Planets and Exoplanets 2009: Exercises to Lecture 1
Due: 22 September 2009 at 11:15

C.U.Keller

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1 Equation of Motion for the Reduced Mass

Using Newton's laws

$$\frac{d}{dt}(m_{1,2}\mathbf{v}_{1,2}) = \mathbf{F}_{1,2} \tag{1}$$

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \tag{2}$$

$$\mathbf{F}_{g12} = -\frac{Gm_1 m_2}{r^2}\hat{\mathbf{r}} \tag{3}$$

with G being the gravitational constant and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $\hat{\mathbf{r}} = \mathbf{r}/r$, show that the following equation of motion holds for the reduced mass $\mu_r = m_1 m_2/(m_1 + m_2)$:

$$\mu_r \frac{d^2}{dt^2}\mathbf{r} = -\frac{G\mu_r M}{r^2}\hat{\mathbf{r}} \tag{4}$$

with the total mass $M = m_1 + m_2$.

2 Kepler's Second Law

Show that a line connecting two bodies sweeps out an area at a constant rate.

Hint: Use polar coordinates to write Kepler's second law as $\frac{d}{dt}\left(\frac{r^2}{2}\frac{d\theta}{dt}\right) = 0$ and express the conservation of angular momentum (vector) $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$: $\frac{d}{dt}\mathbf{L} = 0$ in polar coordinates, too.

3 Kepler's Third Law

Show that the orbital period P_{orb} of pair of bodies about their mutual center of mass is given by

$$P_{\mathrm{orb}}^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)} \tag{5}$$

Hint: Use the fact that an area of an ellipse is given by $A = \pi ab$ where a is the semi-major axis and b is the semi-minor axis and that the magnitude squared of the angular momnetum is given by $l^2 = a\left(1 - \epsilon^2\right) GM$. As the area speed (amount of area swept by the radius vector in unit time) is a constant, the period can be calculated as the ratio between total area and the area speed.

4 Bonus Exercise: Kepler's First Law

Two bodies move along elliptical paths with one focus of each ellipse located at the center of mass r_{cm} of the system.

Hint: Change to a polar coordinate system with coordinates r and θ and express the equation of motion in polar coordinates.