# Today:

## Chapter 1.5,1.6 &1.7
## Stochastic nature of radiation

Relation of ‘special’
functions to Gaussian and
Poissonian distributions
OAF2 chapter 5.1 &
Num Res chapter 6.1 & 6.2

Error propagation
OAF2 chapter 5.2

# What is the size of the fluctuations in the radiation field?

## Bose-Einstein statistics

Particles are distributed in

$h^3$ momentum space boxes

there are $Z \propto 4\pi p^2 dp$ boxes

## For each energy bin i there are $n_i$ particles, $Z_i$ boxes ≡ $Z_i+1$ boundaries of which $Z_i-1$ are "movable"

$$W(n_i) = \frac{(n_i + Z_i - 1)!}{n_i!(Z_i - 1)!}$$

this is also in chapter 6.2 of Lena

WHEN CONSIDERING ALL ENERGIES $i$

$$N = \sum_{i=1}^{\infty} n_i$$ TOTAL NUMBER OF PARTICLES

$$W = \Pi_{i=1}^{\infty} W(n_i)$$ TOTAL NUMBER OF POSSIBLE DISTRIBUTIONS

$$S \equiv k \ln(W)$$

MAXIMISE ENTROPY S

HENCE

$$\frac{d \ln W}{d n_i} = 0$$

# Remember Taylor expansion:

$$W(x+\Delta x) = W(x) + \frac{dW(x)}{dx}\Delta x + \frac{1}{2}\frac{d^2W(x)}{d^2x}\Delta x^2$$

cf. equation 1.28 & 1.37 Lecture notes

Derivation of eq. 1.41 on black board

# DEFINITIONS

$n_i$ NUMBER OF PHOTONS WITH ENERGY $i$

$n_{\nu_k}$ OCCUPATION FRACTION $\frac{\bar{n_i}}{Z_i}$

$N(\nu)$ VOLUME PHOTON DENSITY (PHOTONS PER SECOND PER HERTZ PER UNIT VOLUME) $\bar{N}(\nu)d\nu = g_\nu \bar{n_{\nu_k}} d\nu$

$n(\nu)$ SPECIFIC PHOTON FLUX (PHOTONS PER SECOND PER HERTZ) $\bar{n}(\nu) = 0.5\frac{c}{4\pi}\bar{N}(\nu)A_e\Omega$

$\overline{P}(\nu)$ RADIATION POWER $\overline{P}(\nu) = h\nu\,\overline{n}(\nu)$

# Planck distribution for photons $\epsilon_i = h\nu$

## Fluctuations in the number of photons per s per Hz

$$\Delta n^2(\nu) = n_\nu\left(1 + \frac{1}{exp(\frac{h\nu}{kT}) - 1}\right)$$

Power: $P(\nu) = h\nu\ n(\nu)$ $\qquad \overline{\Delta P^2}(\nu) = (h\nu)^2\overline{\Delta n^2}(\nu)$

## Two limits:

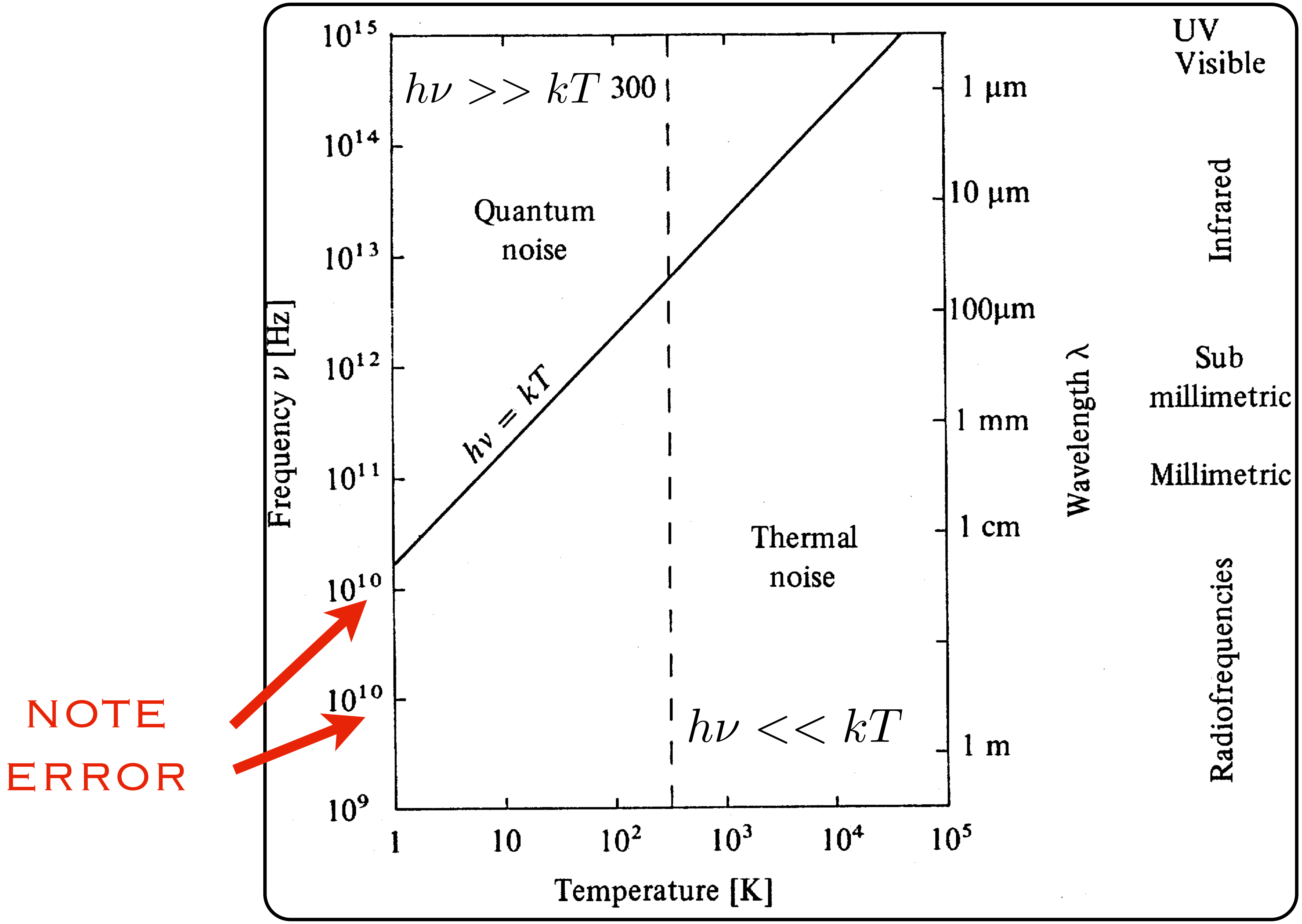
$h\nu >> kT$ Quantum limit

$$n(\nu) = \frac{1}{e^\epsilon - 1} \rightarrow 0 \ \text{ for } \epsilon >> 1 \qquad \overline{\Delta n^2}(\nu) = \bar{n}(\nu)$$

$h\nu << kT$ Thermal limit $\overline{\Delta P^2}(\nu) = \bar{P}^2(\nu)$

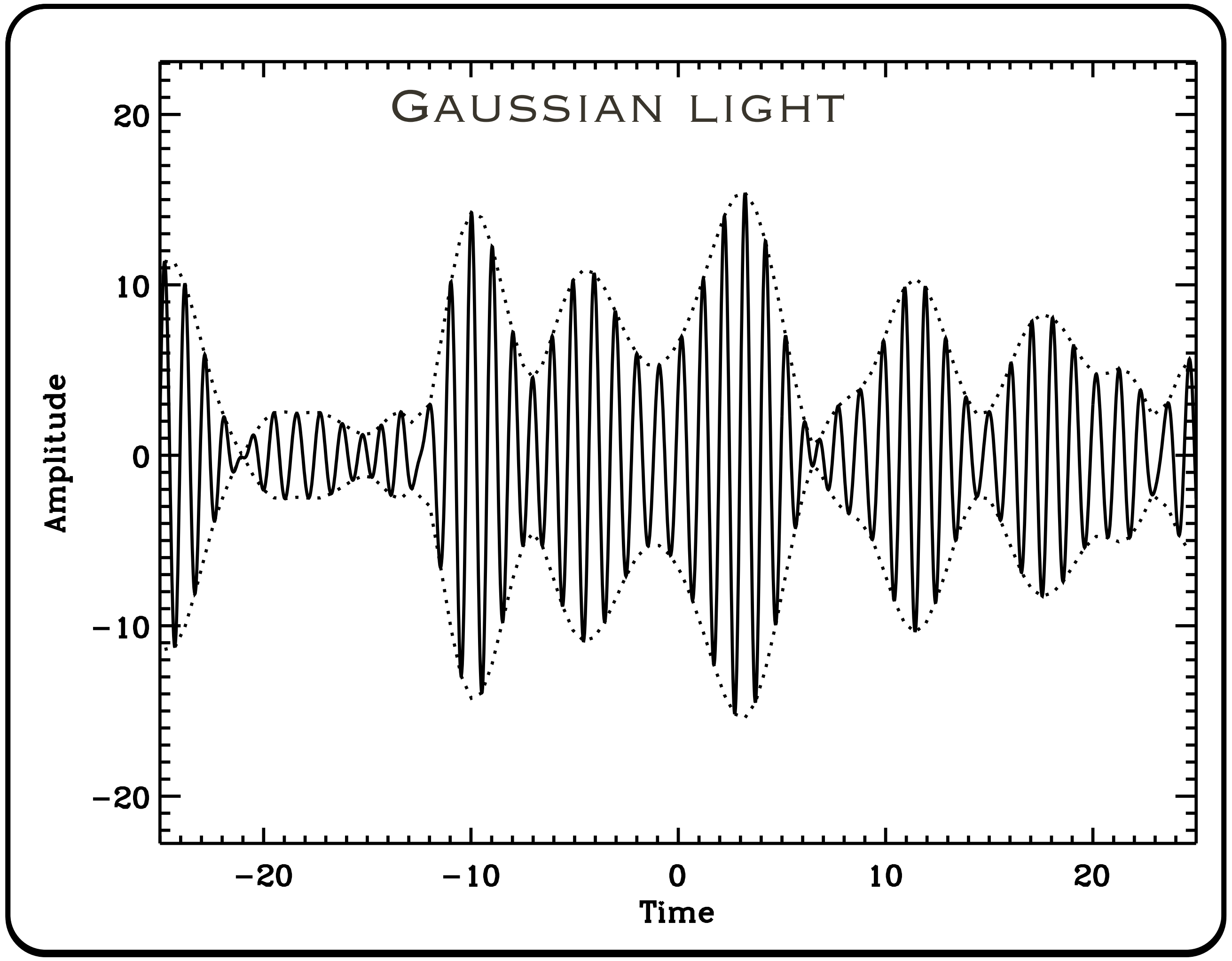
$$n(\nu) = \frac{1}{e^\epsilon - 1} \rightarrow \frac{1}{\epsilon} \ \text{ for } \epsilon << 1 \qquad \overline{P}(\nu) = kT$$

# Difference between thermal and quantum limit explains the difference between the principles behind/limitations of radio and optical/X-ray observations



# Stochastic description of radiation in the thermal limit

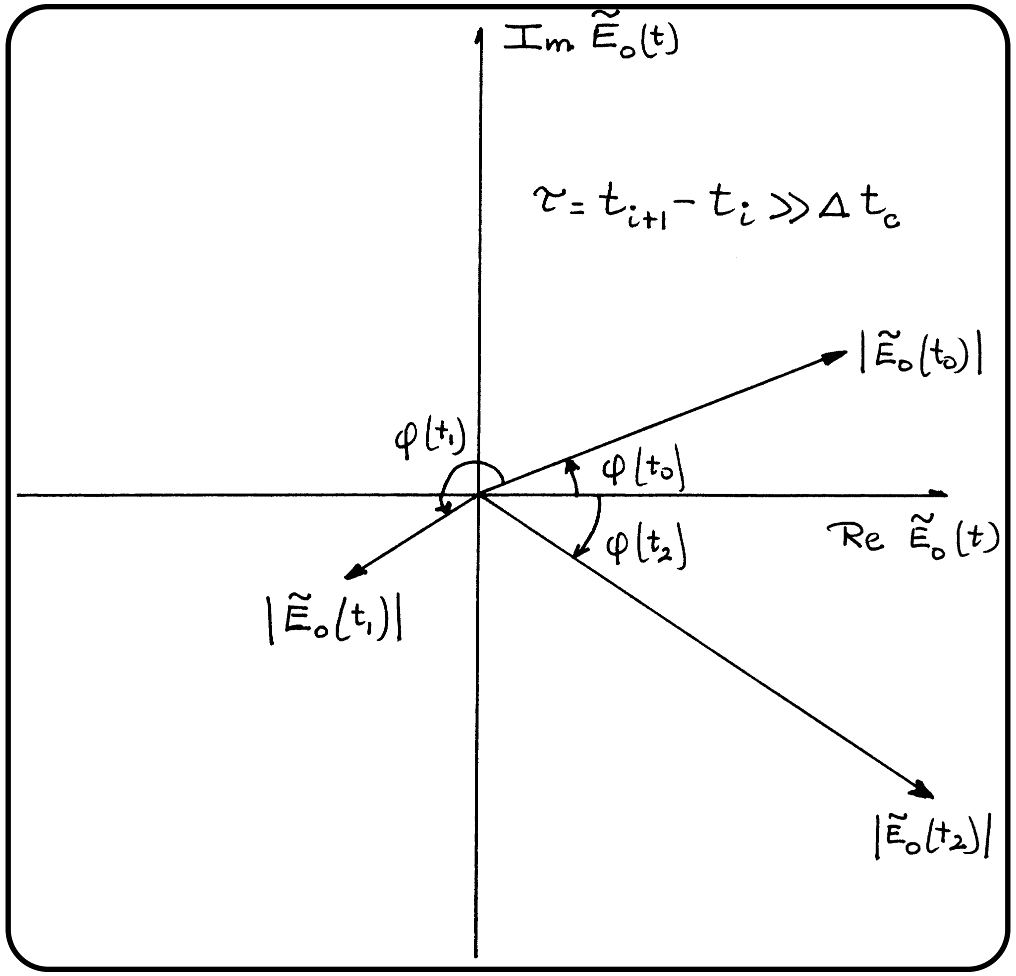
## quasi-monochromatic radiation from a thermal source




Describe electric field by $\tilde{E}(t) = \tilde{E_0}(t)\ e^{2\pi i\bar{\nu}t}$

where $\tilde{E_0}(t)$ is the phasor

The phasor is described by amplitude $|\tilde{E_0}(t)|$ and phase $\phi(t)$

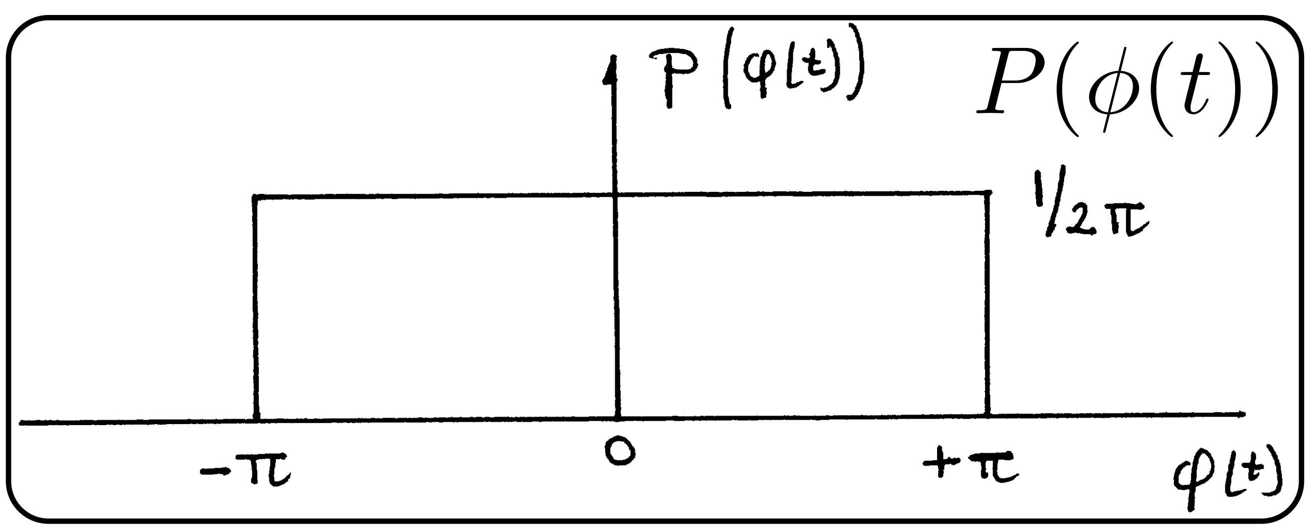



RANDOM VARIATIONS ON TIME SCALES >> THE COHERENCE TIME ASSOCIATED WITH (ATOMIC) TRANSITIONS

$$P(|E_0|(t), \phi(t))d|E_0|d\phi = \frac{E_0}{2\pi\sigma^2} e^{-\frac{E_0^2}{2\sigma^2}} d|E_0|d\phi$$

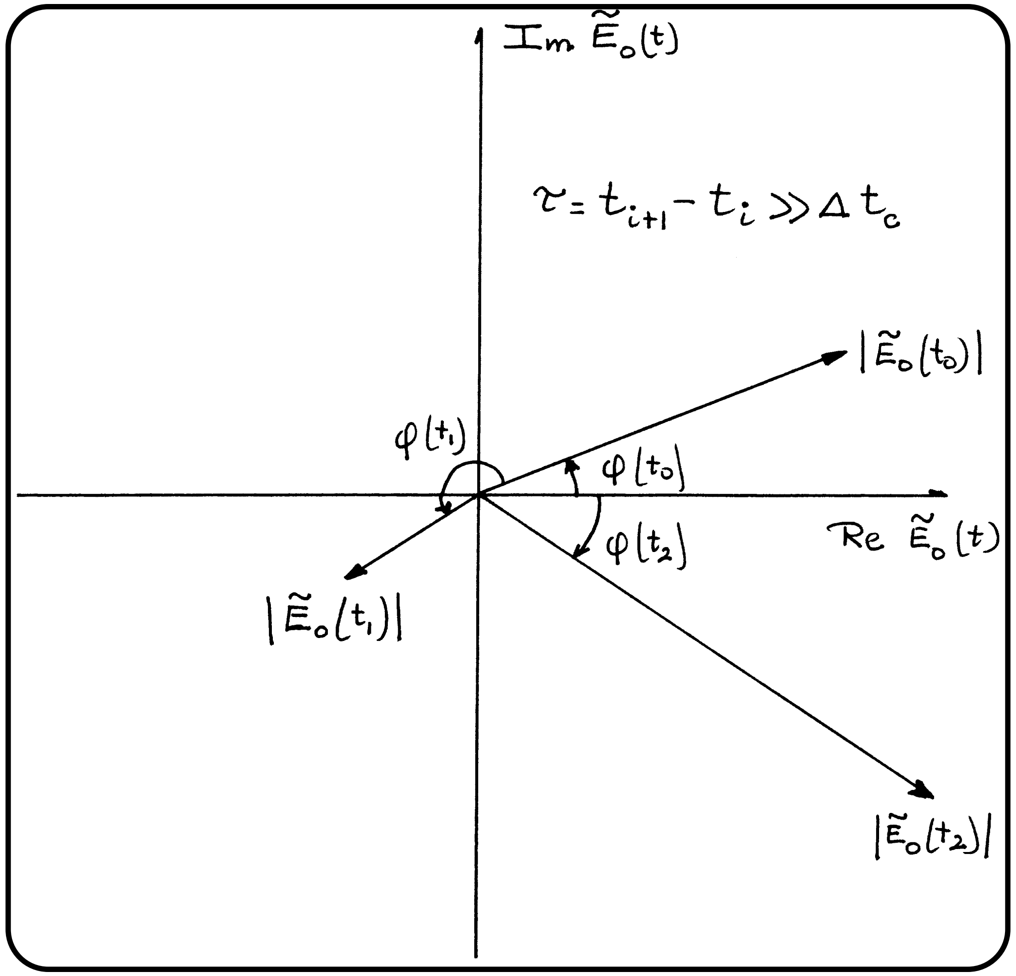
PROBABILITY DENSITY FOR:

for $I(t) = |\tilde{E}_0(t)|^2$




AGAIN ONE FINDS THAT THE FLUCTUATIONS IN THE POWER FLUX DENSITY ARE:

$$\overline{\Delta I^2} = \bar{I}^2$$

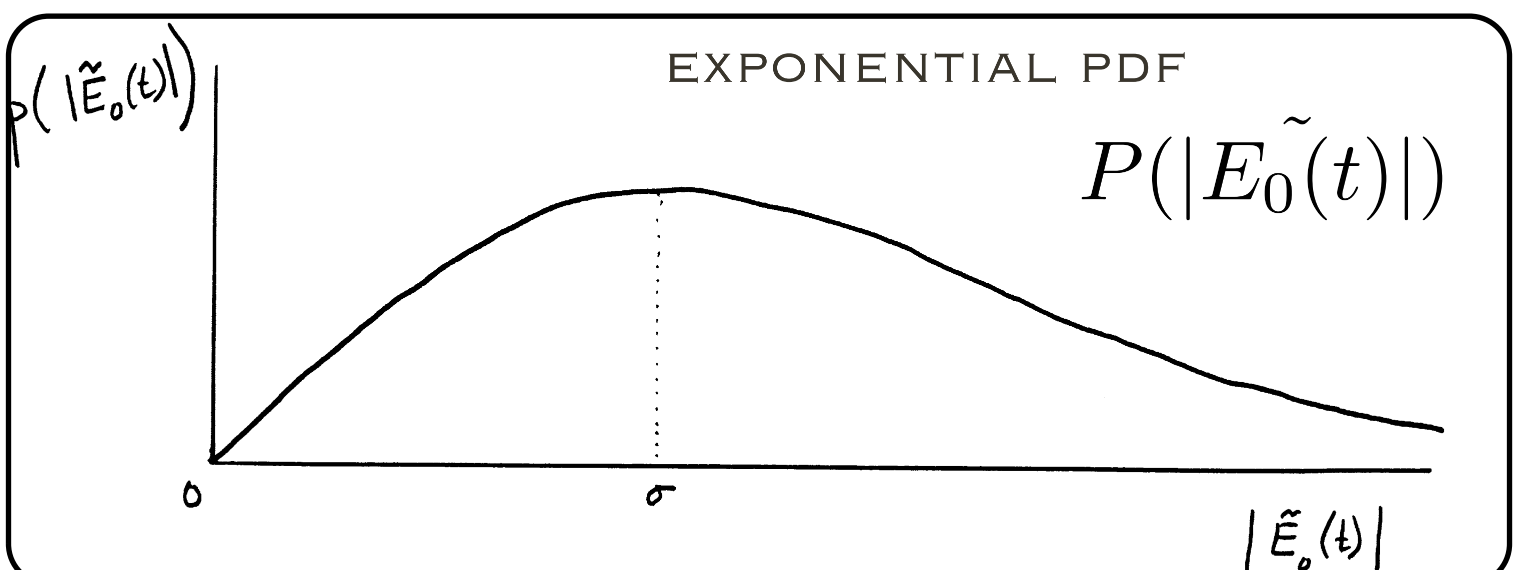



RANDOM VARIATIONS ON TIME SCALES >> THE COHERENCE TIME ASSOCIATED WITH (ATOMIC) TRANSITIONS

$$P(|E_0|(t), \phi(t))d|E_0|d\phi = \frac{E_0}{2\pi\sigma^2}e^{-\frac{E_0^2}{2\sigma^2}}d|E_0|d\phi$$

PROBABILITY DENSITY FOR:

$$\text{for } I(t) = |\tilde{E}_0(t)|^2$$




AGAIN ONE FINDS THAT THE FLUCTUATIONS IN THE POWER FLUX DENSITY ARE:

$$\overline{\Delta I^2} = \bar{I}^2$$

# Three different ways to derive the size of the fluctuation in the thermal limit

- bose-Einstein
- Stochastic description E-M wave
- Thermodynamic See Lena Chapter 6.2

$$\overline{\Delta P^2}(\nu) = (kT)^2$$

# Some (computational) math

stirling's approximation

$$\ln x! = x \ln x - x$$

in code use Gamma Function

$$\Gamma(z+1) = z!$$

$$\Gamma(z+1) = z\Gamma(z)$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

Gamma function has a computationally simple accurate approximation

Numerical Recipes Chap 6.1-6.2

## Incomplete Gamma Functions:

$$P(a,x) \equiv \frac{\gamma(a,x)}{\Gamma(a)} \equiv \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t} dt$$

$$Q(a,x) \equiv 1 - P(a,x) \equiv \frac{\Gamma(\mathrm{a},\mathrm{x})}{\Gamma(\mathrm{a})} \equiv \frac{1}{\Gamma(a)} \int_x^\infty t^{a-1} e^{-t} dt$$

## Error Functions:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad = P(\frac{1}{2}, x^2)$$

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \quad = Q(\frac{1}{2}, x^2)$$

## Cumulative Distribution function

$$F(x) = P\{x \leq y\}$$

## Probability density function

$$\frac{dF(x)}{dx} = f(x)$$

$\hookrightarrow$ Gauss, Poisson, $\chi^2$ etc

## Gaussian or normal distribution and probability density function

$$F(x, \eta, \sigma) = 0.5 + erf\frac{x-\eta}{\sigma^2}$$

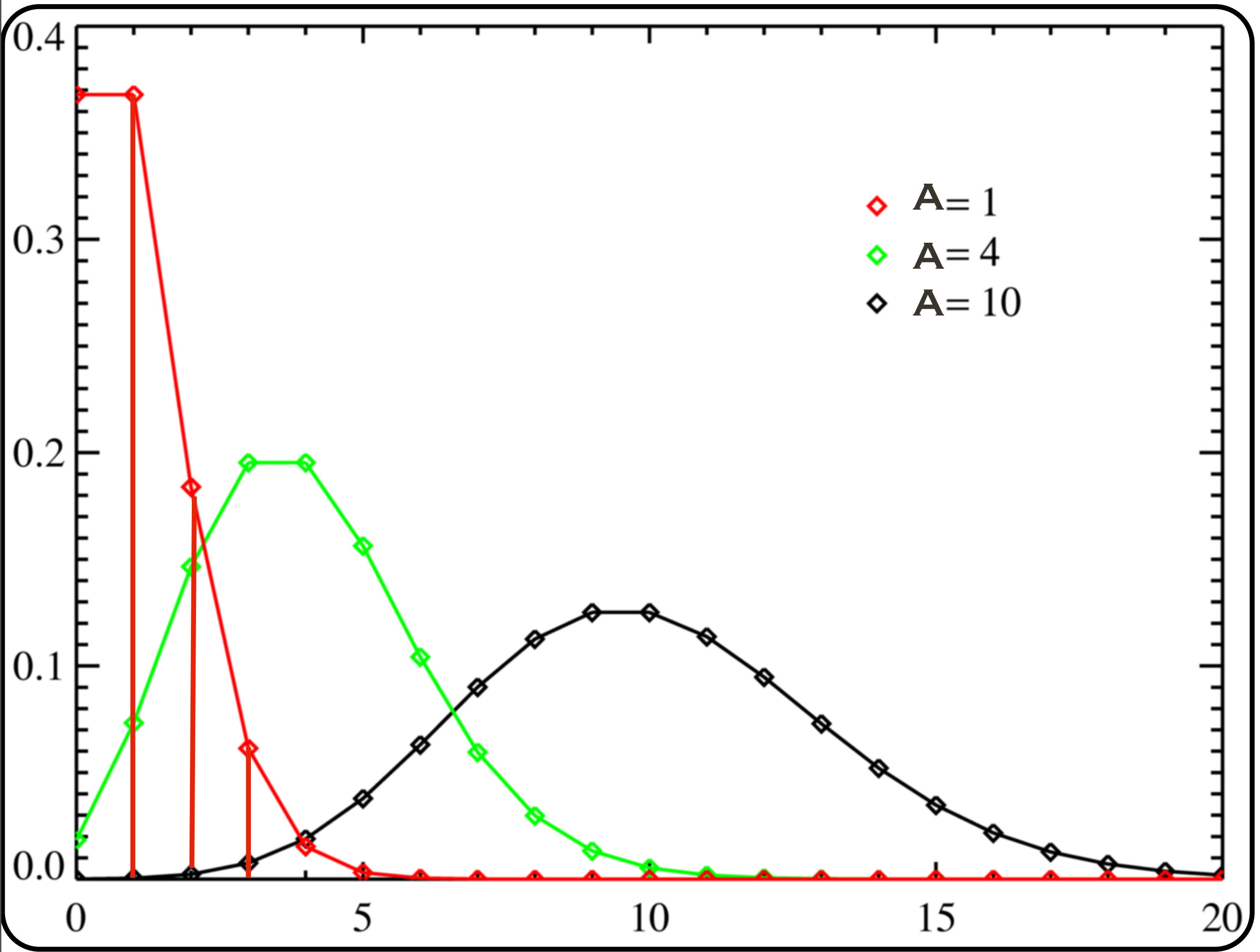
$$f(x) = \frac{1}{\sigma\sqrt{(2\pi)}}exp(-\frac{1}{2}\frac{(x-\eta)^2}{\sigma^2})$$

# Poisson cumulative distribution and probability density function

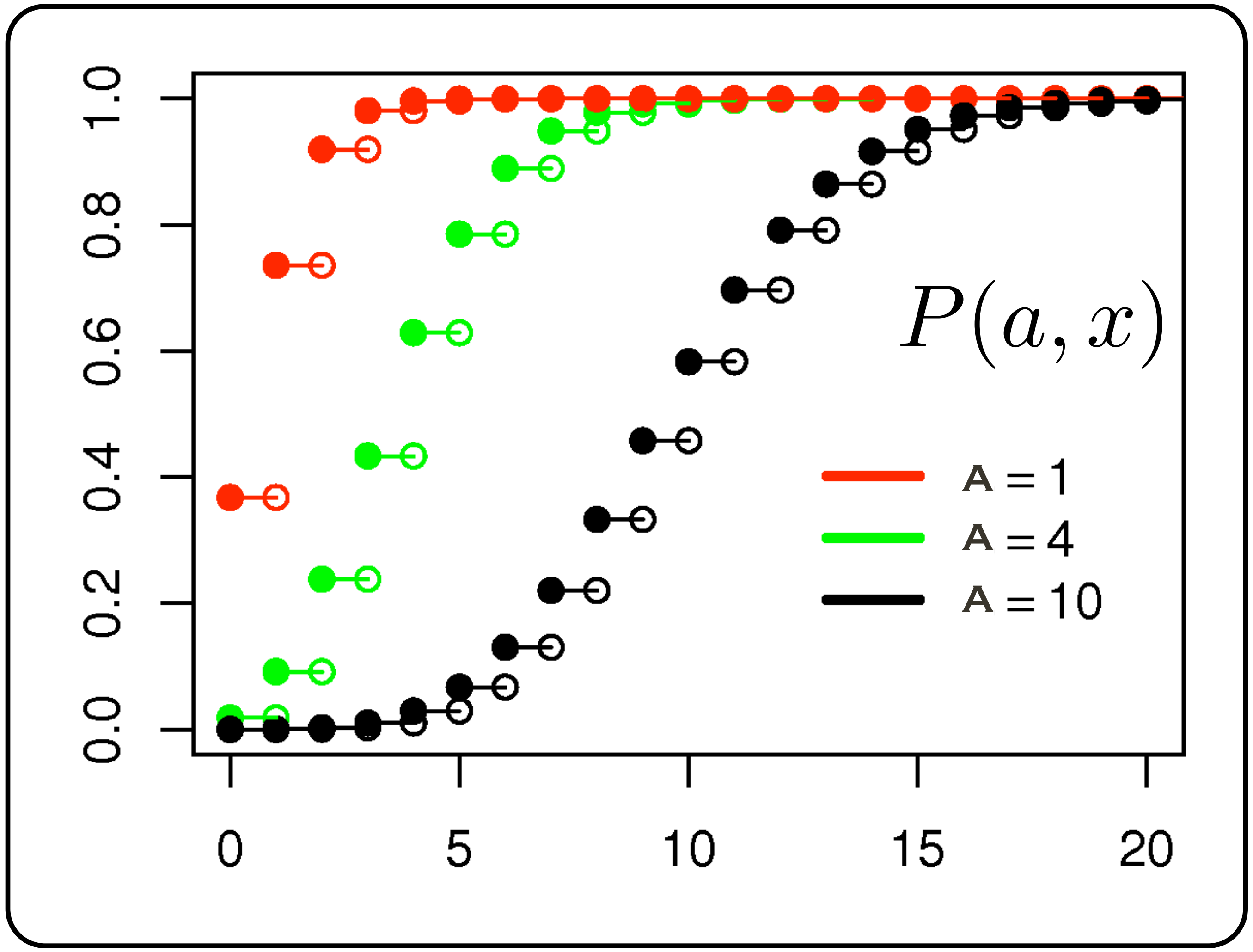
$$F(x) = 1 - P(a,x) \equiv \frac{1}{\Gamma(a)} \int_x^\infty e^{-t} t^{a-1} dt$$

$$f(x) = \frac{a^k}{k!} e^{-a} \rightarrow \ (discrete)\ e^{-a} \sum_{k=0}^{\infty} \frac{a^k}{k!} \delta(x-k)$$

## Probability density




## Cumulative distribution



In computer codes dealing with Poisson and Gaussian distributions incomplete Gamma Functions are used

# Error propagation 

Besides noise intrinsic to the s.p. noise is added due to the detector, background etc.

→ how to determine the resultant variance

Main assumption is that the average of the function f is well represented by the value for f at the averages for the variables

$$\bar{f} = f(\bar{u}, \bar{v}, ..)$$

Taylor expansion to first order around the average for each variable

$$f_i - \bar{f} \approx (u_i - \bar{u})\frac{\partial f}{\partial u} + (v_i - \bar{v})\frac{\partial f}{\partial v} + \ldots$$

REMEMBER THAT THE VARIANCE

$$\sigma^2 = \frac{1}{N-1}\sum_{i=1}^{N}(f_i - \bar{f})^2$$

FILL-IN THE TAYLOR EXPANSION HERE

ASSUME THAT THE VARIABLES ARE INDEPENDENT SUCH THAT THEIR CROSS PRODUCT CANCEL

$$\sigma_f^2 = \sigma_u^2(\frac{\partial f}{\partial u})^2 + \sigma_v^2(\frac{\partial f}{\partial v})^2 + ...$$

$$\text{cross products } \; 2\sigma_{uv}^2 \frac{\partial f}{\partial u}\frac{\partial f}{\partial v}$$