. Assume all of the O, Mg, Si and Fe go into silicate rocks (use Mg_2SiO_4, Fe_2SiO_4, and SiO_2, for simplicity) and that the remaining O, C, and N, go into H_2O, CH_4, and NH_3 (ices).

a. Calculate cosmic ice/rock, He/rock and H/rock mass fractions in Earth masses and use these to fill in the next 3 columns of Table 1 (by approximation!).

b. Look up the current planetary masses and fill these in in the last column of Table 2. Comment on the differences between the original and current masses.

Table 1:

Body	Rock mass	Ice mass	Gas mass	Total original mass	Total current mass
Terrestrial planets	2				
Jupiter	10				
Saturn	10				
Uranus	3				
Neptune	3				
Totals	28				

Table 2:

Atom	Abundance
H	1
He	8×10^{-2}
O	7×10^{-4}
C	4×10^{-4}
N	1×10^{-4}
Mg	4×10^{-5}
Si	4×10^{-5}
Fe	3×10^{-5}

c. Sum up the original masses and the current masses, and express the two totals in solar masses. What was the (estimated) mass of the solar nebula? How much mass was lost? Where, do you think, it has it gone?