Outline

Very Large Array (VLA), New Mexico, USA

Image courtesy of NRAO/AUI

Cygnus A at 6 cm

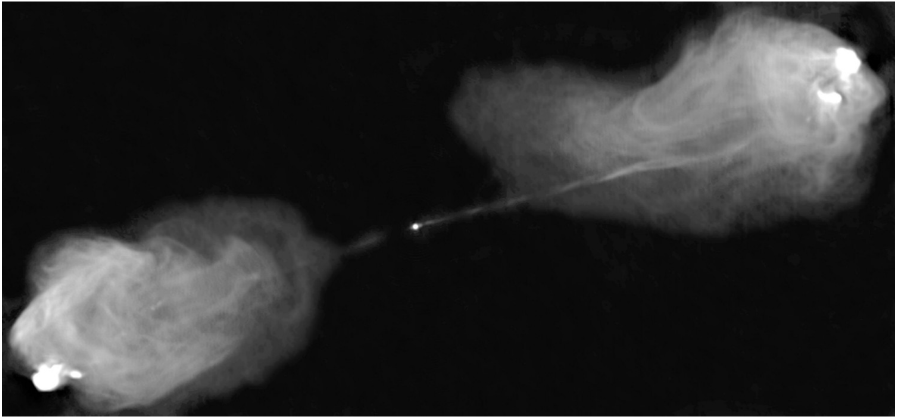


Image courtesy of NRAO/AUI

Plane-Wave Solutions

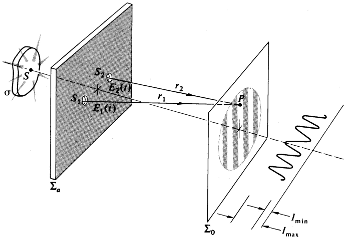
Plane Vector Wave ansatz: $\vec{E} = \vec{E}_0 e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$

- $\vec{k}$ spatially and temporally constant *wave vector*
- $\vec{k}$ normal to surfaces of constant phase
- $|\vec{k}|$ *wave number*
- $\vec{x}$ spatial location
- ω *angular frequency* ($2\pi\times$ frequency)
- t time
- $\vec{E}_0$ a (generally complex) vector independent of time and space
- real electric field vector given by real part of $\vec{E}$

Scalar Wave

- electric field at position $\vec{r}$ at time t is $\tilde{E}(\vec{r}, t)$
- complex notation to easily express amplitude and phase
- real part of complex quantity is the physical quantity

Young's Double Slit Experiment



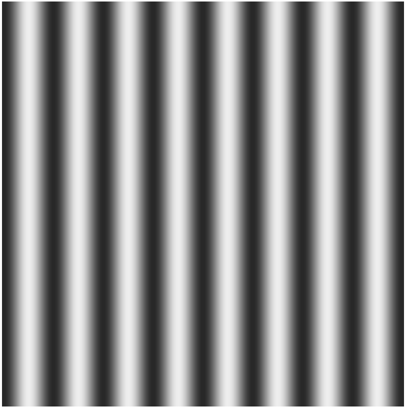
- monochromatic wave
- infinitely small holes (pinholes)
- source S generates fields $\tilde{E}(\vec{r}_1, t) \equiv \tilde{E}_1(t)$ at S_1 and $\tilde{E}(\vec{r}_2, t) \equiv \tilde{E}_2(t)$ at S_2
- two spherical waves from pinholes interfere on screen
- electrical field at P

$$\tilde{E}_P(t) = \tilde{C}_1 \tilde{E}_1(t - t_1) + \tilde{C}_2 \tilde{E}_2(t - t_2)$$

- $t_1 = r_1/c$, $t_2 = r_2/c$
- r_1, r_2: path lengths to P from S_1, S_2
- propagators $\tilde{C}_{1,2} = \frac{i}{\lambda}$

no tilt

tilt by 0.5 λ/d

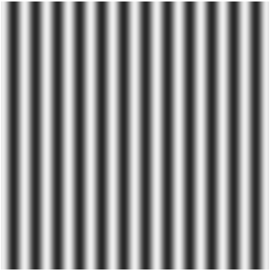


Change in Angle of Incoming Wave

- phase of fringe pattern changes, but not fringe spacing
- tilt of λ/d produces identical fringe pattern

long wavelength

short wavelength

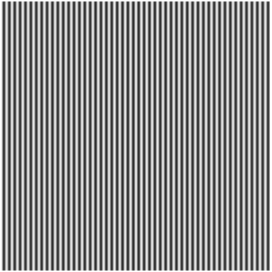


wavelength average

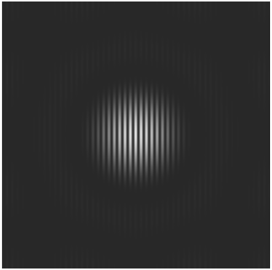
Change in Wavelength

- fringe spacing changes, central fringe broadens
- integral over 0.8 to 1.2 of central wavelength
- intergral over wavelength makes fringe envelope

2 pinholes



2 small holes



2 large holes

Finite Hole Diameter

- fringe spacing only depends on separation of holes and wavelength
- the smaller the hole, the larger the 'illuminated' area
- fringe envelope is Airy pattern (diffraction pattern of a single hole)

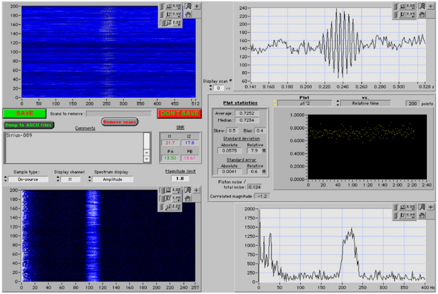
Visibility

- “quality” of fringes described by **Visibility function**

$$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$$

- $I_{\max}$, $I_{\min}$ are maximum and adjacent minimum in fringe pattern

First Fringes from VLT Interferometer



'First Fringes' from Sirius with VLTI

ESO PR Photo 10a/01 (18 March 2001)

Mutual Coherence

- total field in point P

$$\tilde{E}_P(t) = \tilde{C}_1 \tilde{E}_1(t - t_1) + \tilde{C}_2 \tilde{E}_2(t - t_2)$$

- irradiance at P, averaged over time

$$I = \mathbf{E}|\tilde{E}_P(t)|^2 = \mathbf{E}\left\{\tilde{E}_P(t)\tilde{E}_P^*(t)\right\}$$

- writing out all the terms

$$\begin{aligned} I = {} & \tilde{C}_1 \tilde{C}_1^* \mathbf{E}\left\{\tilde{E}_1(t - t_1)\tilde{E}_1^*(t - t_1)\right\} + \tilde{C}_2 \tilde{C}_2^* \mathbf{E}\left\{\tilde{E}_2(t - t_2)\tilde{E}_2^*(t - t_2)\right\} \\ & + \tilde{C}_1 \tilde{C}_2^* \mathbf{E}\left\{\tilde{E}_1(t - t_1)\tilde{E}_2^*(t - t_2)\right\} + \tilde{C}_1^* \tilde{C}_2 \mathbf{E}\left\{\tilde{E}_1^*(t - t_1)\tilde{E}_2(t - t_2)\right\} \end{aligned}$$

Mutual Coherence (continued)

- as before

$$I = \tilde{C}_1\tilde{C}_1^*\mathbf{E}\left\{\tilde{E}_1(t-t_1)\tilde{E}_1^*(t-t_1)\right\} + \tilde{C}_2\tilde{C}_2^*\mathbf{E}\left\{\tilde{E}_2(t-t_2)\tilde{E}_2^*(t-t_2)\right\}$$
$$+\tilde{C}_1\tilde{C}_2^*\mathbf{E}\left\{\tilde{E}_1(t-t_1)\tilde{E}_2^*(t-t_2)\right\} + \tilde{C}_1^*\tilde{C}_2\mathbf{E}\left\{\tilde{E}_1^*(t-t_1)\tilde{E}_2(t-t_2)\right\}$$

- *stationary* wave field, time average independent of absolute time

$$I_{S_1} = \mathbf{E}\left\{\tilde{E}_1(t)\tilde{E}_1^*(t)\right\},\ I_{S_2} = \mathbf{E}\left\{\tilde{E}_2(t)\tilde{E}_2^*(t)\right\}$$

- irradiance at P is now

$$I = \tilde{C}_1\tilde{C}_1^*I_{S_1} + \tilde{C}_2\tilde{C}_2^*I_{S_2}$$
$$+\tilde{C}_1\tilde{C}_2^*\mathbf{E}\left\{\tilde{E}_1(t-t_1)\tilde{E}_2^*(t-t_2)\right\} + \tilde{C}_1^*\tilde{C}_2\mathbf{E}\left\{\tilde{E}_1^*(t-t_1)\tilde{E}_2(t-t_2)\right\}$$

Mutual Coherence (continued)

- as before

$$I = \tilde{C}_1 \tilde{C}_1^* I_{S_1} + \tilde{C}_2 \tilde{C}_2^* I_{S_2}$$
$$+\tilde{C}_1 \tilde{C}_2^* \mathbf{E}\left\{\tilde{E}_1(t-t_1)\tilde{E}_2^*(t-t_2)\right\} + \tilde{C}_1^* \tilde{C}_2 \mathbf{E}\left\{\tilde{E}_1^*(t-t_1)\tilde{E}_2(t-t_2)\right\}$$

- time difference $\tau = t_2 - t_1 \Rightarrow$ last two terms become

$$\tilde{C}_1 \tilde{C}_2^* \mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_2^*(t)\right\} + \tilde{C}_1^* \tilde{C}_2 \mathbf{E}\left\{\tilde{E}_1^*(t+\tau)\tilde{E}_2(t)\right\}$$

- equivalent to

$$2\ Re\left[\tilde{C}_1 \tilde{C}_2^* \mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_2^*(t)\right\}\right]$$

- propagators $\tilde{C}$ purely imaginary: $\tilde{C}_1 \tilde{C}_2^* \ = \ \tilde{C}_1^* \tilde{C}_2 \ = \ |\tilde{C}_1||\tilde{C}_2|$
- cross-term becomes $2|\tilde{C}_1||\tilde{C}_2| Re\left[\mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_2^*(t)\right\}\right]$

Mutual Coherence (continued)

- irradiance at P

$$\begin{aligned} I &= \tilde{C}_1 \tilde{C}_1^* I_{S_1} + \tilde{C}_2 \tilde{C}_2^* I_{S_2} \\ &\quad + 2|\tilde{C}_1||\tilde{C}_2| Re \left[\mathbf{E} \left\{ \tilde{E}_1(t+\tau) \tilde{E}_2^*(t) \right\} \right] \end{aligned}$$

- **mutual coherence function** of wave field at S_1 and S_2

$$\tilde{\Gamma}_{12}(\tau) = \mathbf{E} \left\{ \tilde{E}_1(t+\tau) \tilde{E}_2^*(t) \right\}$$

- therefore $I = |\tilde{C}_1|^2 I_{S_1} + |\tilde{C}_2|^2 I_{S_2} + 2|\tilde{C}_1||\tilde{C}_2| \; Re \, \tilde{\Gamma}_{12}(\tau)$
- $I_1 = |\tilde{C}_1|^2 I_{S_1}$, $I_2 = |\tilde{C}_2|^2 I_{S_2}$: irradiances at P from single aperture

$$I = I_1 + I_2 + 2|\tilde{C}_1||\tilde{C}_2| \; Re \, \tilde{\Gamma}_{12}(\tau)$$

Self-Coherence

- $S_1 = S_2 \Rightarrow$ mutual coherence function = autocorrelation

$$\tilde{\Gamma}_{11}(\tau) = \tilde{R}_1(\tau) = \mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_1^*(t)\right\}$$
$$\tilde{\Gamma}_{22}(\tau) = \tilde{R}_2(\tau) = \mathbf{E}\left\{\tilde{E}_2(t+\tau)\tilde{E}_2^*(t)\right\}$$

- autocorrelation functions are also called *self-coherence functions*
- for $\tau = 0$

$$I_{S_1} = \mathbf{E}\left\{\tilde{E}_1(t)\tilde{E}_1^*(t)\right\} = \Gamma_{11}(0) = \mathbf{E}\left\{|\tilde{E}_1(t)|^2\right\}$$
$$I_{S_2} = \mathbf{E}\left\{\tilde{E}_2(t)\tilde{E}_2^*(t)\right\} = \Gamma_{22}(0) = \mathbf{E}\left\{|\tilde{E}_2(t)|^2\right\}$$

- autocorrelation function with zero lag ($\tau = 0$) represent (average) irradiance (power) of wave field at S_1, S_2

Complex Degree of Coherence

- using selfcoherence functions

$$|\tilde{C}_1||\tilde{C}_2| = \frac{\sqrt{I_1}\sqrt{I_2}}{\sqrt{\Gamma_{11}(0)}\sqrt{\Gamma_{22}(0)}}$$

- normalized mutual coherence defines the **complex degree of coherence**

$$\tilde{\gamma}_{12}(\tau) \equiv \frac{\tilde{\Gamma}_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}} = \frac{\mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_2^*(t)\right\}}{\sqrt{\mathbf{E}\left\{|\tilde{E}_1(t)|^2\right\}\mathbf{E}\left\{|\tilde{E}_2(t)|^2\right\}}}$$

- irradiance in point P as *general interference law for a partially coherent radiation field*

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\, Re\, \tilde{\gamma}_{12}(\tau)$$

Spatial and Temporal Coherence

- complex degree of coherence

$$\tilde{\gamma}_{12}(\tau) \equiv \frac{\tilde{\Gamma}_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}} = \frac{\mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_2^*(t)\right\}}{\sqrt{\mathbf{E}\left\{|\tilde{E}_1(t)|^2\right\}\mathbf{E}\left\{|\tilde{E}_2(t)|^2\right\}}}$$

- measures both
  - *spatial coherence* at S_1 and S_2
  - *temporal coherence* through time lag τ
- $\tilde{\gamma}_{12}(\tau)$ is a complex variable and can be written as:

$$\tilde{\gamma}_{12}(\tau) = |\tilde{\gamma}_{12}(\tau)|e^{i\psi_{12}(\tau)}$$

- $0 \leq |\tilde{\gamma}_{12}(\tau)| \leq 1$
- phase angle $\psi_{12}(\tau)$ relates to
  - phase angle between fields at S_1 and S_2
  - phase angle difference in P resulting in time lag τ

Coherence of Quasi-Monochromatic Light

- quasi-monochromatic light, mean wavelength $\overline{\lambda}$, frequency $\overline{\nu}$, phase difference ϕ due to optical path difference:

$$\phi \;=\; \frac{2\pi}{\overline{\lambda}}(r_2 - r_1) \;=\frac{2\pi}{\overline{\lambda}}c(t_2 - t_1) \;=\; 2\pi\overline{\nu}\tau$$

- with phase angle $\alpha_{12}(\tau)$ between fields at pinholes S_1, S_2

$$\psi_{12}(\tau) = \alpha_{12}(\tau) - \phi$$

- and

$$Re\,\tilde{\gamma}_{12}(\tau) \;=\; |\tilde{\gamma}_{12}(\tau)| \cos\left[\alpha_{12}(\tau) - \phi\right]$$

- intensity in P becomes

$$I \;=\; I_1 \;+\; I_2 \;+\; 2\sqrt{I_1 I_2}\,|\tilde{\gamma}_{12}(\tau)| \cos\left[\alpha_{12}(\tau) - \phi\right]$$

Visibility of Quasi-Monochromatic, Partially Coherent Light

- intensity in P

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\,|\tilde{\gamma}_{12}(\tau)|\cos\left[\alpha_{12}(\tau) - \phi\right]$$

- maximum, minimum I for $\cos(...) = \pm 1$
- visibility V at position P

$$V = \frac{2\sqrt{I_1}\sqrt{I_2}}{I_1 + I_2}|\tilde{\gamma}_{12}(\tau)|$$

- for $I_1 = I_2 = I_0$

$$\begin{aligned} I &= 2I_0\left\{1 + |\tilde{\gamma}_{12}(\tau)|\cos\left[\alpha_{12}(\tau) - \phi\right]\right\} \\ V &= |\tilde{\gamma}_{12}(\tau)| \end{aligned}$$

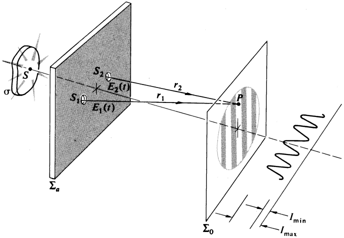
Interpretation of Visibility

- for $I_1 = I_2 = I_0$

$$I = 2I_0 \left\{1 + |\tilde{\gamma}_{12}(\tau)| \cos\left[\alpha_{12}(\tau) - \phi\right]\right\}$$
$$V = |\tilde{\gamma}_{12}(\tau)|$$

- *modulus of complex degree of coherence = visibility of fringes*
- modulus can therefore be measured
- shift in location of central fringe (no optical path length difference, $\phi = 0$) is measure of $\alpha_{12}(\tau)$
- measurements of visibility and fringe position yield amplitude and phase of complex degree of coherence

Fringe Pattern



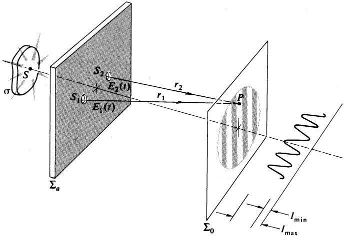
- for $I_1 = I_2 = I_0$

$$I = 2I_0 \left\{1 + |\tilde{\gamma}_{12}(\tau)| \cos\left[\alpha_{12}(\tau) - \phi\right]\right\} \quad V = |\tilde{\gamma}_{12}(\tau)|$$

- source S on central axis, fully coherent waves from two holes

$$I = 2I_0(1 + \cos\phi) = 4I_0 \cos^2\frac{\phi}{2}$$

Fringe Pattern (continued)



$$I = 4 I_0 \cos^2 \frac{\phi}{2}$$

$$\phi = \frac{2\pi}{\bar{\lambda}}(r_2 - r_1) = 2\pi\bar{\nu}\tau$$

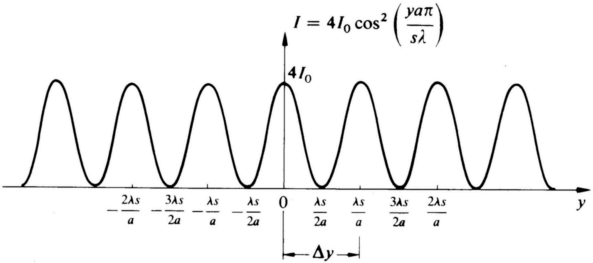
- distance *a* between pinholes
- distance *s* to observation plane Σ_O, $s \gg a$
- path difference $(r_2 - r_1)$ in equation for ϕ in good approximation

$$r_2 - r_1 = a\theta = \frac{a}{s} y$$

- and therefore

$$I = 4 I_0 \cos^2 \frac{\pi a y}{s\bar{\lambda}}$$

Interference Fringes from Monochromatic Point Source



- irradiance as a function of the y-coordinate of the fringes in observation plane Σ_O
- irradiance vs. distance distribution is Point-Spread Function (PSF) of ideal two-element interferometer

Finite Apertures

- non-ideal two-element interferometer with finite apertures using *pupil function* concept (Observational Astrophysics 1)
- optical transfer function (OTF)

$$OTF = 2\left(\frac{\lambda}{R}\right)^2 \left[\delta(\vec{\zeta}) + \frac{1}{2}\delta\left(\vec{\zeta} - \vec{s}/\lambda\right) + \frac{1}{2}\delta\left(\vec{\zeta} + \vec{s}/\lambda\right)\right]$$

- pair of pinholes transmits three spatial frequencies
  - DC-component $\delta(\vec{0})$
  - two high frequencies related to length of baseline vector $\vec{s}$ at $\pm\vec{s}/\lambda$
- 3 spatial frequencies represent three-point sampling of the **uv-plane** in 2-d spatial frequency space
- complete sampling of uv-plane provides sufficient information to completely reconstruct original brightness distribution

Point-Spread Function (PSF)

- PSF is Fourier Transform of OTF

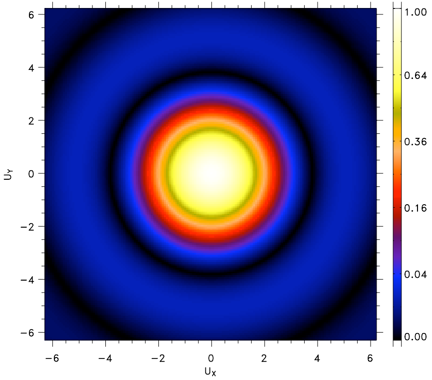
$$\begin{aligned} \delta(\vec{\zeta}) &\Leftrightarrow 1 \\ \delta\left(\vec{\zeta} - \vec{s}/\lambda\right) &\Leftrightarrow e^{i2\pi\vec{\theta}\cdot\vec{s}/\lambda} \\ \delta\left(\vec{\zeta} + \vec{s}/\lambda\right) &\Leftrightarrow e^{-i2\pi\vec{\theta}\cdot\vec{s}/\lambda} \end{aligned}$$

- Point-Spread Function of 2-element interferometer

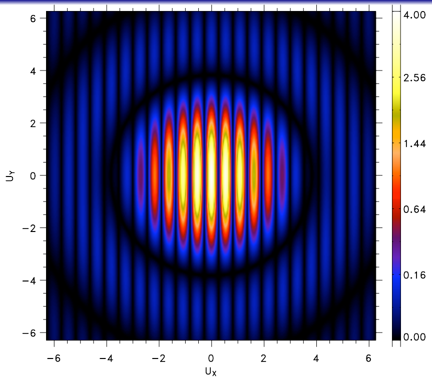
$$\left(\frac{\lambda}{R}\right)^2 \left[2(1 + \cos 2\pi\vec{\theta} \cdot \vec{s}/\lambda)\right] = 4\left(\frac{\lambda}{R}\right)^2 \cos^2 \pi\vec{\theta} \cdot \vec{s}/\lambda$$

- $\vec{\theta}$: 2-d angular coordinate vector
- attenuation factor $(\lambda/R)^2$ from spherical expansion

2-d Brightness Distribution



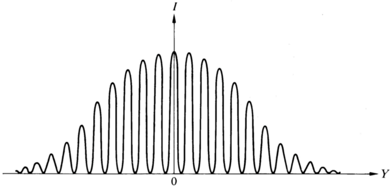
PSF of single circular aperture



PSF of two-element interferometer, aperture diameter $d = 25$ m, length of baseline vector $|\vec{s}| = 144$ m

- double beam interference fringes showing modulation effect of diffraction by aperture of a single pinhole

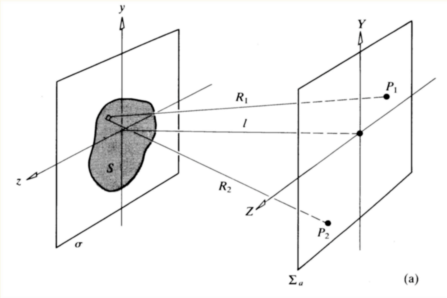
Modulation Effect of Aperture



- typical one-dimensional cross-section along $u_y = 0$ of the central part of the interferogram
- visibilities are equal to one, because $I_{min} = 0$
- $|\tilde{\gamma}_{12}(\tau)| = 1$ for all values of τ and any pair of spatial points, if and only if the radiation field is *strictly monochromatic*

Van Cittert-Zernike Theorem

The Problem



- relates brightness distribution of extended source and phase correlation between two points in radiation field
- extended source S incoherent, quasi-monochromatic
- positions P_1 and P_2 in observers plane Σ

$$\tilde{E}_1(t)\tilde{E}_2^*(t) = \mathbf{E}\{\tilde{E}_1(t)\tilde{E}_2^*(t)\} = \tilde{\Gamma}_{12}(0)$$

The Solution

- $I(\vec{\Omega})$ is intensity distribution of extended source as function of unit direction vector $\vec{\Omega}$ as seen from observation plane Σ
- $\tilde{\Gamma}(\vec{r})$ is coherence function in Σ-plane
- vector $\vec{r}$ represents arbitrary baseline
- van Cittert-Zernike theorem

$$\tilde{\Gamma}(\vec{r}) = \int\int_{\text{source}} I(\vec{\Omega}) e^{\frac{2\pi i \vec{\Omega}.\vec{r}}{\lambda}} d\vec{\Omega}$$

$$I(\vec{\Omega}) = \lambda^{-2} \int\int_{\Sigma\text{-plane}} \tilde{\Gamma}(\vec{r}) e^{-\frac{2\pi i \vec{\Omega}.\vec{r}}{\lambda}} d\vec{r}$$

- $\tilde{\Gamma}(\vec{r})$ and $I(\vec{\Omega})$ are linked through Fourier transform, except for scaling with wavelength λ
- "true" Fourier transform with *conjugate variables* $\vec{\Omega}$, $\vec{r}/\lambda$,
- Fourier pair: $I(\vec{\Omega}) \quad \Leftrightarrow \quad \tilde{\Gamma}(\vec{r}/\lambda)$

Overview

- positions 1, 2 in observation plane Σ not pointlike, but finite aperture with diameter D
- single aperture has diffraction-sized beam of λ/D
- Van Cittert-Zernike relation needs to be "weighted" with telescope element (single dish) transfer function $H(\vec{\Omega})$
- circular dish antenna: $H(\vec{\Omega})$ is *Airy brightness function*
- The Van Cittert-Zernike relations now become:

$$\tilde{\Gamma}'(\vec{r}) = \int \int_{\text{source}} I(\vec{\Omega}) H(\vec{\Omega}) e^{\frac{2\pi i \vec{\Omega}.\vec{r}}{\lambda}} d\vec{\Omega}$$

$$I(\vec{\Omega}) H(\vec{\Omega}) = \lambda^{-2} \int \int_{\Sigma\text{-plane}} \tilde{\Gamma}'(\vec{r}) e^{-\frac{2\pi i \vec{\Omega}.\vec{r}}{\lambda}} d\vec{r}$$

- field of view scales with λ/D; if λ decreases, synthesis resolution improves but field-of-view reduces proportionally!

Overview (continued)

- aperture synthesis: incoming beams from antenna dish 1 and antenna dish 2 are fed into a *correlator (multiplier)* producing as output $\tilde{E}_1(t)\tilde{E}_2^*(t)$
- output subsequently fed into *integrator/averager* producing

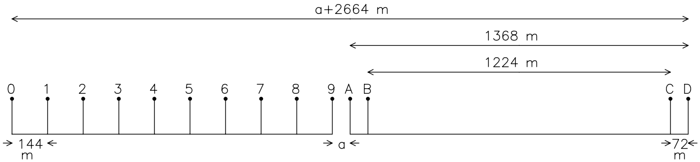
$$\mathbf{E}\left\{\tilde{E}_1(t)\tilde{E}_2^*(t)\right\} = \tilde{\Gamma}'(\vec{r})$$

- applying Fourier transform and correcting for beam profile of single dish $H(\vec{\Omega}) \Rightarrow$ reconstruct source brightness distribution $I(\vec{\Omega})$
- limited to measuring image details within *single pixel* of an individual telescope element, dish

Earth-Rotation Aperture Synthesis

- due to rotation of Earth, baseline vectors $k \cdot \vec{s}/\lambda$ of N-element array scan the YZ-plane if X-axis is lined up with North polar axis
- principal maxima or 'grating lobes' in PSF are concentric annuli around central source peak at angular distances $k \cdot \lambda/|\vec{s}\,|$
- if circular scans in YZ-plane are too widely spaced ($|\vec{s}\,|$ is larger than single dish diameter), the Nyquist criterion is not respected and undersampling of spatial frequency uv-plane (=YZ-plane) occurs
- consequently, grating lobes will show up within the field of view defined by the single-dish beam profile
- can be avoided by decreasing sampling distance $|\vec{s}\,|$

Westerbork Radio Synthesis Telescope (WSRT)

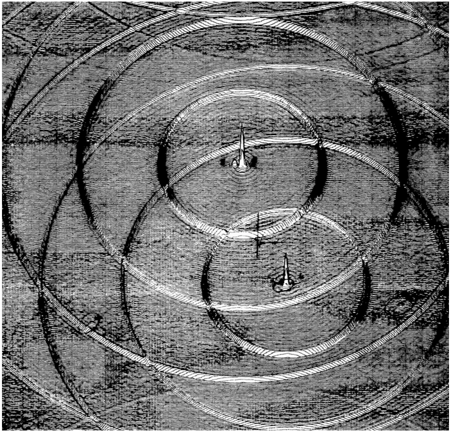


- 14 parabolic antennae, diameters $D = 25$ m
- lined up along East-West direction over ≈ 2750 m
- 10 antennae have fixed mutual distance of 144 m
- 4 antennae can be moved collectively with respect to fixed array
- 14 antennae comprise 40 simultaneously operating interferometers
- array is rotated in plane containing Westerbork perpendicular to Earth's rotation axis
- limited to sources near the North polar axis
- standard distance *a* between 9 and A equals 72 meters

WSRT (continued)

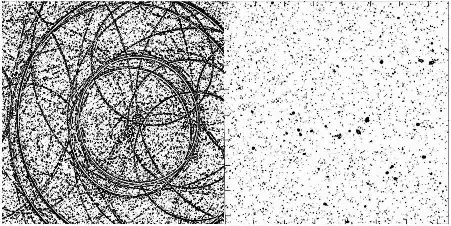
- after 12 hours, 38 concentric semi-circles with radii ranging from $L_{min} = 72$ meters to $L_{max} = 2736$ meters in increments of $\triangle L = 72$ meters
- correlators integrate over 10 s, sampling of semi-circles every 1/24 degrees
- other half can be found by mirroring the first half since $I(\vec{\Omega})$ is a real function

Imaging



- brightness distribution $I(\vec{\Omega})$ by inverse Fourier transform
- reconstructed $\hat{I}(\vec{\Omega})$ needs to be corrected for single dish response function $H(\vec{\Omega})$

CLEAN



- undersampling of uv-plane, grating lobes within field of view
- decrease distance between antennas 9 and A during second half of rotation for 36 meter increment coverage
- four half rotations in 48 hours can increase coverage to 18 meter increments ⇒ complete uv coverage
- incomplete coverage of uv-plane ⇒ coherence function $\tilde{\Gamma}(\vec{r})$ are zero in some places ⇒ erroneous results
- apply CLEAN method for improving dirty radio maps