Recap lecture 4

thermal limit of stochastic radiation processes

→ $h\nu << kT$ Thermal limit $\overline{\Delta P^2}(\nu) = \bar{P}^2(\nu)$

→ Use incomplete gamma function to calculate Poisson and Gauss cumulative distribution function

→ Propagation of errors under the assumption of independent variables

$$\bar{f} = f(\bar{u}, \bar{v}, ..)$$

$$\sigma_f^2 = \sigma_u^2 (\frac{\partial f}{\partial u})^2 + \sigma_v^2 (\frac{\partial f}{\partial v})^2 + ...$$

Today:

Comparing data with a model:
Least-squares fitting, maximum likelihood method: Gaussian data

Monte Carlo simulations

"real" maximum likelihood method: Poissonian data

OAF2 chapter 5.3+5.4
see also Num Res Chapter 15

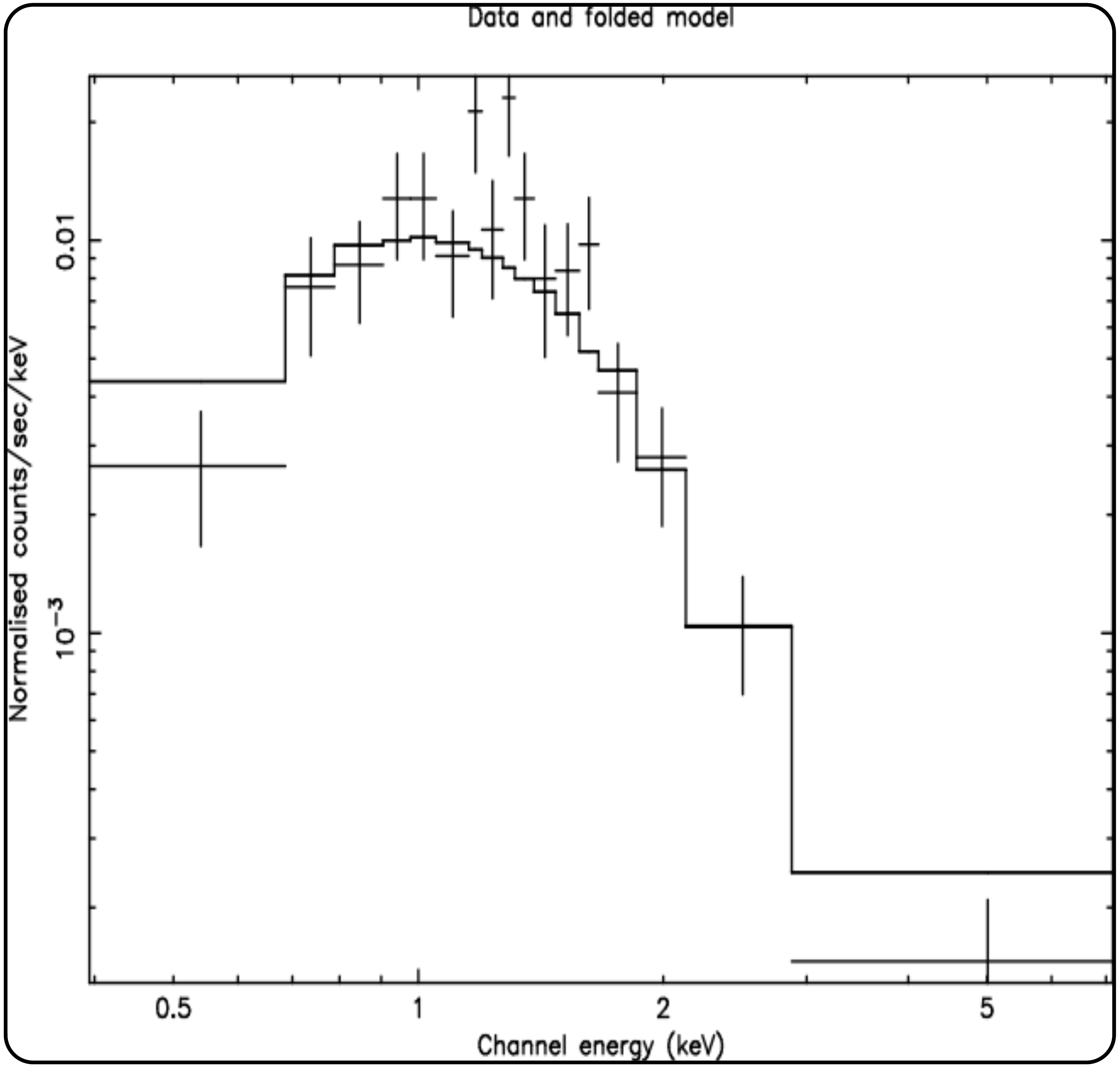
Compare data with a model

Describe data in terms of a continuous function

Compare observations (data) with theoretical model prediction

Describe the data in a few parameters

Example: continuous model through discrete data & model prediction



X-ray binary in quiescence neutron star atmosphere model

Maximum likelihood: most likely outcome is assumed to be the 'correct' one

Method of least squares

$$dQ_i = P_i dx$$

Probability density function $\frac{dQ_i}{dx} = P_i$

$\hookrightarrow$ Gauss, Poisson

$$P(y_i)\Delta y = \frac{1}{\sqrt{2\pi}\sigma_i} exp(-\frac{1}{2}\frac{(y_i - y_m)^2}{\sigma_i^2})\Delta y$$

note: y_m= model value not mean here!

Method of least squares

$$P \propto \prod_{i=1}^{N} \{exp[-\frac{1}{2}(\frac{y_i - y_m}{\sigma_i})^2]\}$$

$$\propto exp[-\frac{1}{2}\sum_{i=1}^{N}(\frac{y_i - y_m}{\sigma_i})^2]\}$$

MINIMISE: $\chi^2 \equiv \sum_{i=1}^{N}(\frac{y_i - y_m}{\sigma_i})^2$

MINIMISATION: ROOT FINDING PROBLEM

1D: $\frac{\partial}{\partial y_m}\sum_{i=1}^{N}(\frac{y_i - y_m}{\sigma_i})^2 = 0$

More about χ^2

Drawn from normal distribution

Distribution of χ_i^2 is a χ^2 distribution

For N measurements described by M variables, there are N-M Degrees of Freedom (d.o.f.)

Probability of obtaining a χ^2 as *bad* as observed

or higher by chance

$$P(\chi^2_{obs}) = \mathrm{gammq}(\frac{N-M}{2}, \frac{\chi^2_{obs}}{2})$$

χ^2 fitting provides:

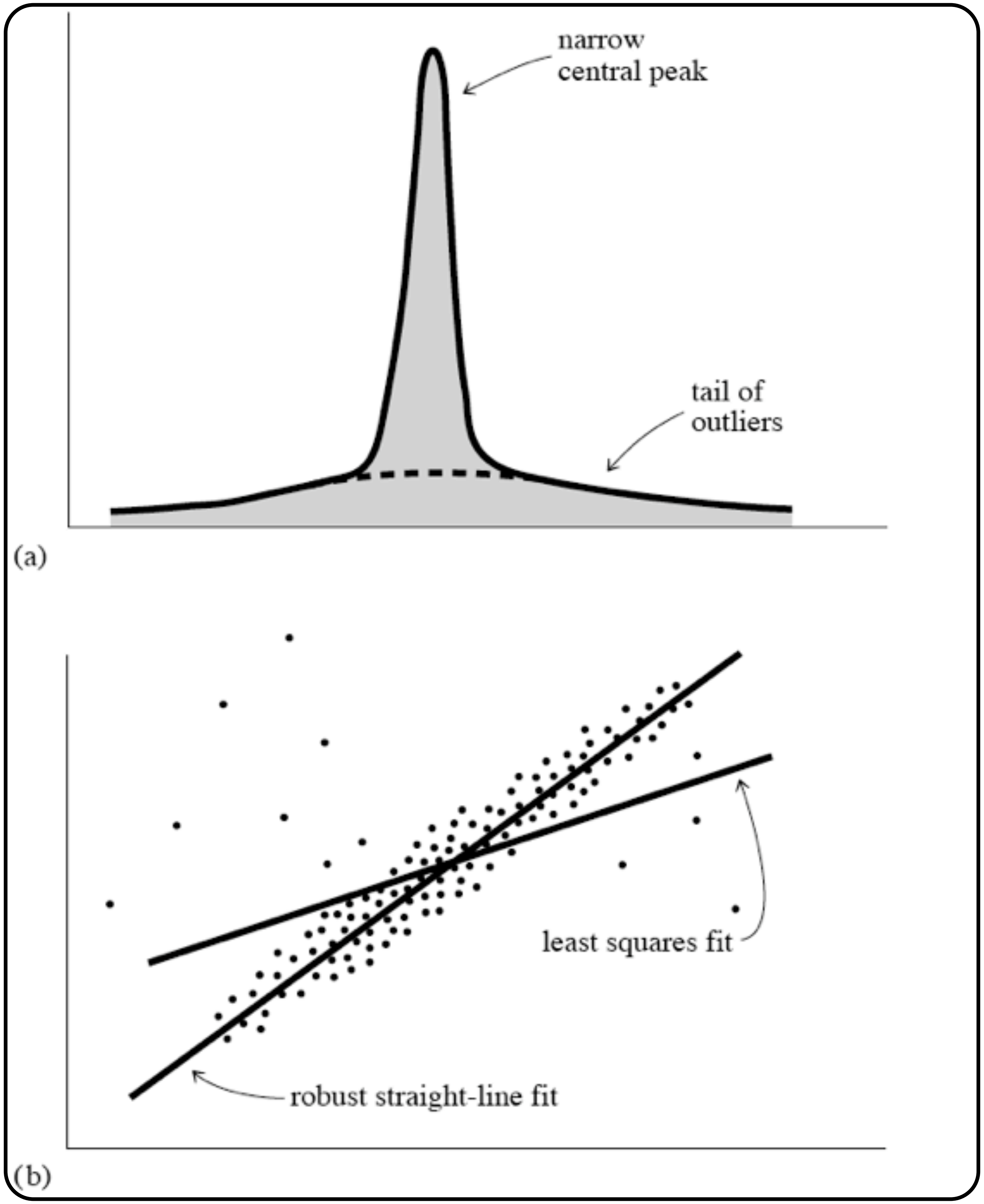
Best-fitting parameters

An error estimate of the uncertainty of the fitted parameters

A probability that the data is drawn from a parent population described by the model parameters

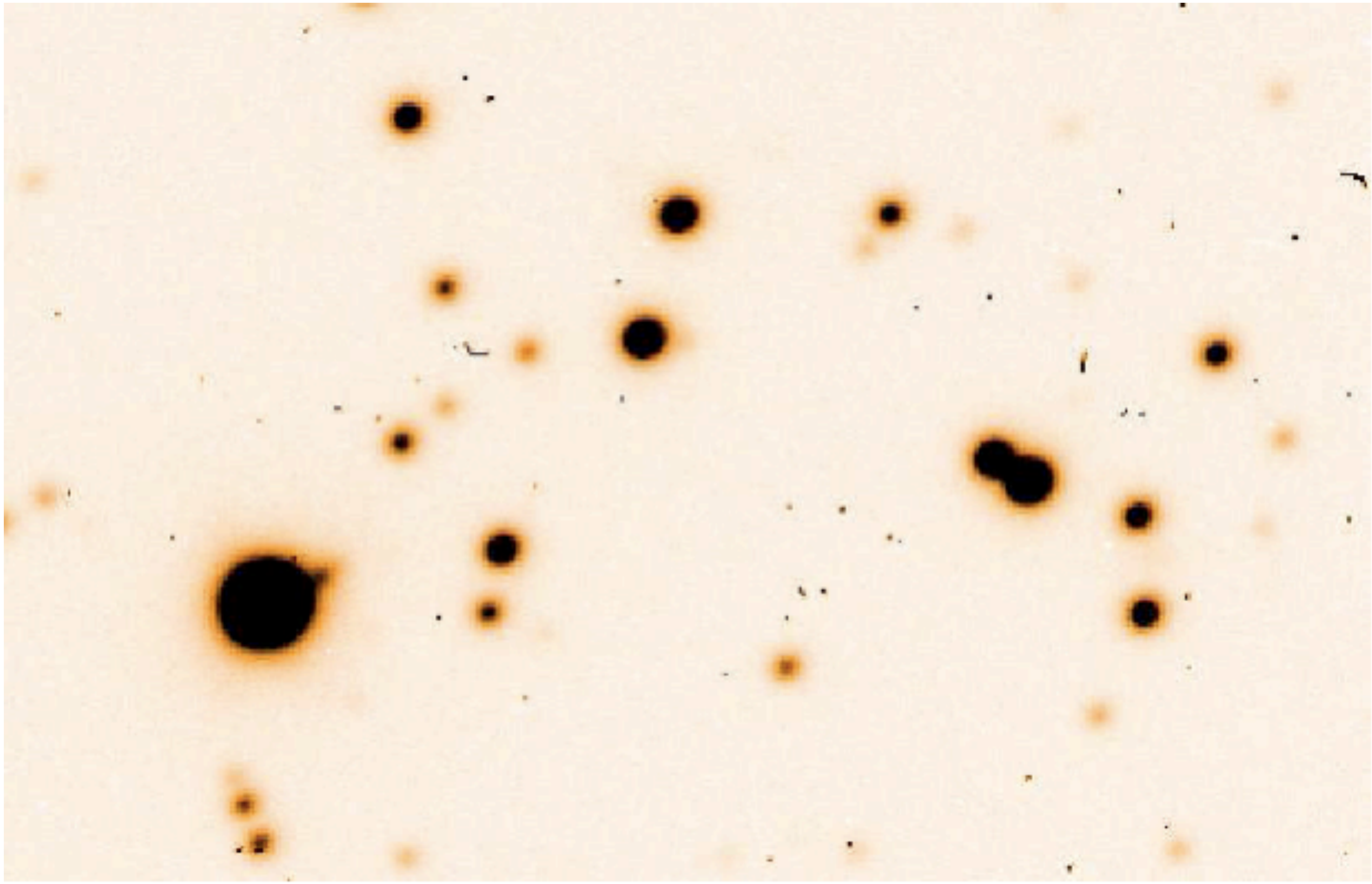
Note that outliers make this probability generally low

Be aware of non-gaussian distributions



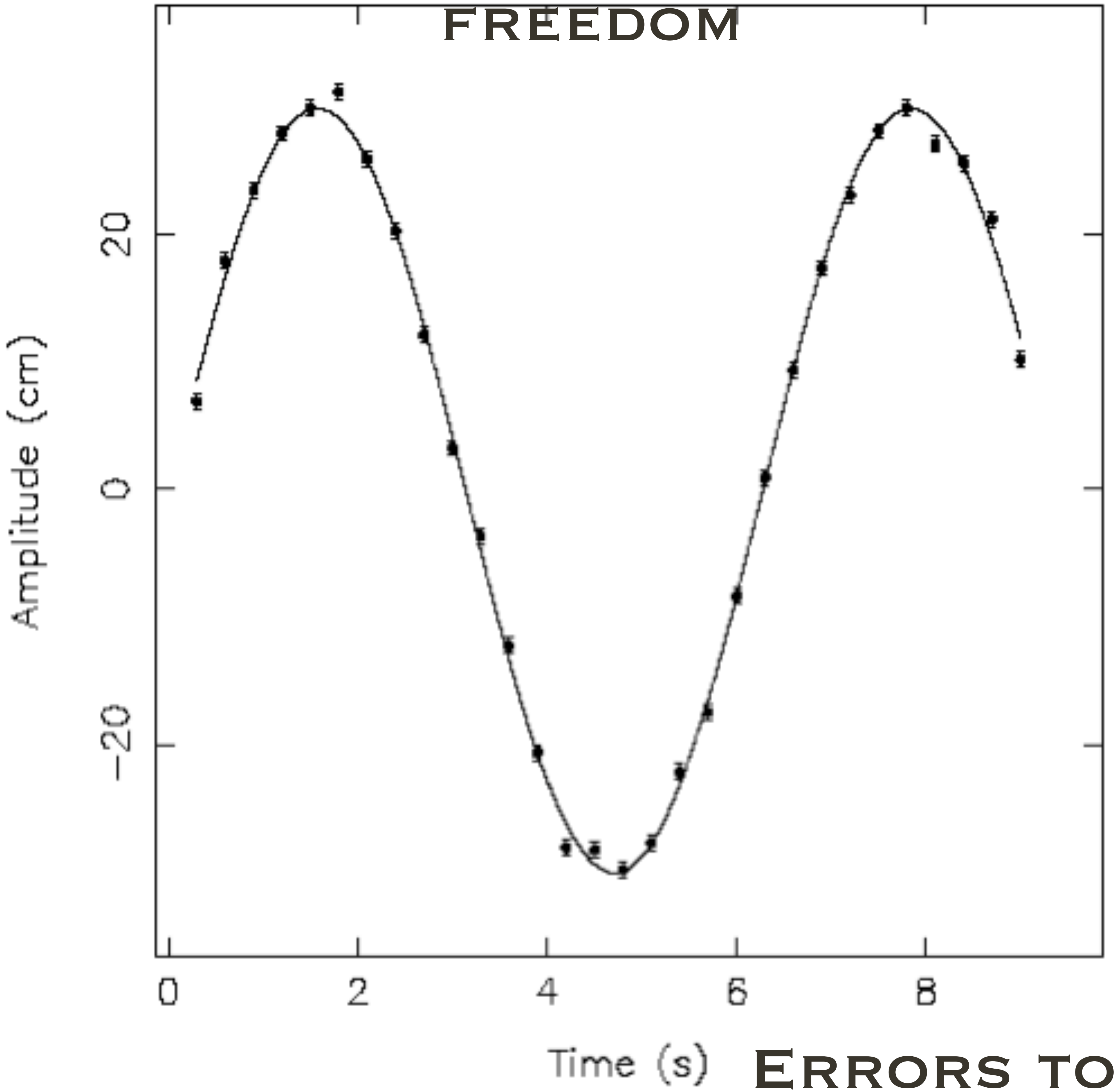
from Num Res page 695

Part of U-band image VLT



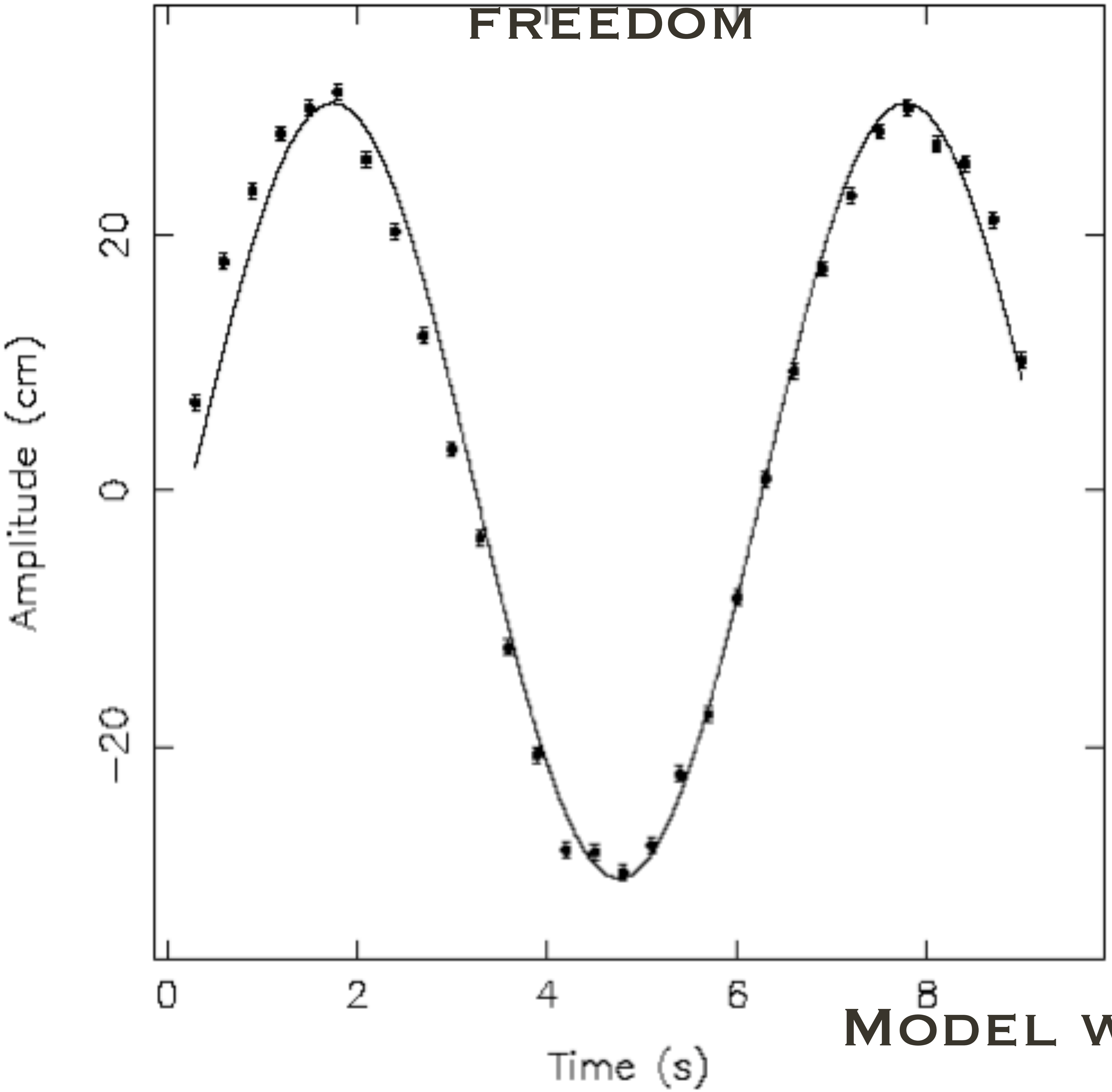
Remove outliers via (sigma) clipping

Sinusoid: chi**2=81.2 for 26 degrees of freedom



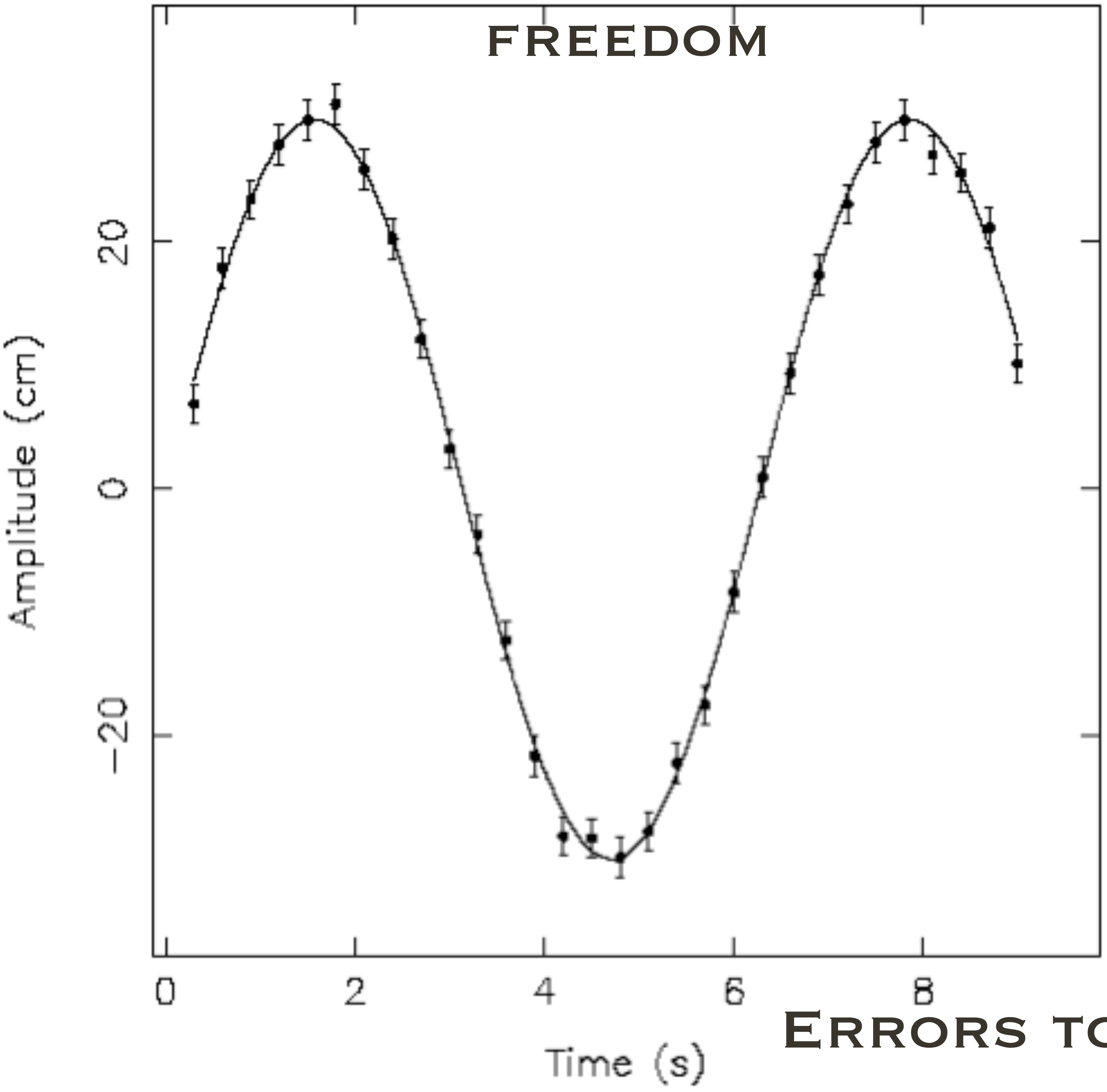
Errors too small

Sinusoid: chi**2=580 for 28 degrees of freedom



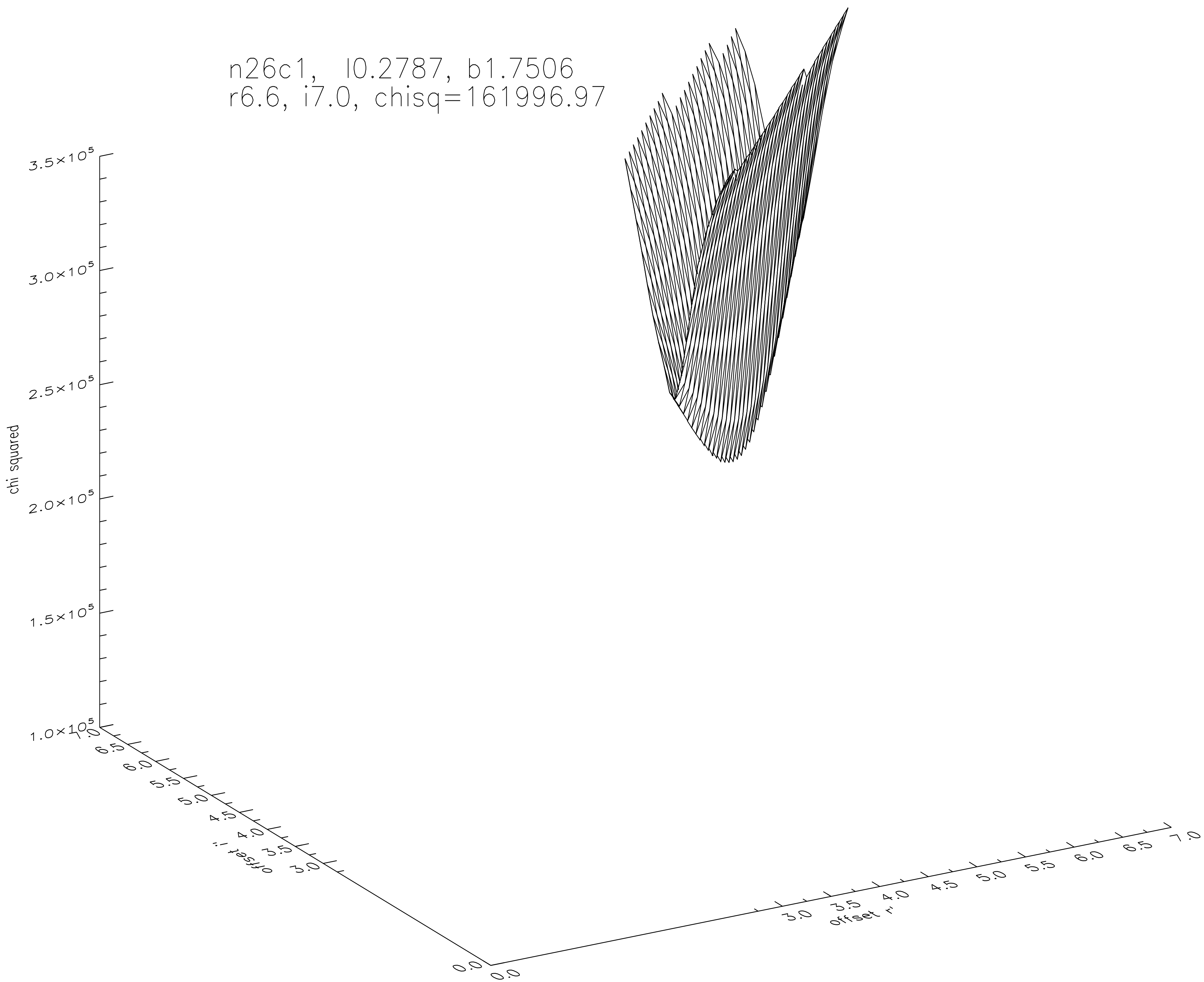
Model wrong

Sinusoid: chi**2=12 for 27 degrees of freedom



Errors too large

χ^2 surface



not always this smooth!

Think about what to fit!

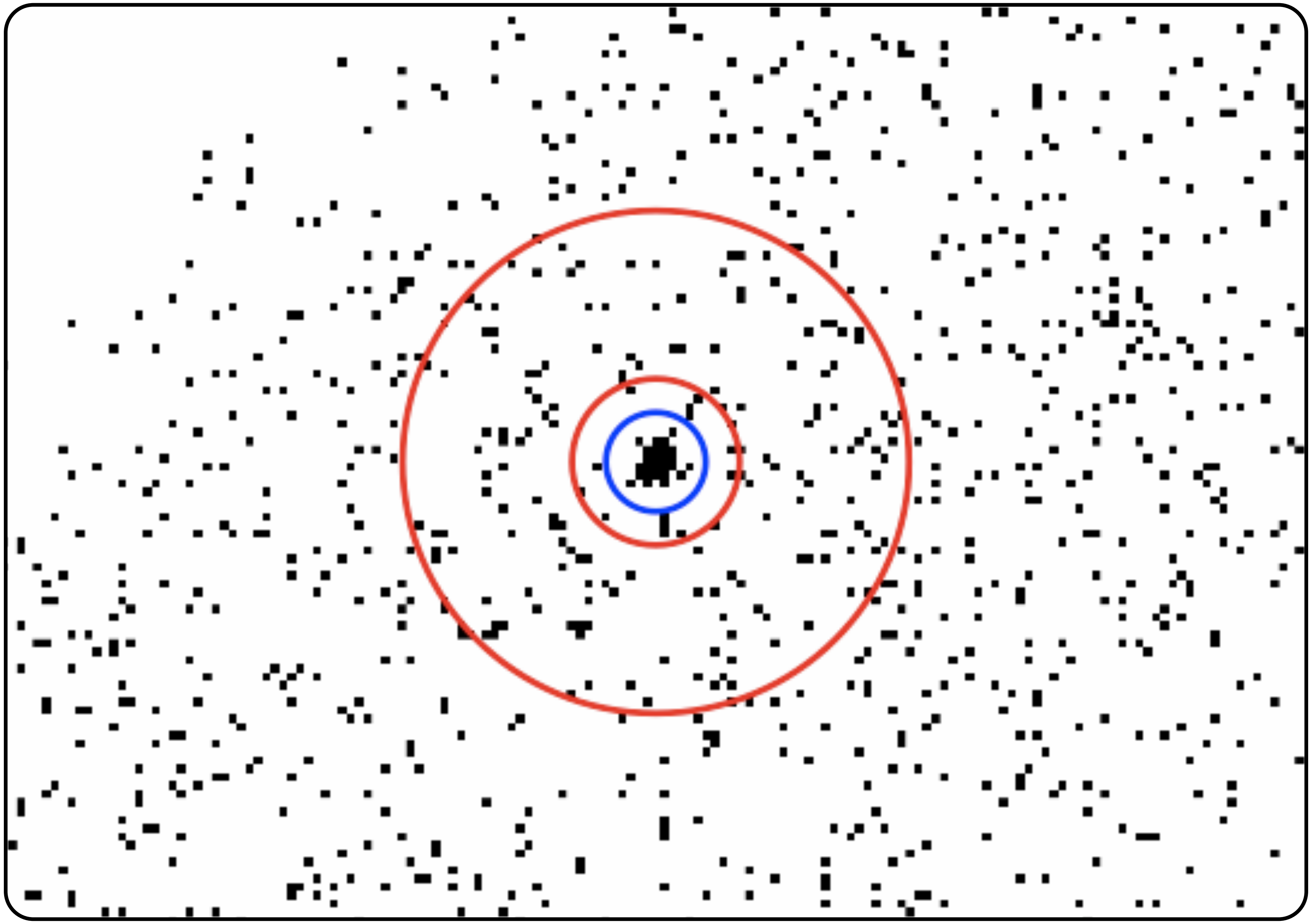
$y(x) = ae^{bx+c}$ a and c are degenerate

$y(x) = ae^{-bx}$ is equal to $\log[y(x)] = c - bx$

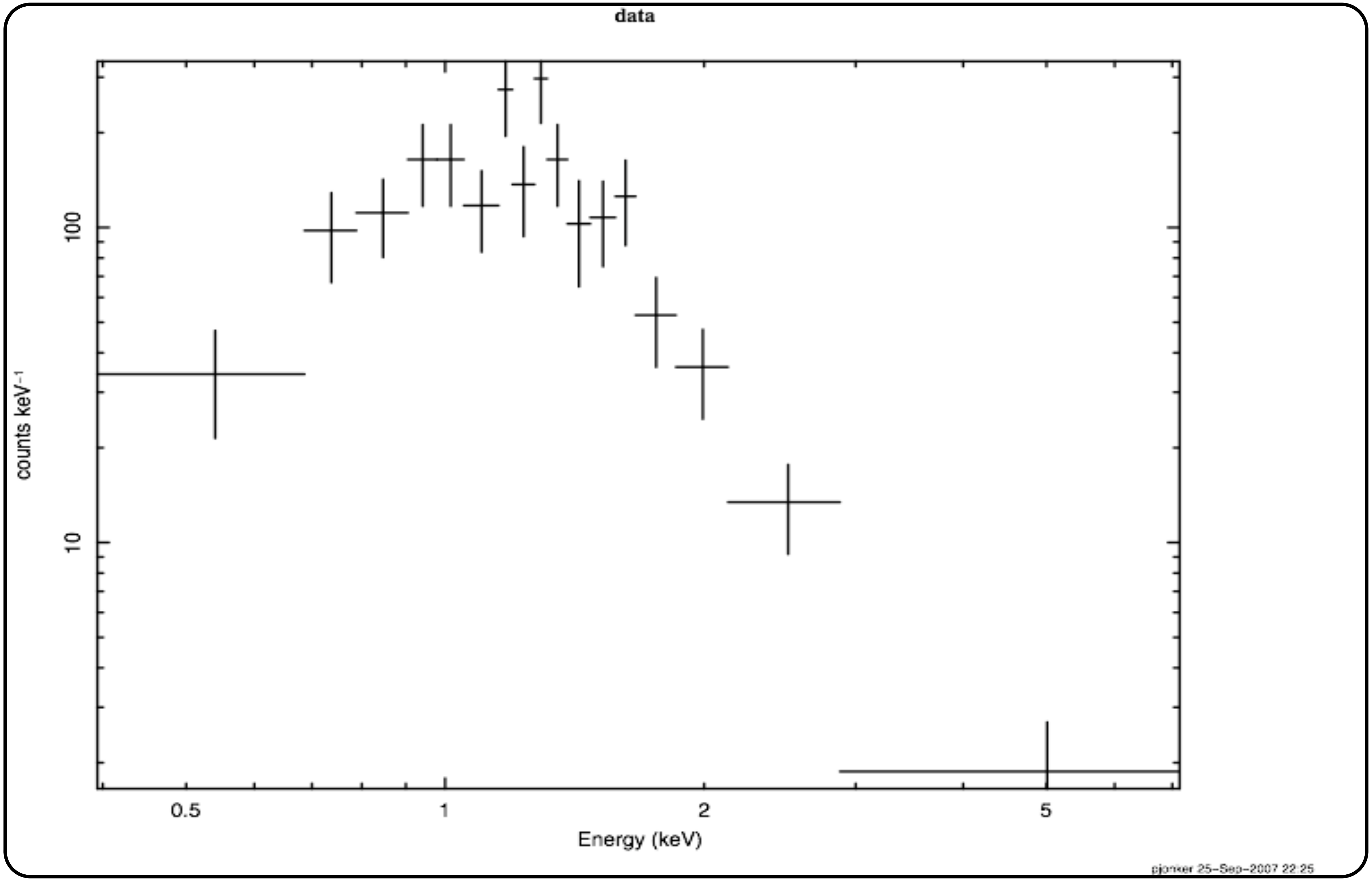
which is linear in fitting c and b
but the errors are no longer Gaussian

E.g. computer exercise 3

Chandra CCD (ACIS) observation of an X-ray binary

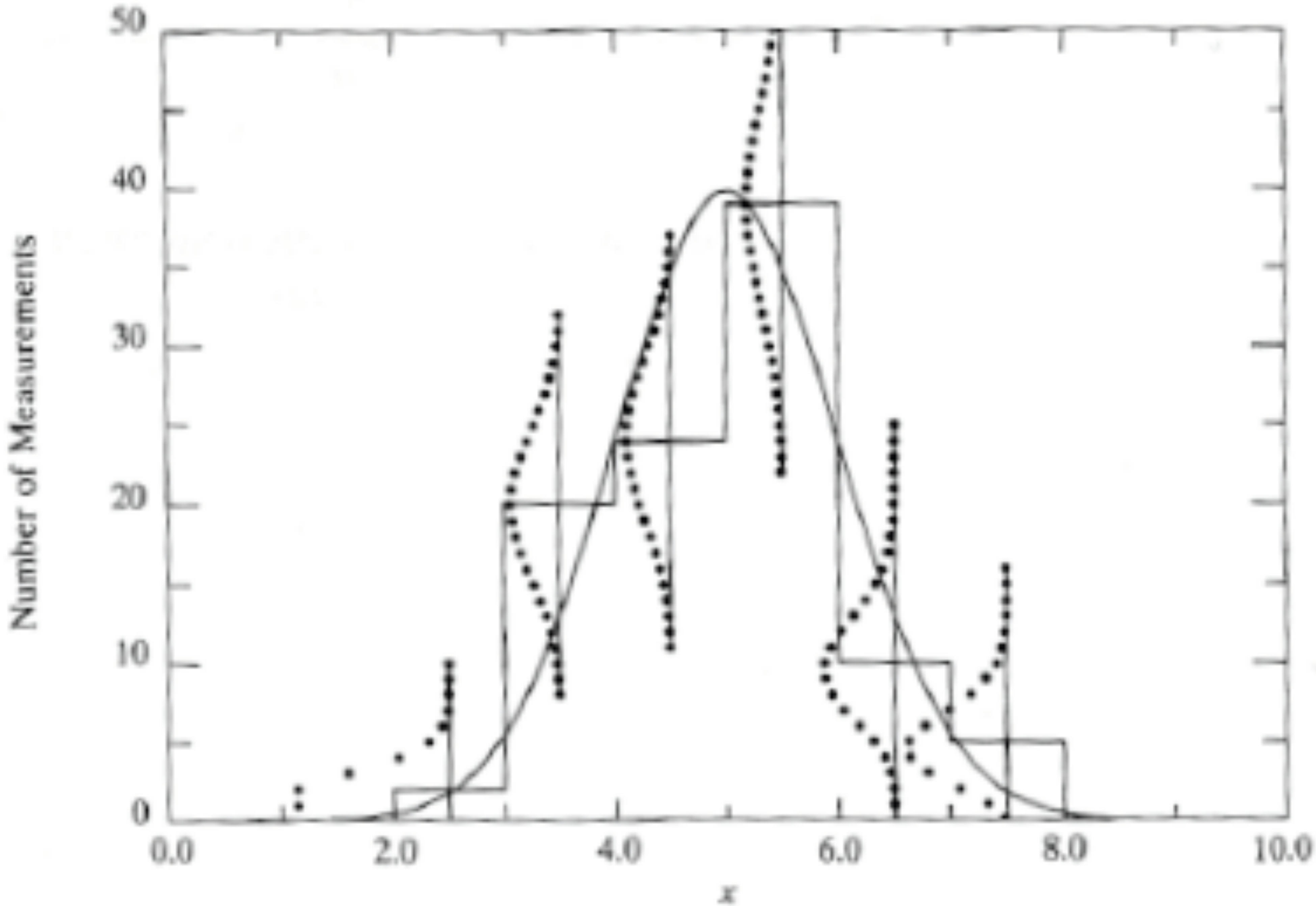


Same data as before



Gaussian approximation for errors but at low counts Gauss and Poisson errors differ

Value and Poisson errors



Example from Bevington & Robertson 1992

Fitting a straight line to the data

$$y_m(x_i, a, b) = a + bx_i$$

minimise χ_i^2 to find best-fitting parameters

$$\frac{\partial \sum_{i=1}^{N} \left(\frac{y_i - a - bx_i}{\sigma_i}\right)^2}{\partial a} = 0$$

$$\sum \frac{y_i}{\sigma_i^2} - a\sum \frac{1}{\sigma_i^2} - b\sum \frac{x_i}{\sigma_i^2} = 0$$

&

$$\sum \frac{x_i y_i}{\sigma_i^2} - a\sum \frac{x_i}{\sigma_i^2} - b\sum \frac{x_i^2}{\sigma_i^2} = 0$$

$$a = \frac{\sum_i \frac{x_i^2}{\sigma_i^2} \sum_i \frac{y_i}{\sigma_i^2} - \sum_i \frac{x_i}{\sigma_i^2} \sum_i \frac{x_i y_i}{\sigma_i^2}}{\sum_i \frac{1}{\sigma_i^2} \sum_i \frac{x_i^2}{\sigma_i^2} - (\sum_i \frac{x_i}{\sigma_i^2})^2}$$

DETERMINE ERRORS ON THE BEST-FITTING PARAMETERS

REMEMBER $$\sigma_f^2 = \sigma_u^2(\frac{\partial f}{\partial u})^2 + \sigma_v^2(\frac{\partial f}{\partial v})^2 + ...$$

$$\sigma_a^2 = \sum_{i=1}^{N}[\sigma_i^2 \frac{\partial a}{\partial y_i}]^2$$

$\partial u \,\&\, \partial v$ ETC ARE THE DIFFERENT MEASUREMENT VALUES y_i

$$\sigma_a^2 = \frac{\sum_i \frac{x_i^2}{\sigma_i^2}}{\sum_i \frac{1}{\sigma_i^2} \sum_i \frac{x_i^2}{\sigma_i^2} - (\sum_i \frac{x_i}{\sigma_i^2})^2}$$

SIMILARLY

$$\sigma_b^2 = \frac{\sum_i \frac{1}{\sigma_i^2}}{\sum_i \frac{1}{\sigma_i^2} \sum_i \frac{x_i^2}{\sigma_i^2} - (\sum_i \frac{x_i}{\sigma_i^2})^2}$$

Finally calculate the probability of obtaining the χ^2 by chance

$$P(\chi^2_{obs}) = \text{gammq}(\frac{N-M}{2}, \frac{\chi^2_{obs}}{2})$$

for the straight line fit M=2

$\nu = N - M$ degrees of freedom

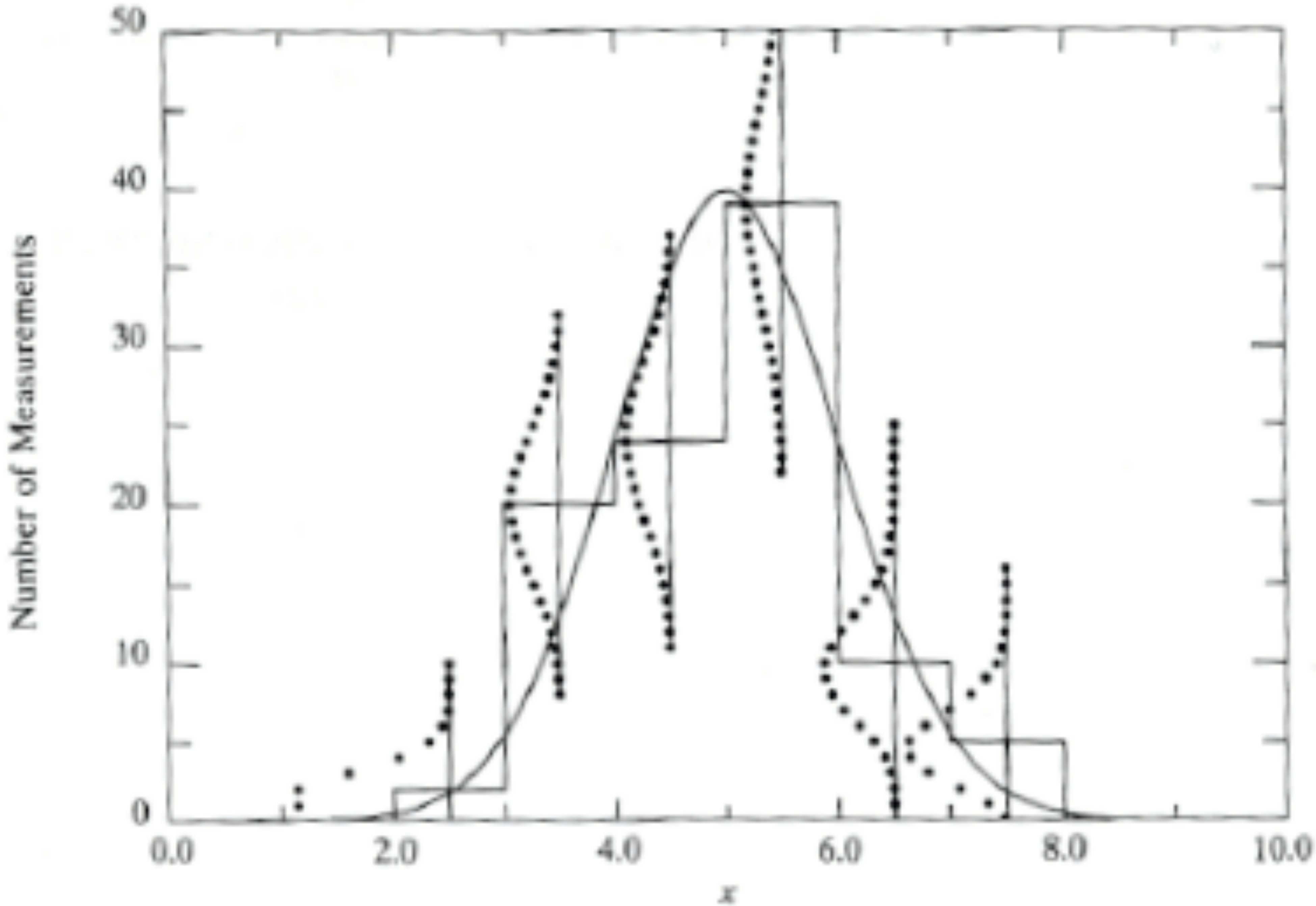
reduced χ^2_ν

$$\chi^2_\nu \equiv \frac{\chi^2}{\nu}$$

for data fitting: $\chi^2_\nu \sim 1$ $\chi^2 \approx \nu$

$$\sigma_{\chi^2} = \sqrt{2\nu}$$

Value and Poisson errors



Example from Bevington & Robertson 1992

Estimating confidence regions via Monte Carlo

Monte Carlo simulations

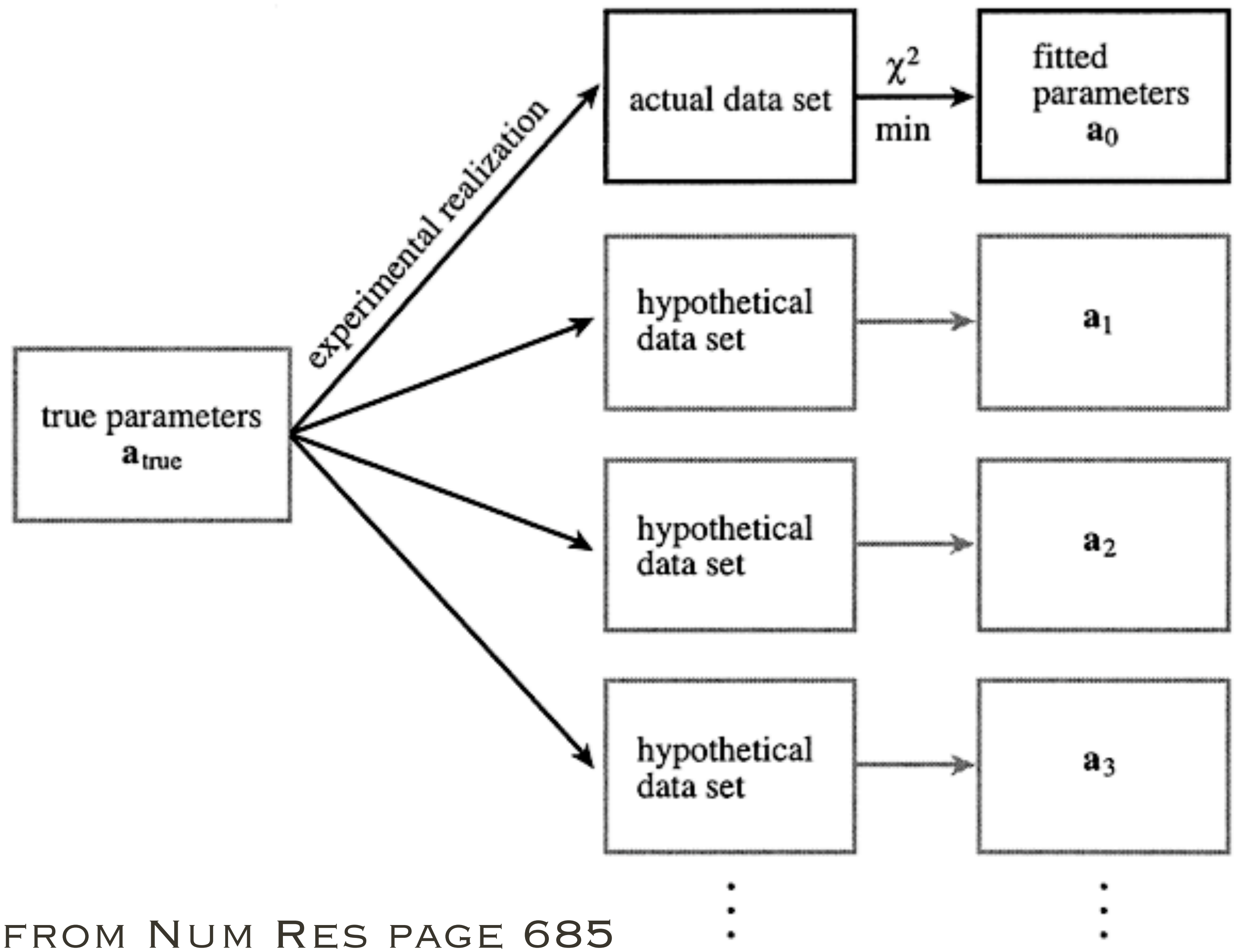
1 Replace the observed values by another random selected value from the range y+- sigma

2 Repeat fitting (chi**2 minimisation etc)

3 Repeat 1 & 2 N times to build up a distribution
in the determined parameters and from that determine the mean, variance etc

Estimating confidence limits

Monte Carlo simulations



from Num Res page 685

$\vec{a}_0$

Measurement one draw from the distribution of a's

assume that the distribution of

$$a_i - a_0$$

is close to the probability distribution

$$a_i - a_{true}$$

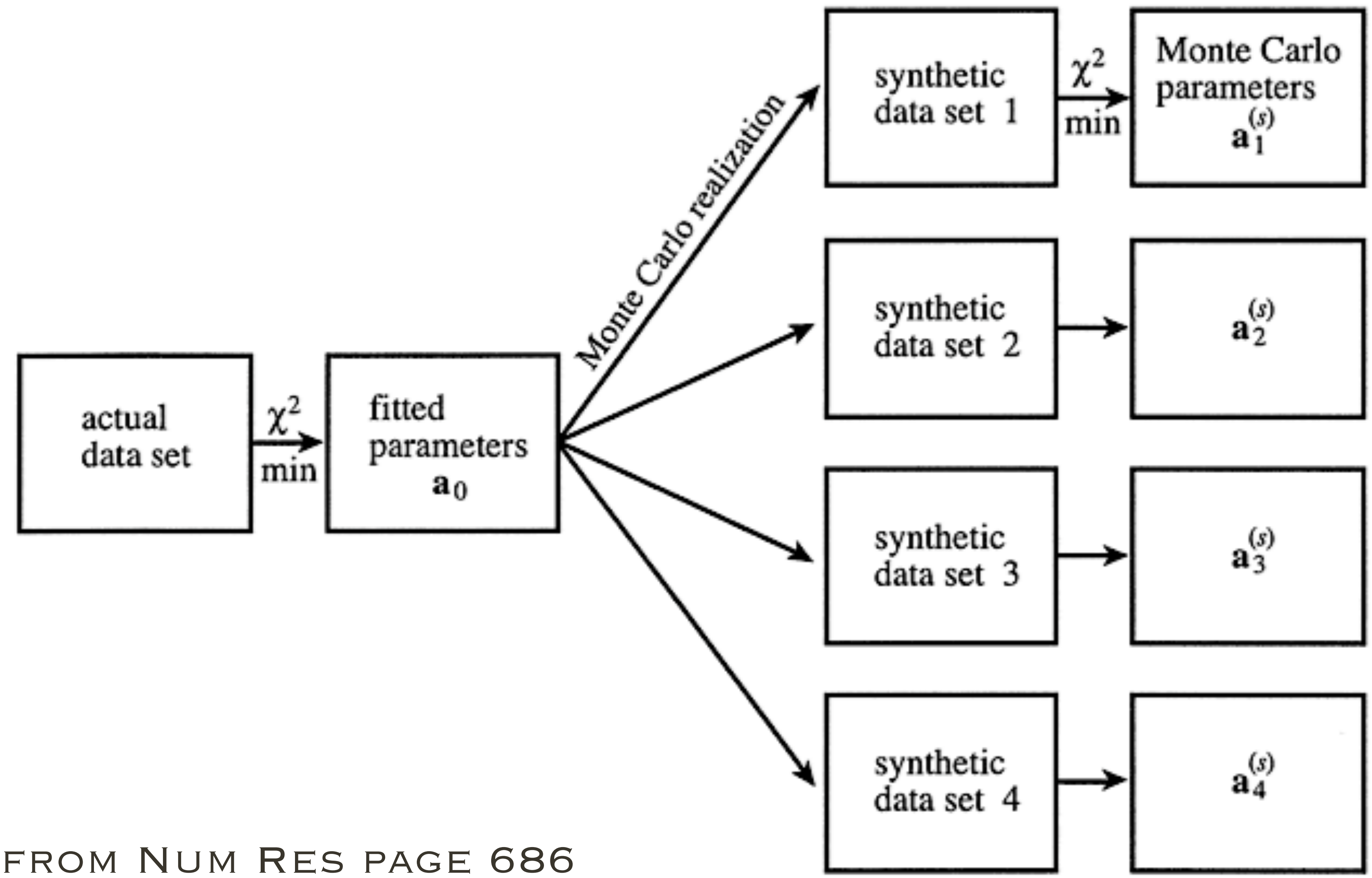
$a_i - a_0$ distribution
we can determine via Monte Carlo simulations

Many Monte Carlo methods

Basis: (pseudo) random draws

Also simulate an experiment!
a.o. useful for proposal writing
Computer exercise

Monte Carlo simulations



from Num Res page 686

calculate distribution of $a_i - a_0$

by simulating many sets of data and using χ^2 fitting to determine a_i

Special MC: bootstrapping

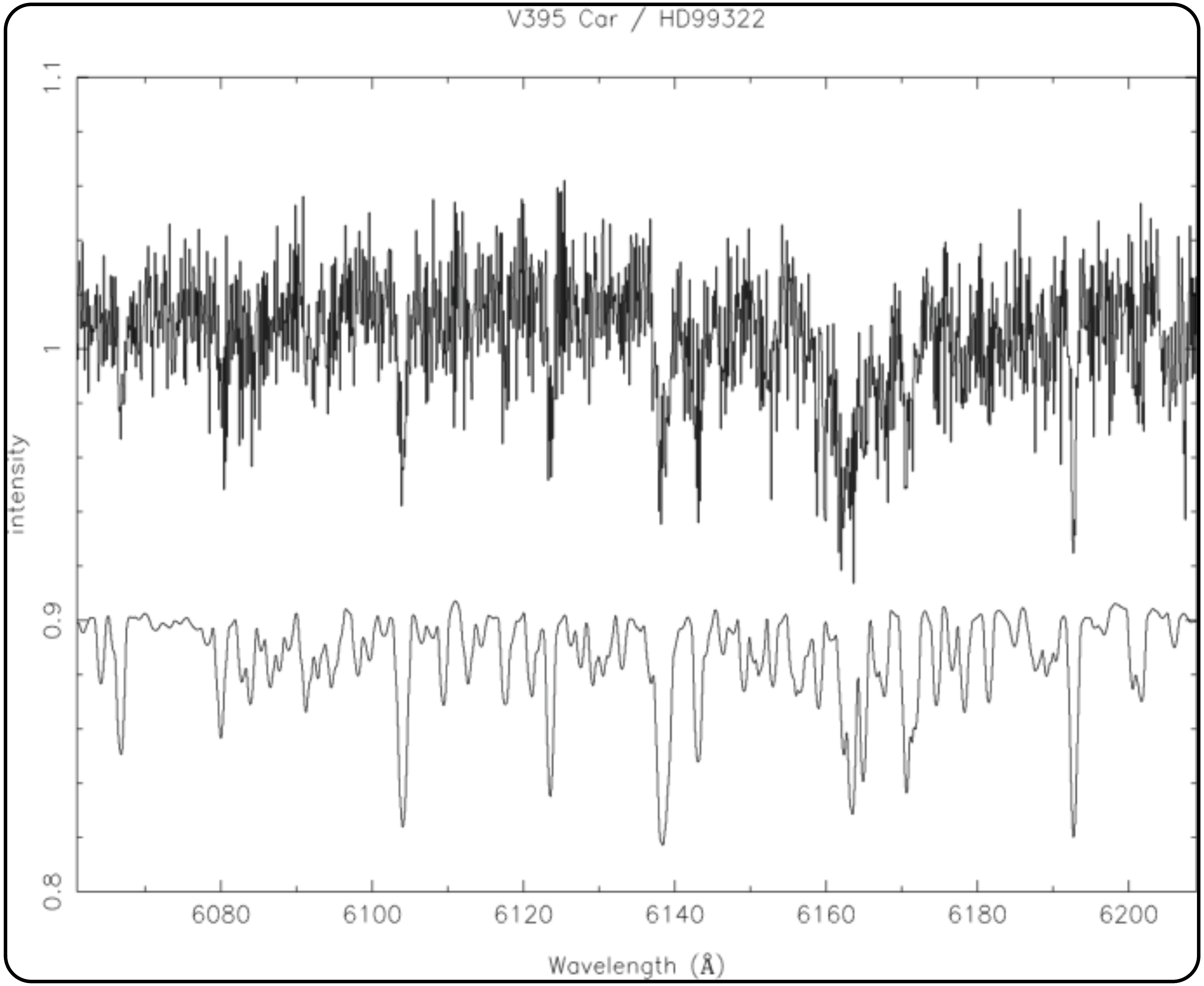
1 Replace a random number of observed values
by another random selected observed value

2 Repeat fitting (chi**2 minimisation etc)

3 Repeat 1 & 2 N times to build up a distribution
in the determined parameters and from that determine the mean, variance etc

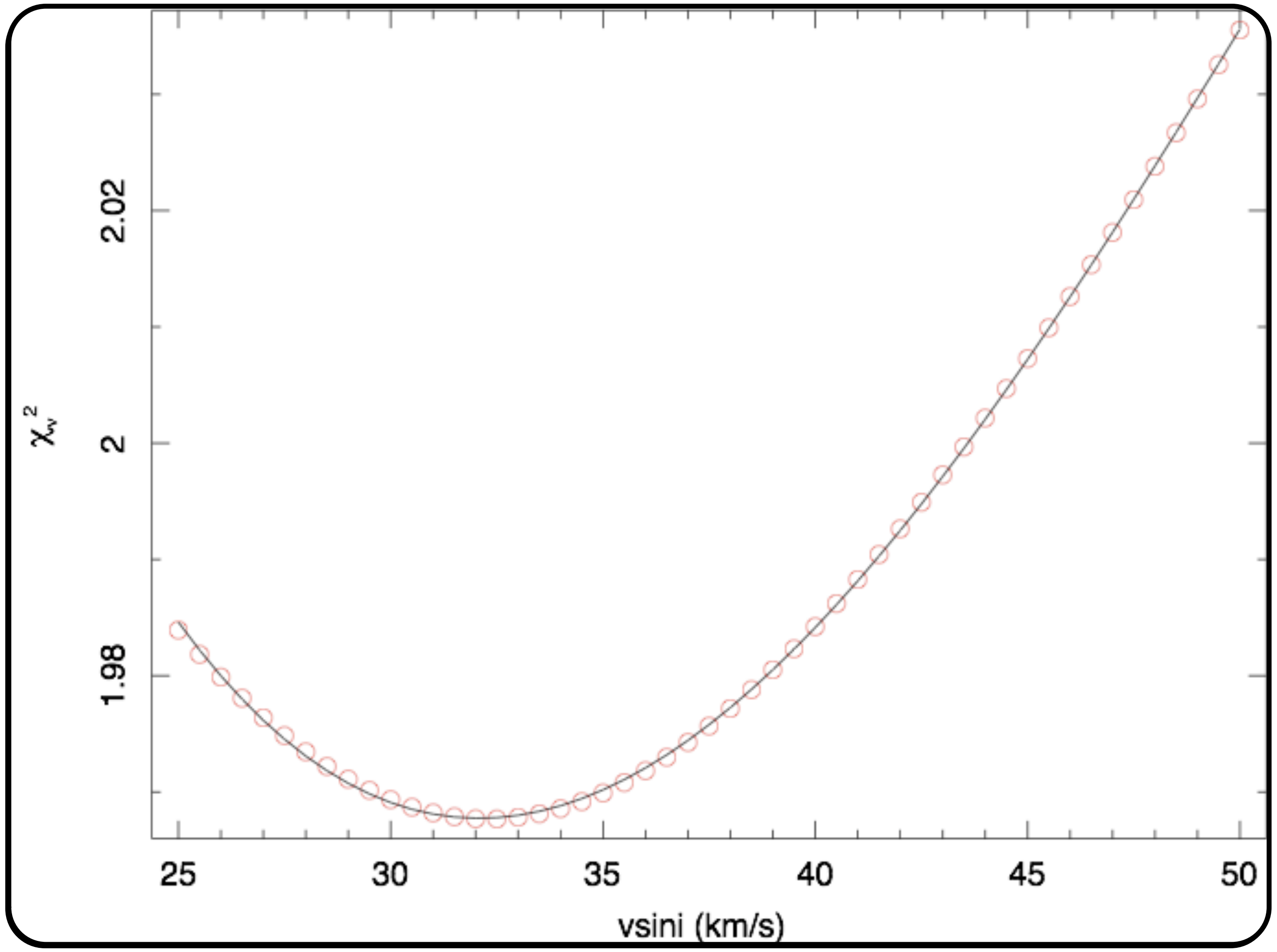
BOOTSTRAP METHOD AND APPLICATION

X-ray binary V395 Car

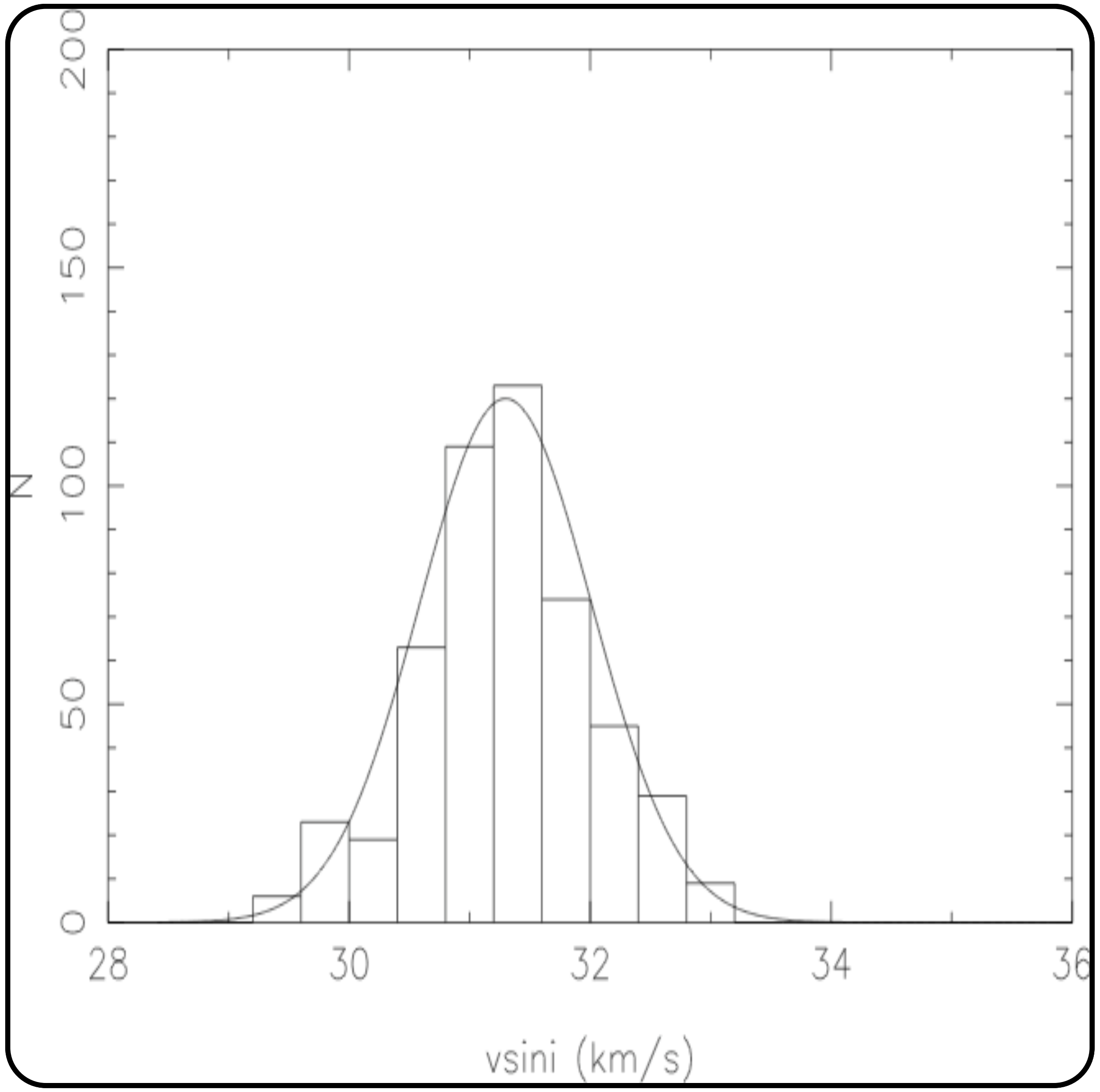


from Steeghs & Jonker 2007

Broadening and optimal subtraction

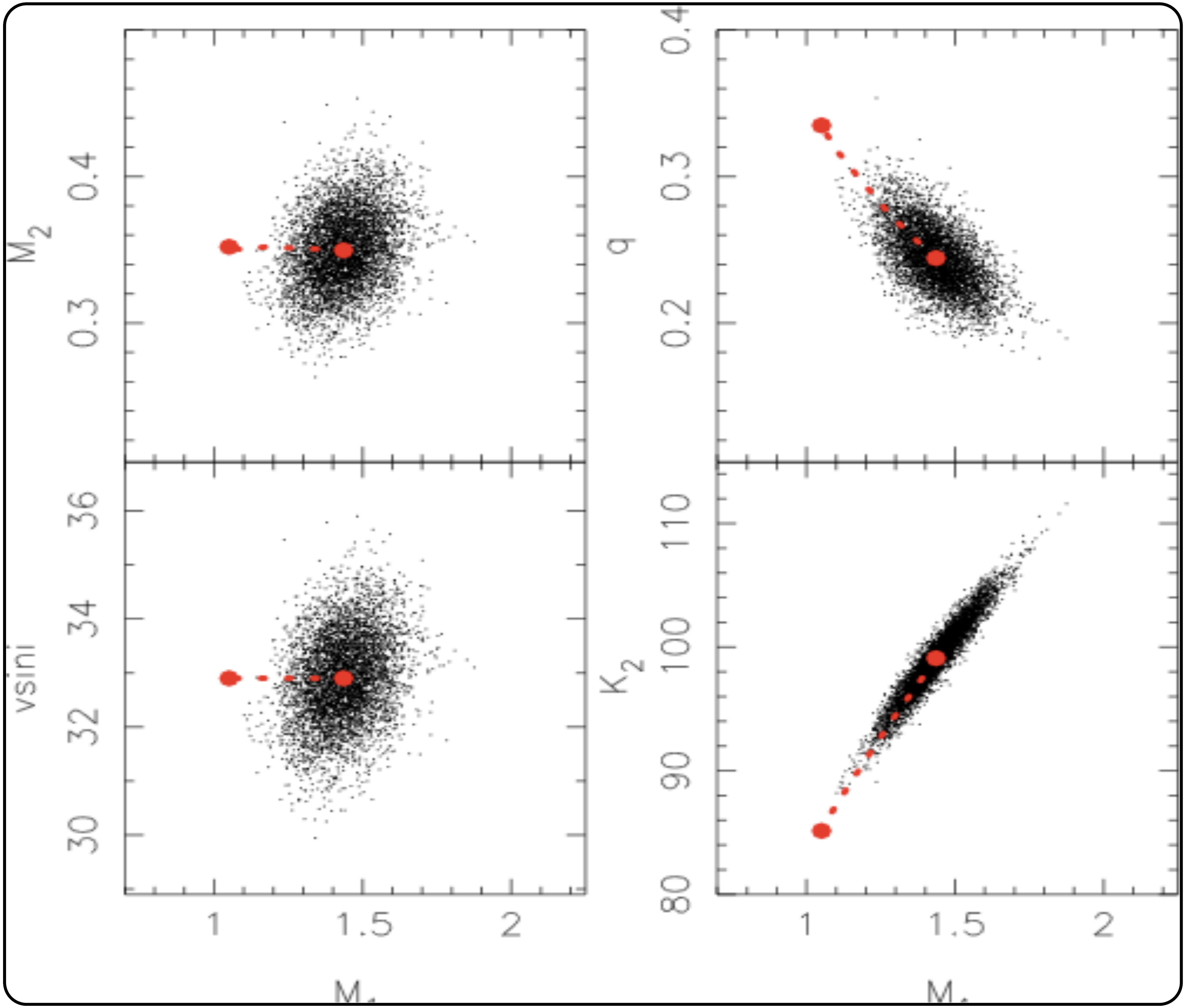


NOTE REDUCED χ_ν^2



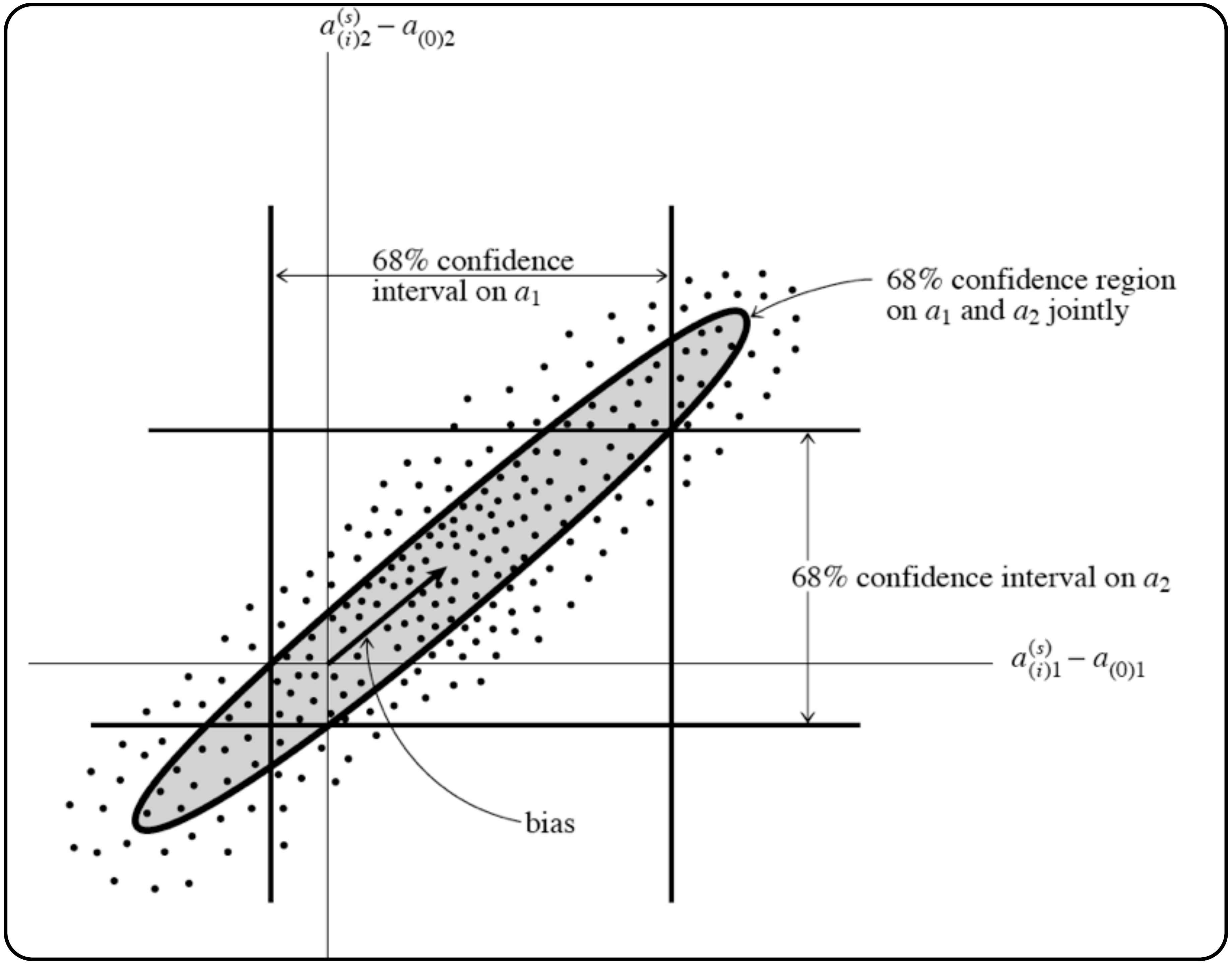
Bootstrap determined rotational velocity

BOOTSTRAP METHOD: AN APPLICATION



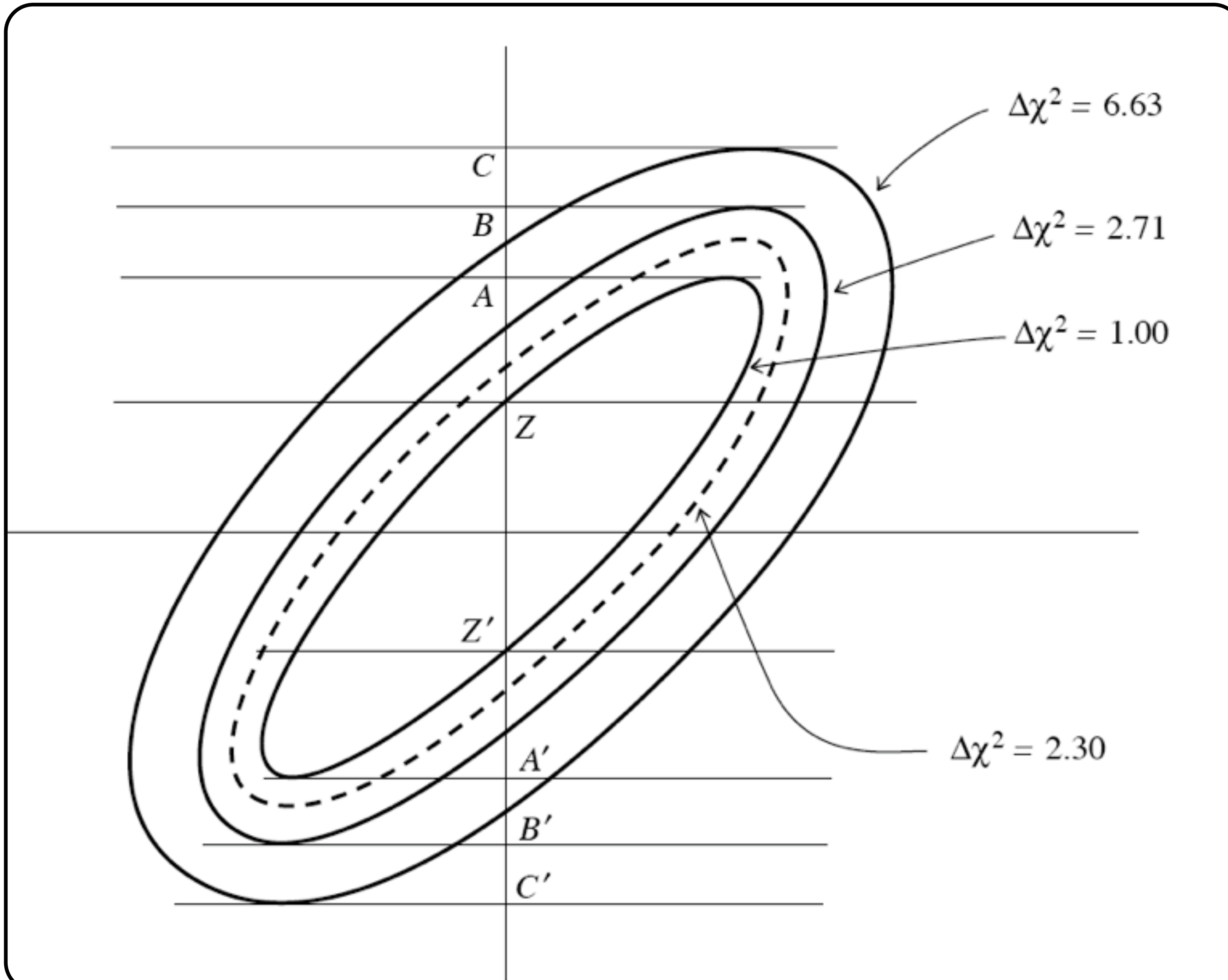
Confidence limits

single vs. multiple parameter confidence region



from Num Res page 688

Projections



from Num Res page 689

Maximum likelihood method
(Poisson noise, unbinned data)

probability to find n_i photons when m_i expected

for each pixel i in an image $$P_i = \frac{m_i^{n_i} e^{-m_i}}{n_i!}$$

total probability $$L' \equiv \prod_i P_i$$

$$\ln L' \equiv \sum_i \ln P_i = \sum_i n_i \ln m_i - \sum_i m_i - \sum_i \ln n_i!$$

minimise $$\ln L \equiv -2(\sum_i n_i \ln m_i - \sum_i m_i)$$

Detection of a constant background, A, plus a source of strength B of which a fraction f_i falls on pixel i

$$-0.5 \ln L = \sum_i n_i \ln(A + Bf_i) - \sum_i (A + Bf_i)$$

again search for the minimum of L for variations in A and B

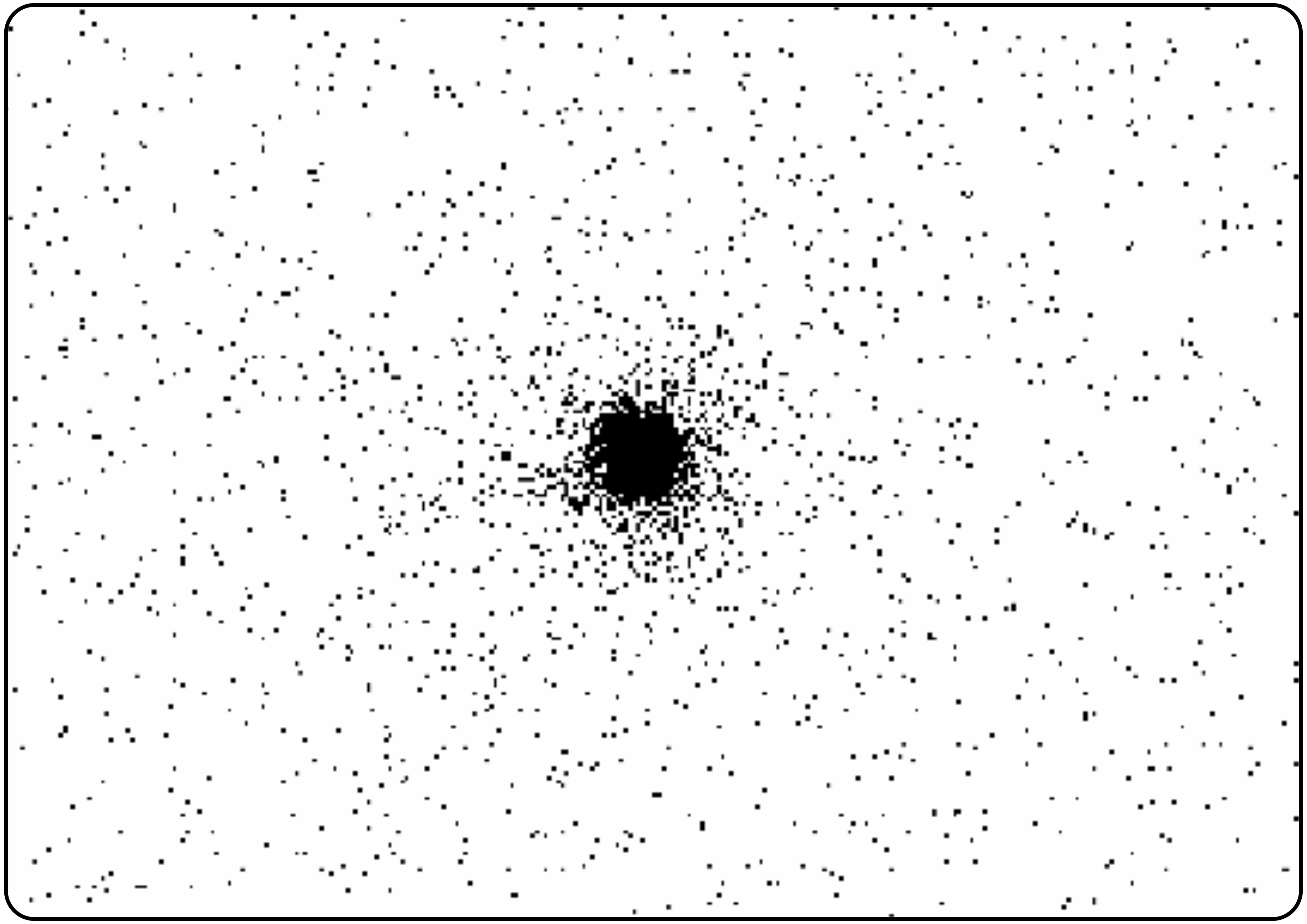
determined independently in some cases

total pixels Z

$$\frac{\partial \ln L}{\partial A} = 0 \Rightarrow \sum_i \frac{n_i}{A + Bf_i} - \sum_i (1) = \sum_i \frac{n_i}{A + Bf_i} - Z = 0$$

$$\frac{\partial \ln L}{\partial B} = 0 \Rightarrow \sum_i \frac{n_i f_i}{A + Bf_i} - \sum_i (f_i) = \sum_i \frac{n_i f_i}{A + Bf_i} - 1 = 0$$

Maximum likelihood method (application X-ray binary Cir X-1, a jet present?)

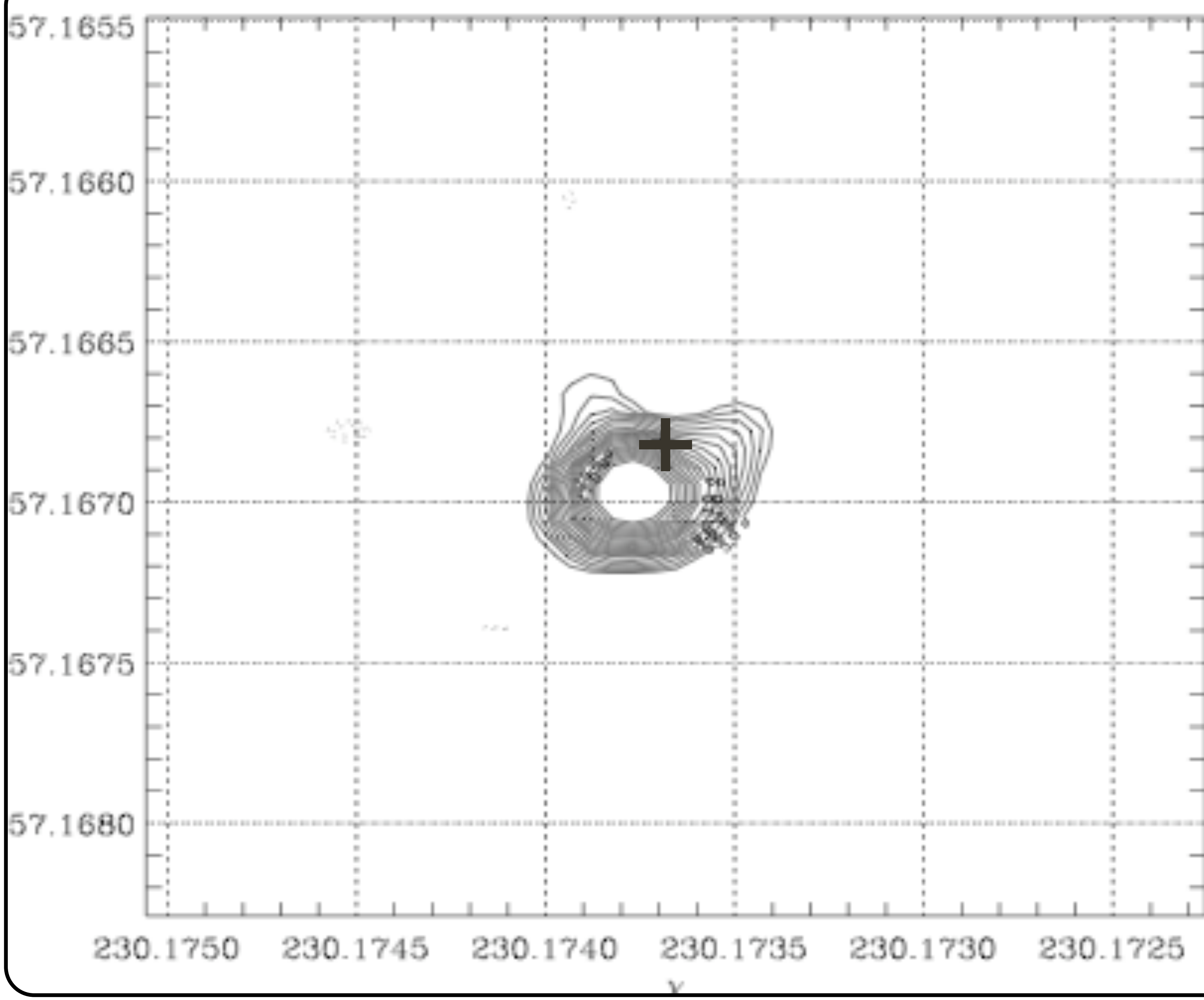


Chandra HRC observation

model and subsequently subtract PSF

only close to the source the assumption of a constant background is valid

APPLICATION MAXIMUM LIKELIHOOD METHOD X-RAY BINARY CIR X-1



ONE SOURCE SUBTRACTED