Today's course

Chapter 1.3, 1.4 & 2.2 OAF-2
Numerical Recipes Chapter 13.3

Topics:

- Fourier transformations
- Aliasing & Nyquist frequency
- (Optimal) Filtering
- measuring moments of a stochastic process

Recap lecture 1

- detection of astronomical signals (s.p.) plus noise is convoluted by instrument transfer function and data sampling
- statistical moments characterise the signal (plus noise)
- Noise can be due to the detector, background, and/or intrinsic to the signal
- Assume WSS s.p. (mean does not depend on time, or much slower than measuring process, auto-correlation depends on offset only)
- Convolutions and cross-correlations

Continuous Fourier Transformations

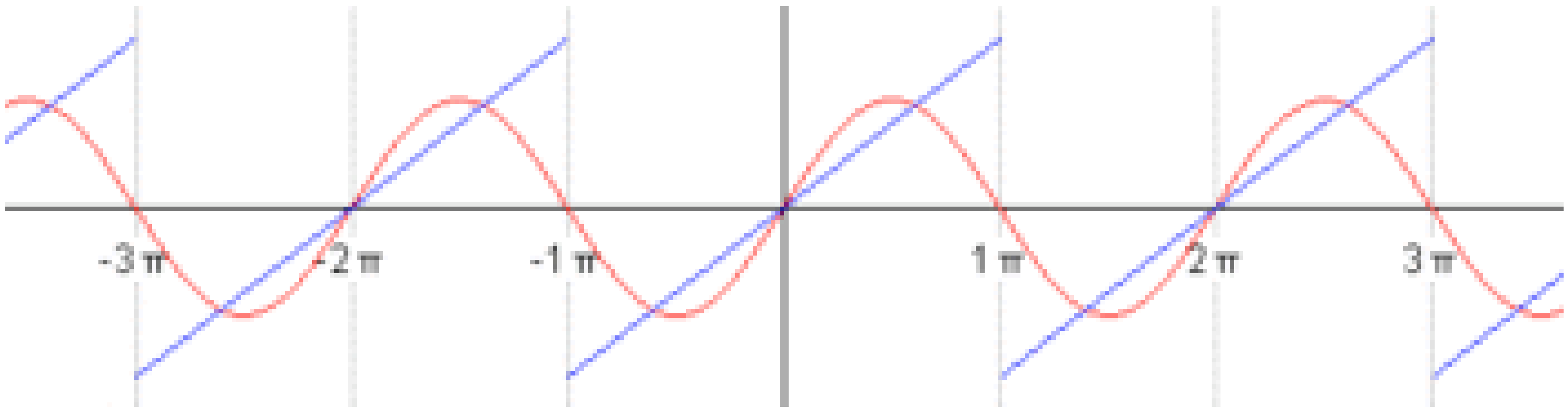


Figure from Wikipedia

$$F(t) \Leftrightarrow f(x)$$

$$F(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x t} dx$$

$$f(x) = \int_{-\infty}^{\infty} F(t) e^{2\pi i x t} dt$$

$$Euler's\ relation: \quad e^{ix} = cos x + i sin x$$

Used in restoration and/or spectral analysis of the signal

Convolution using FTs in practice

Convolution

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x) g(x_1 - x) dx$$

Convolution always broaden the input function

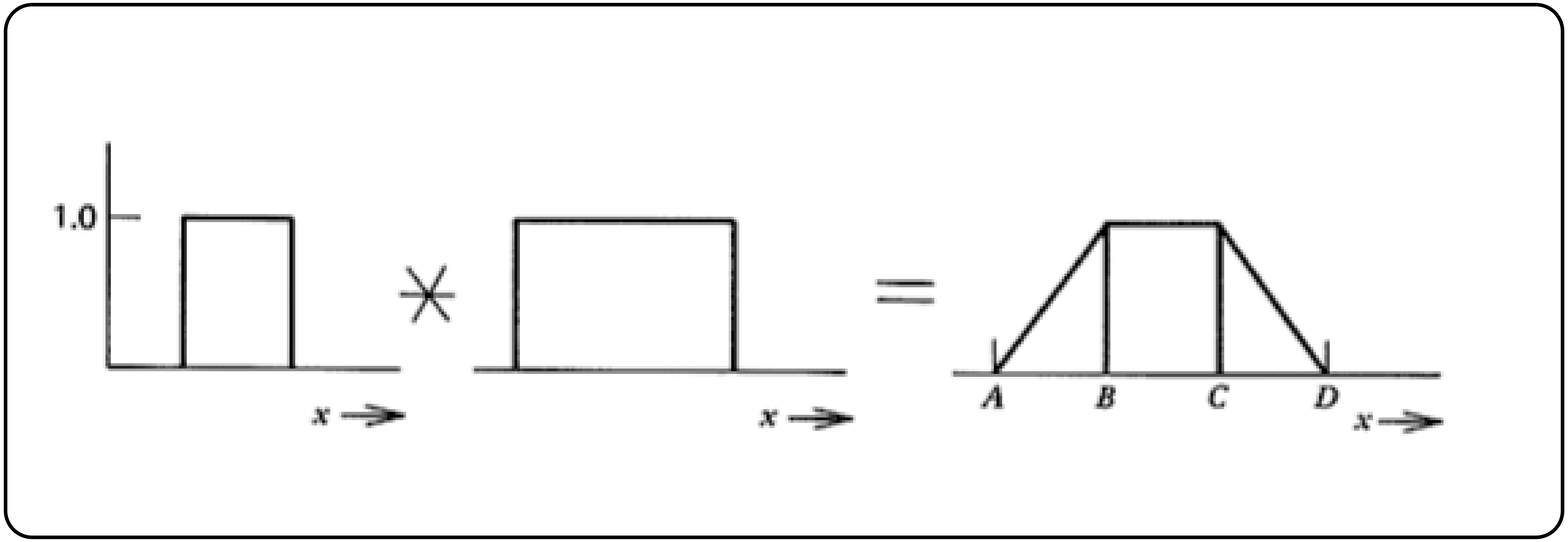


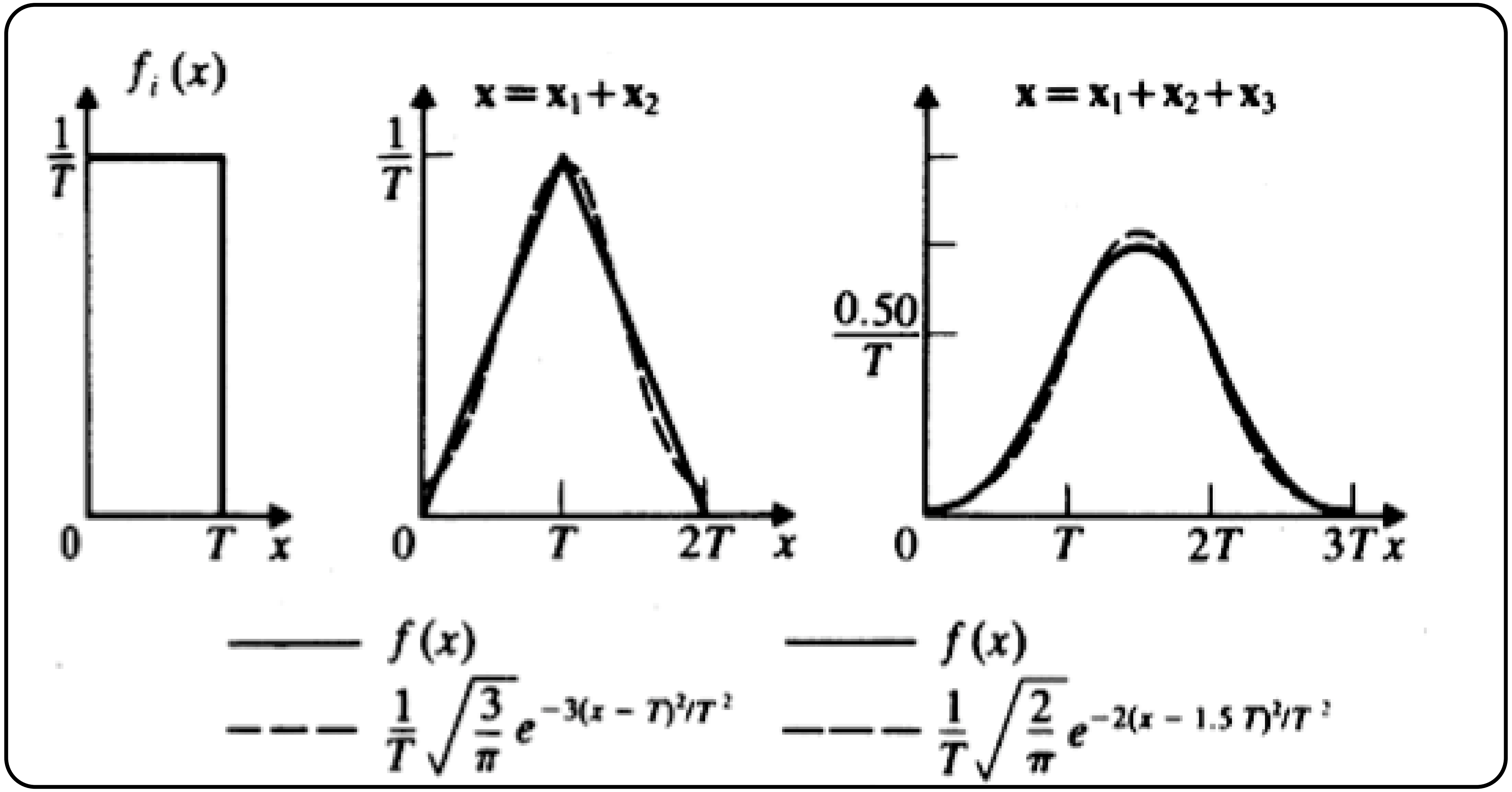
Figure from Gray

Central Limit Theorem

many convolutions → smoothing until Gaussian PDF

$$p_X(x) = p_{X_1}(x) * p_{X_2}(x) * p_{X_3}(x) * \cdots p_{X_n}(x)$$

$$\lim_{n\to\infty} p_X(x) = \frac{1}{\sqrt{(2\pi)}\sigma} exp - \frac{(x-\eta)^2}{2\sigma^2}$$



many physical processes/measurements yield a Gaussian probability density function

Convolution using Fourier transformations

Convolution theorem

$$M(\lambda) = S(\lambda) * R(\lambda)$$

$$F(M(\lambda)) = F(S(\lambda)) \cdot F(R(\lambda))$$

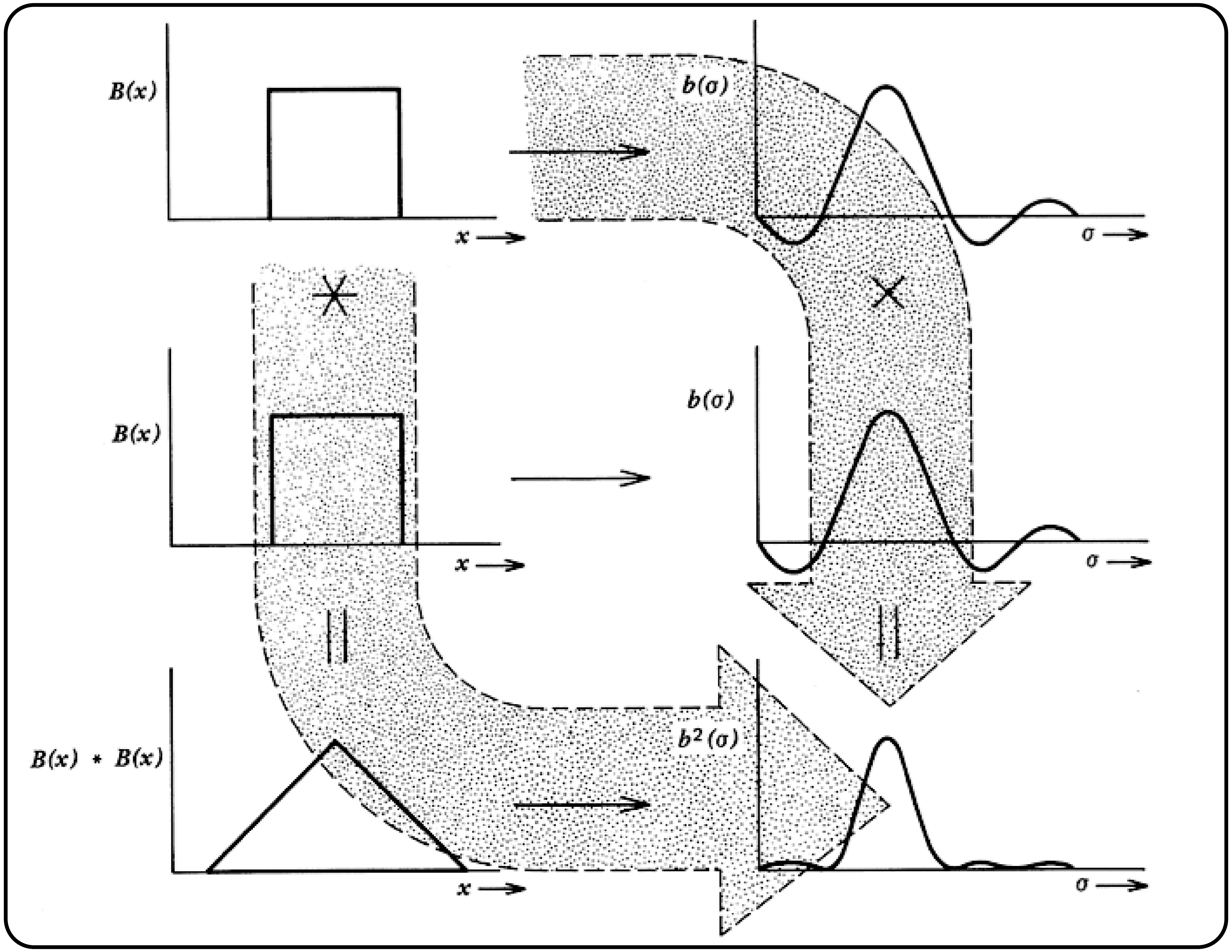


figure from Gray

Reconstruction of the input=source spectrum

$$M(\lambda) = S(\lambda) * R(\lambda)$$

Convolution theorem

$$F(M(\lambda)) = F(S(\lambda)) \cdot F(R(\lambda))$$

$$F(M(\lambda)) \equiv M(s)\,(etc)$$

$$M(s) = S(s) \cdot R(s)$$

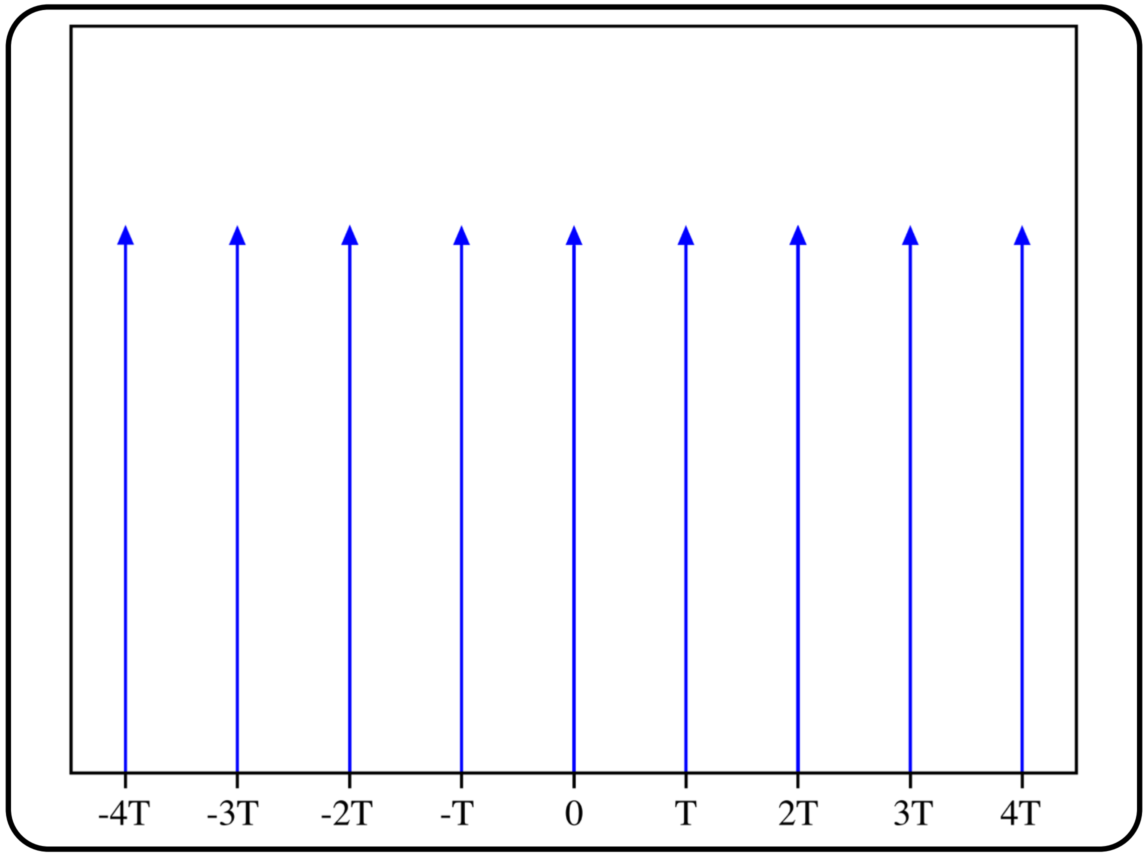
$$S(s) = \frac{M(s)}{R(s)}$$

$$S(\lambda) = F^{-1}\left(\frac{M(s)}{R(s)}\right)$$

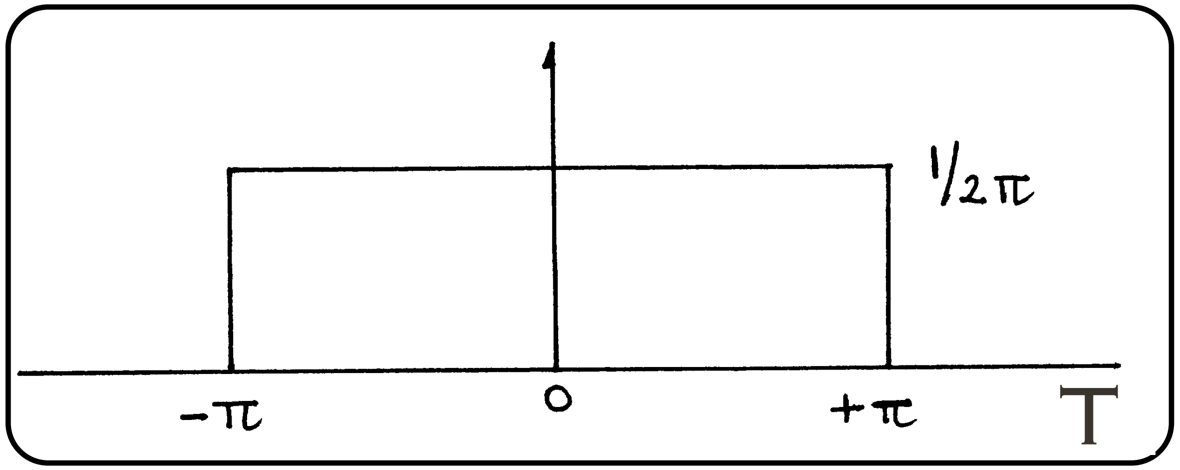
Some special functions:

Shah's function/Dirac comb

$$III(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



Box/window function

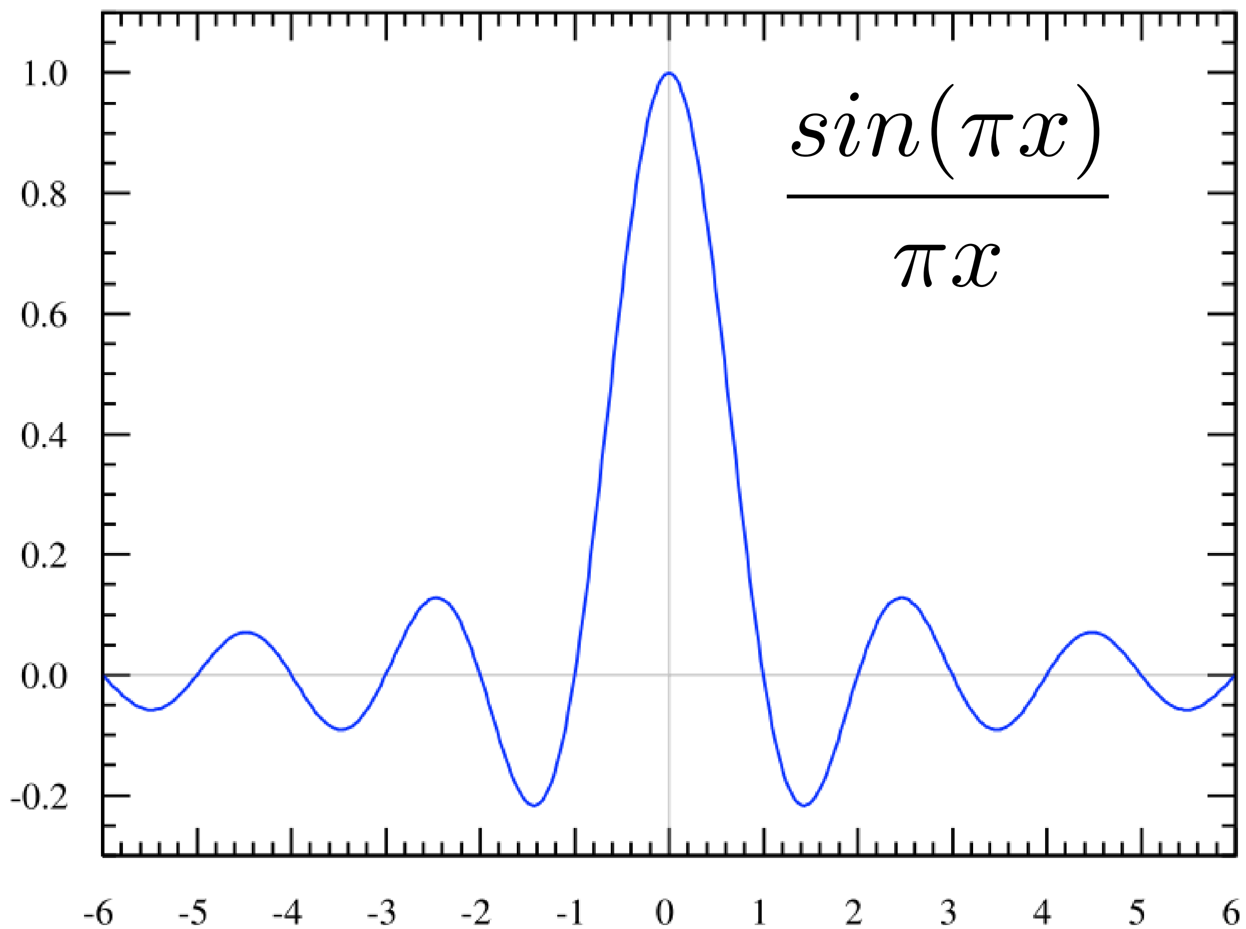


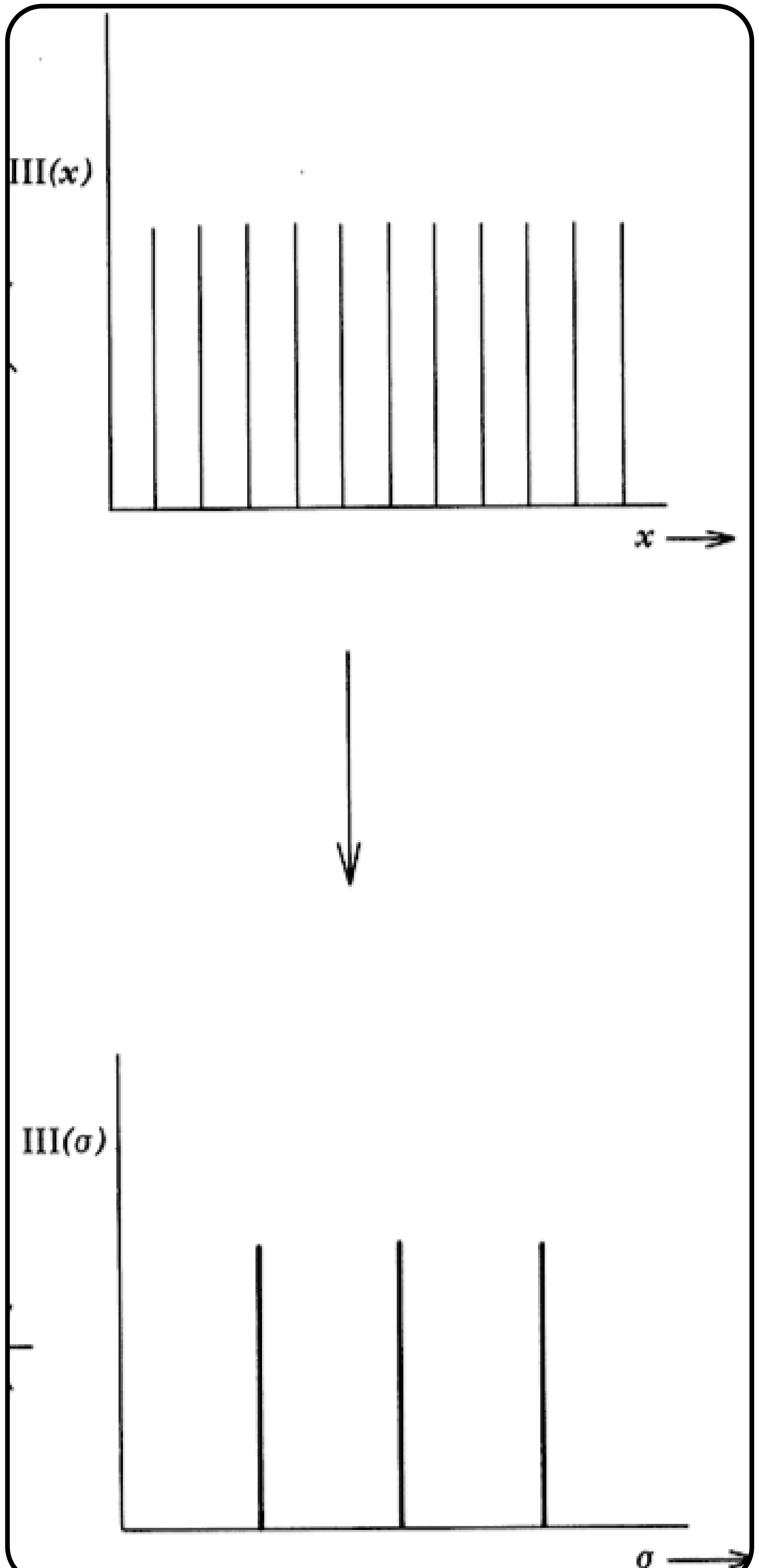
$$B(t) = 0 \ for \ -\frac{W}{2} > t > \frac{W}{2}$$

$$B(t) = 1 \ for \ -\frac{W}{2} < t < \frac{W}{2}$$

Fourier transformations of these special functions

Sinc function





$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \Leftrightarrow \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{T})$$

A sharp narrow signal needs more/ higher frequencies to be described in the Fourier Transform than broad shallow signal

cf. the number of sin+cos necessary to describe the signal

Optical spectra: bandwidth set by the width of the spectral lines

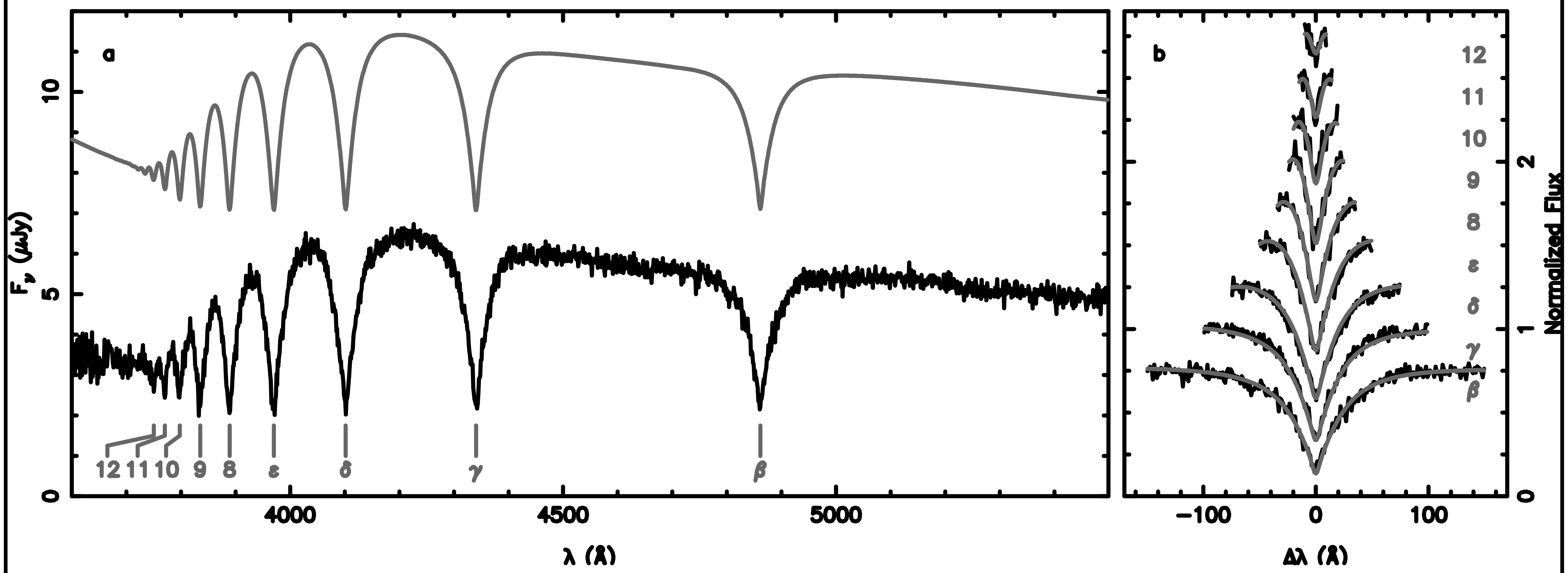
Sampling theorem

Sampling: no loss of information if the input process has no frequencies > $\frac{1}{2\Delta t_{crit}}$

Continuous signal h(t) fully described by the samples

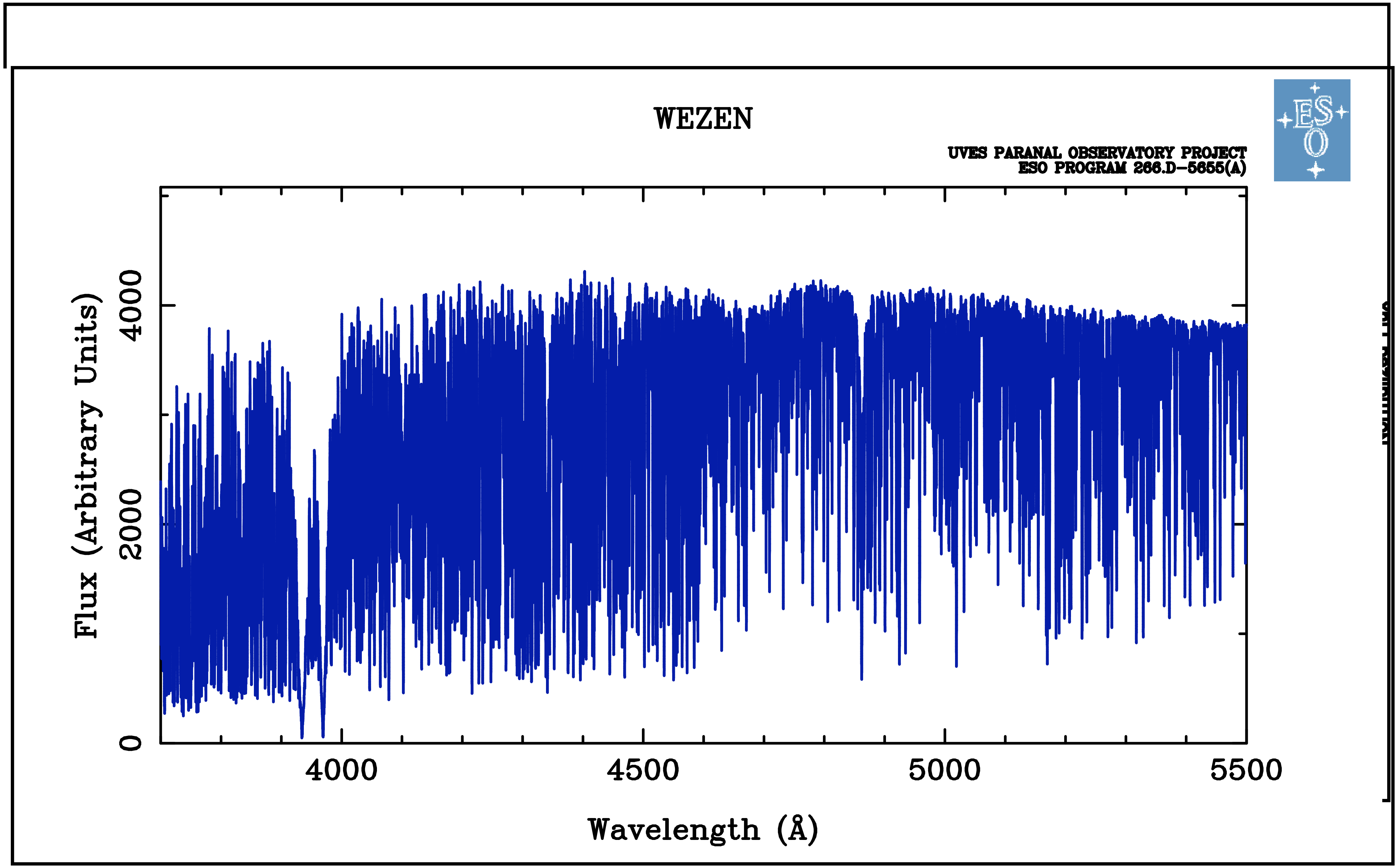
Optical spectra: bandwidth set by the width of the spectral lines

Low-mass White Dwarf spectrum

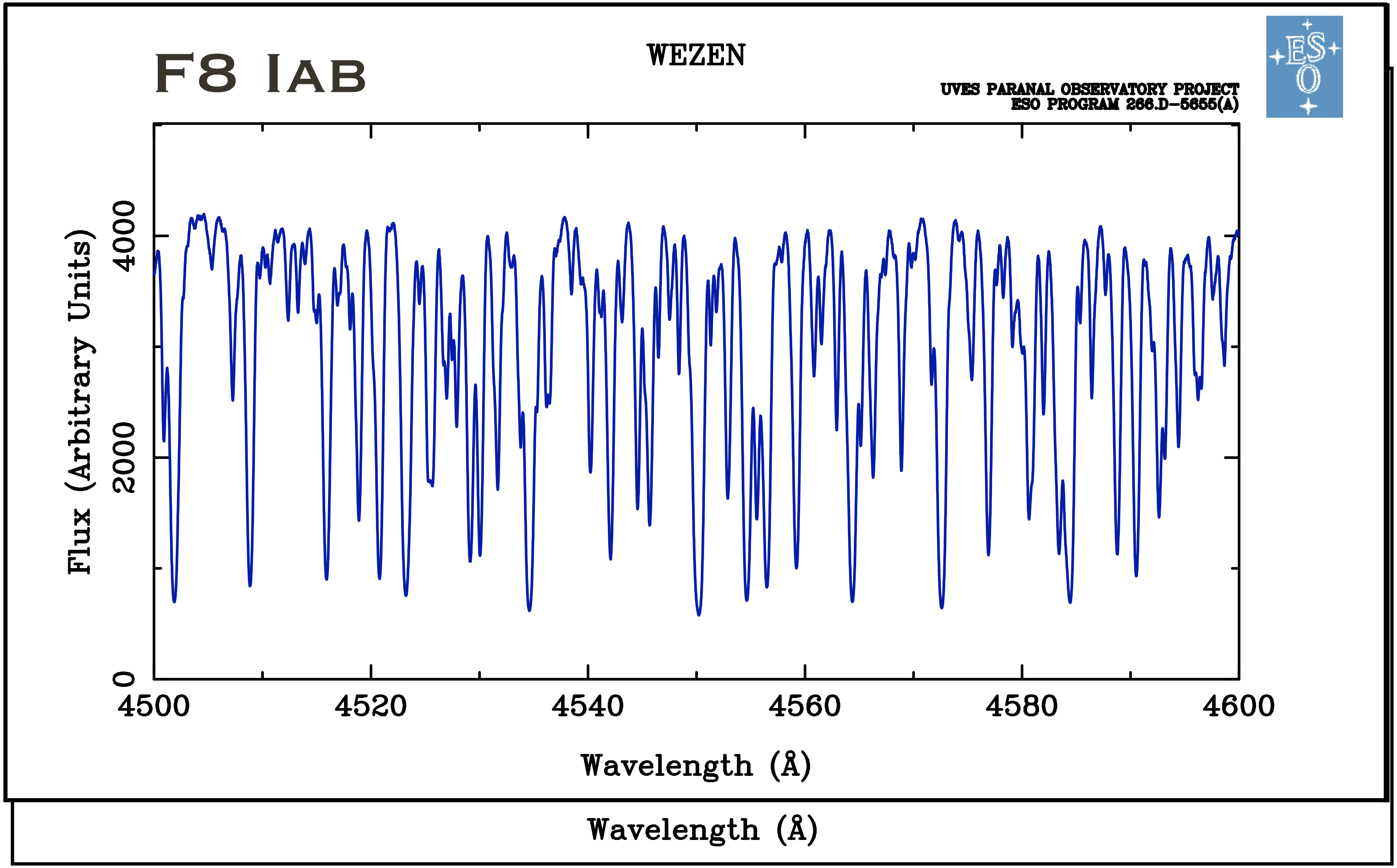


Bassa et al. 2006

Optical spectra: bandwidth set by the width of the spectral lines



Optical spectra: bandwidth set by the width of the spectral lines



Another math tool
Power Spectral Density

(∝ amplitude of individual sinusoids)

(will return in more depth in Chapter 6)

Continuous FT: $$F(f) = \int_{-\infty}^{\infty} f(t)\ e^{-2\pi i f t} dt$$

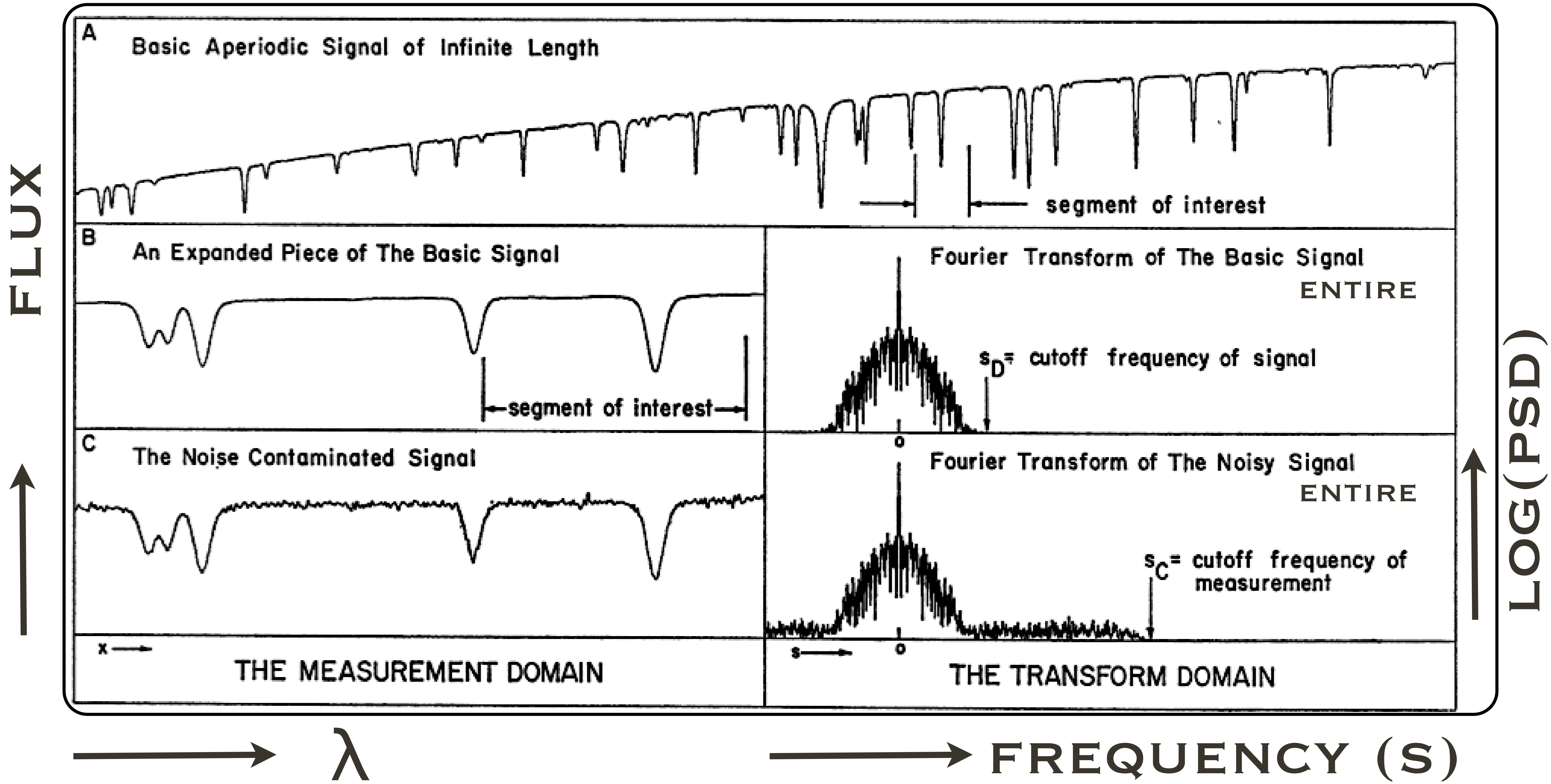
Continuous PSD: $$P(f) = \tilde{F(f)} \tilde{F(f)}^*$$

for WSS signals: $$P(f) = \int_{-\infty}^{\infty} R(\tau) e^{-2\pi i f \tau} d\tau$$

hence:

$$\widetilde{F(f)} \widetilde{F(f)} = |F(f)|^2 = \int_{-\infty}^{\infty} R(\tau) e^{-2\pi i f \tau} d\tau$$

Important concept in PSD=Nyquist theorem



Windowing & noise, Brault & White 1971, A&A

Data sampling

Data is discrete not continuous

Time domain multiply S.P. with shah function

$$m_{samp,n} = m_s(x) = m(x)\frac{1}{\tau}\text{III}(\frac{x}{\tau}) = \sum_n m(n\tau)\delta(x - n\tau)$$

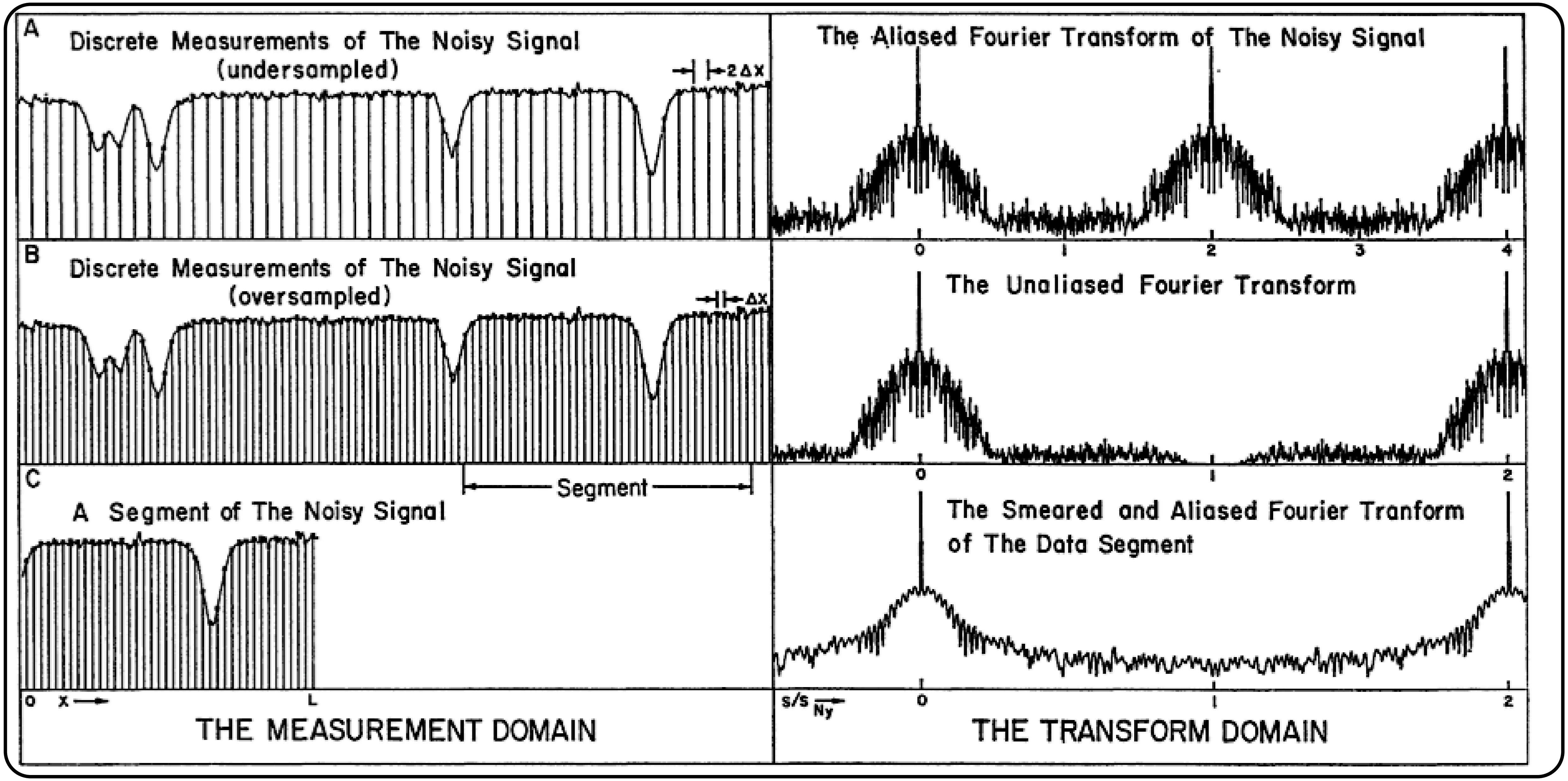
Discrete FT: $$M_{samp,k} = \sum_{n=0}^{N-1} m_{samp,n}\ e^{2\pi ink/N}$$

Discrete PSD: $$P_j = \frac{2}{a_0}|a_j|^2$$ power ∝ amplitude squared:

$$a_0 = M_{samp,k=0} = \sum_{n=0}^{N-1} m_{samp,n} \equiv N_0$$

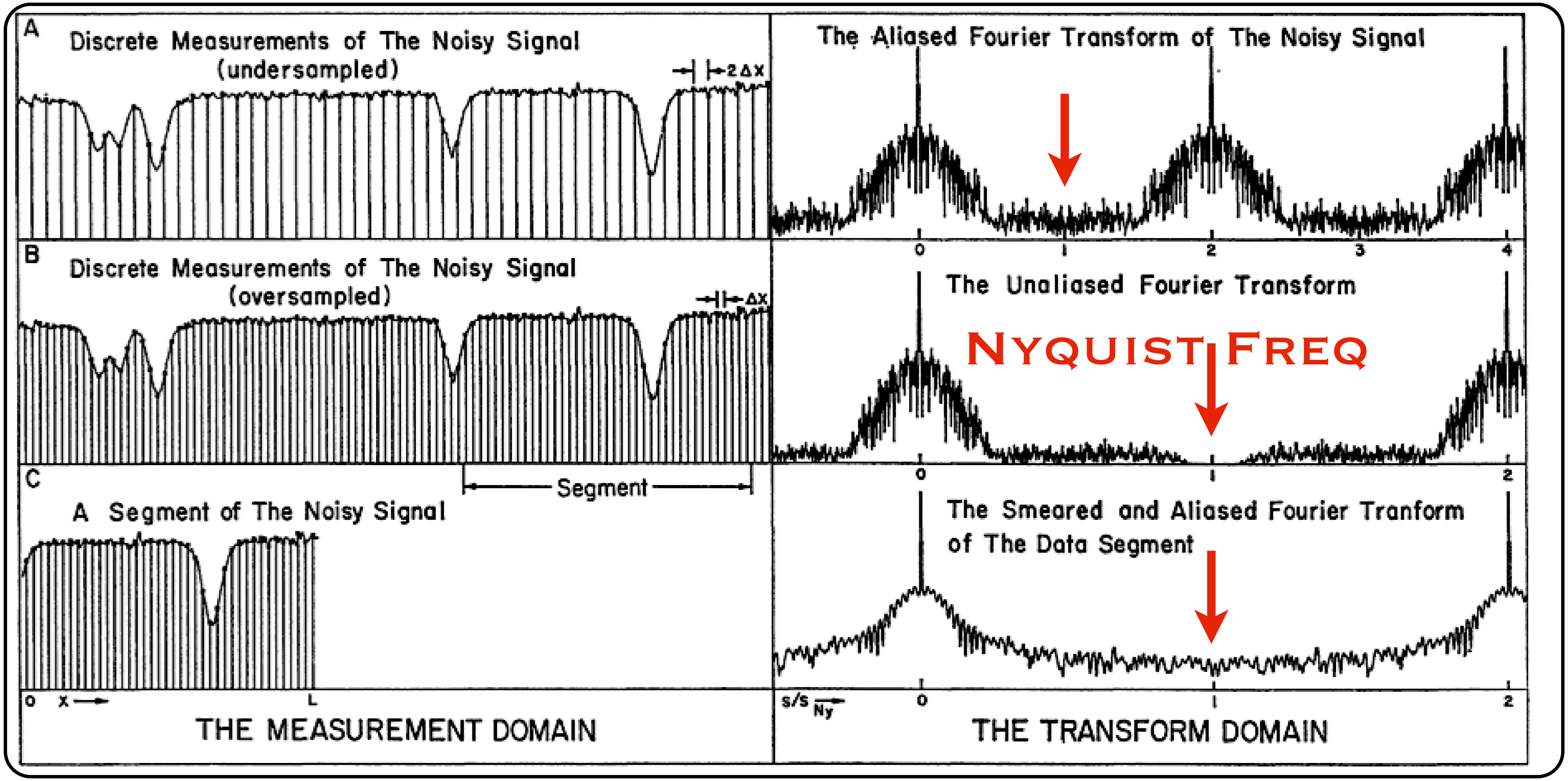
$$a_k = M_{samp,k} = \sum_{n=0}^{N-1} m_{samp,n}\ e^{2\pi ink/N}$$

Nyquist theorem: cont'd



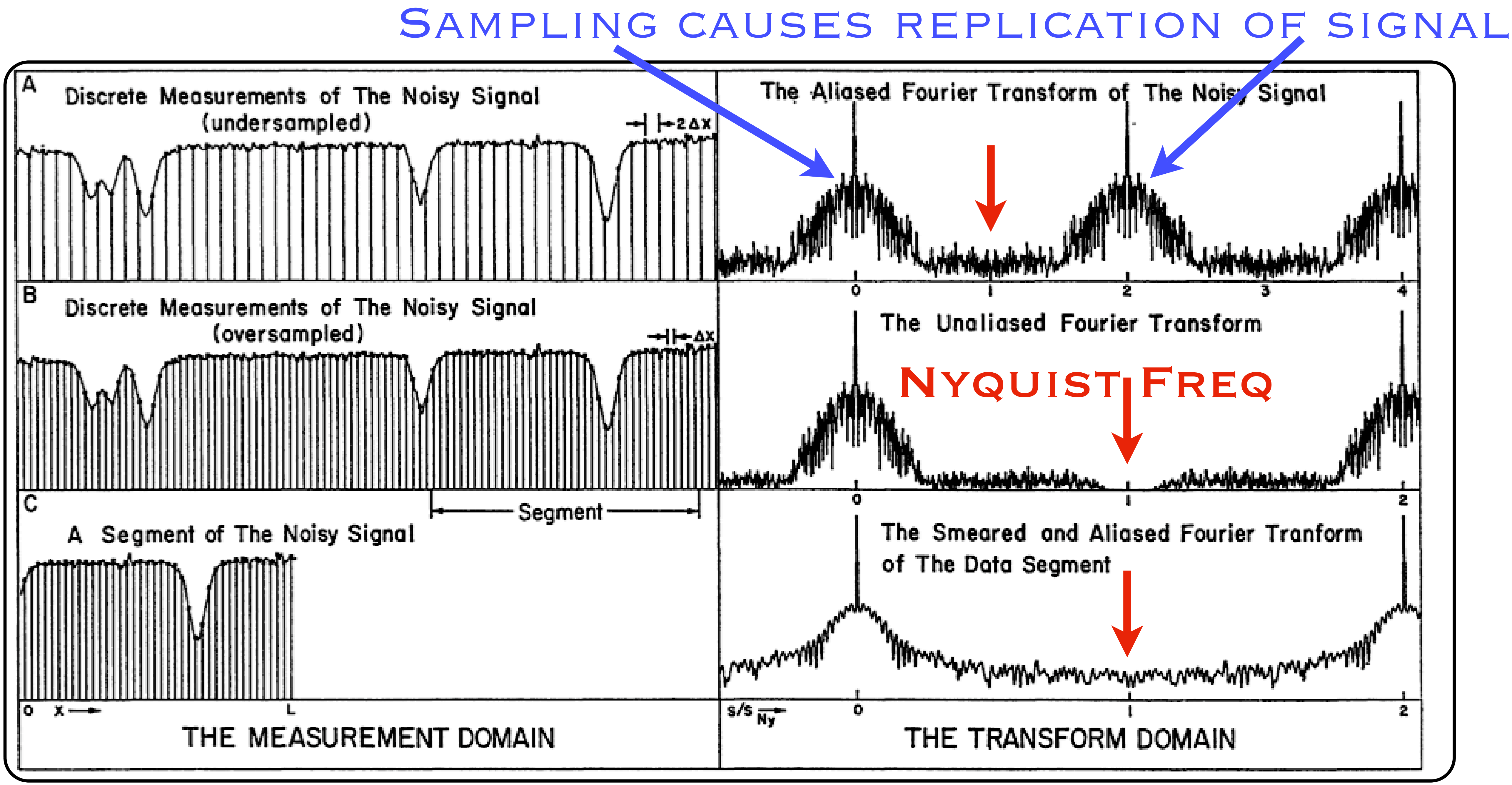
Sampling; Brault & White 1971, A&A, 13, 169 (in list of presentation papers!)

Nyquist theorem: cont'd



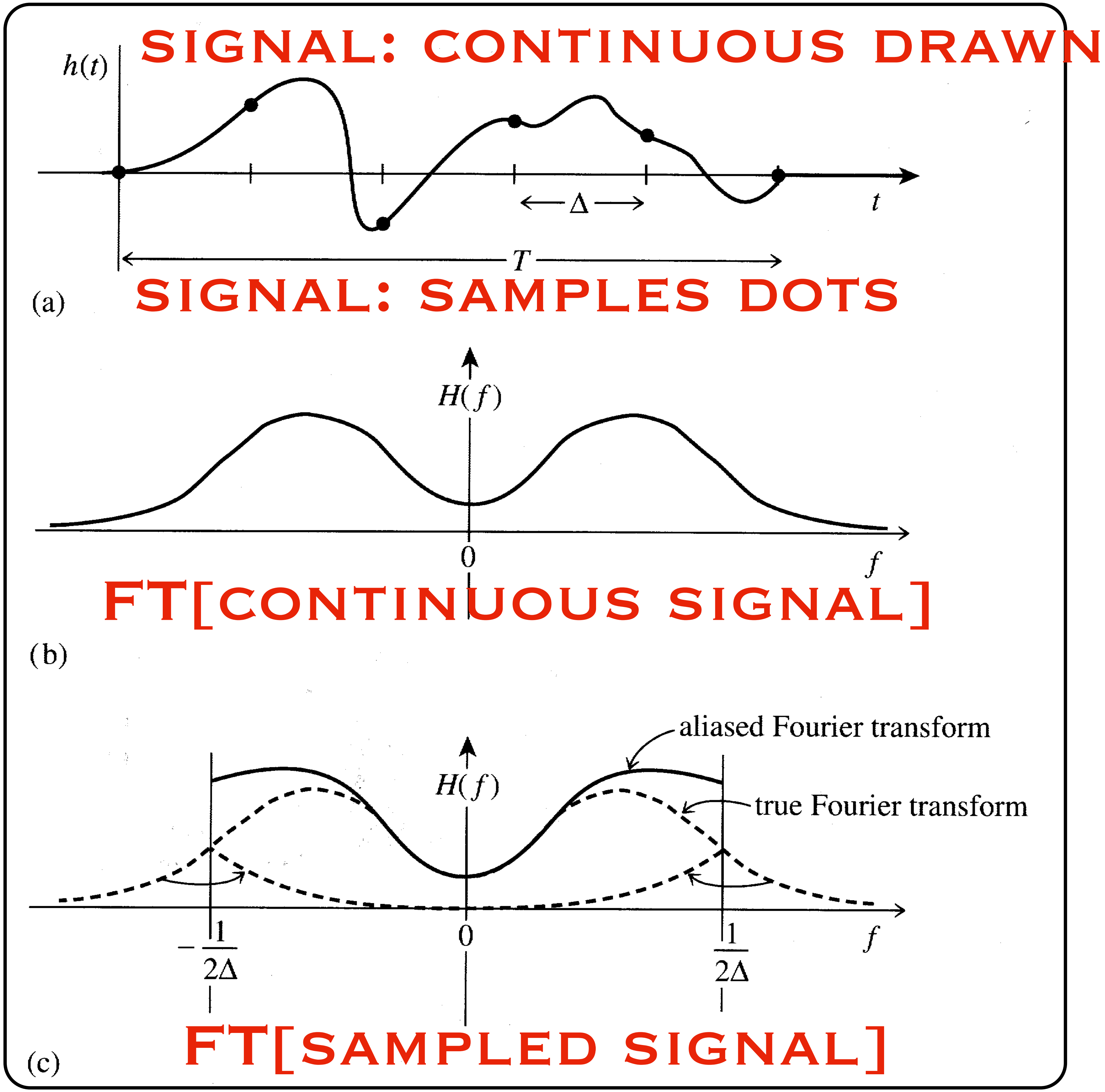
Sampling; Brault & White 1971, A&A, 13, 169 (in list of presentation papers!)

Nyquist theorem: cont'd

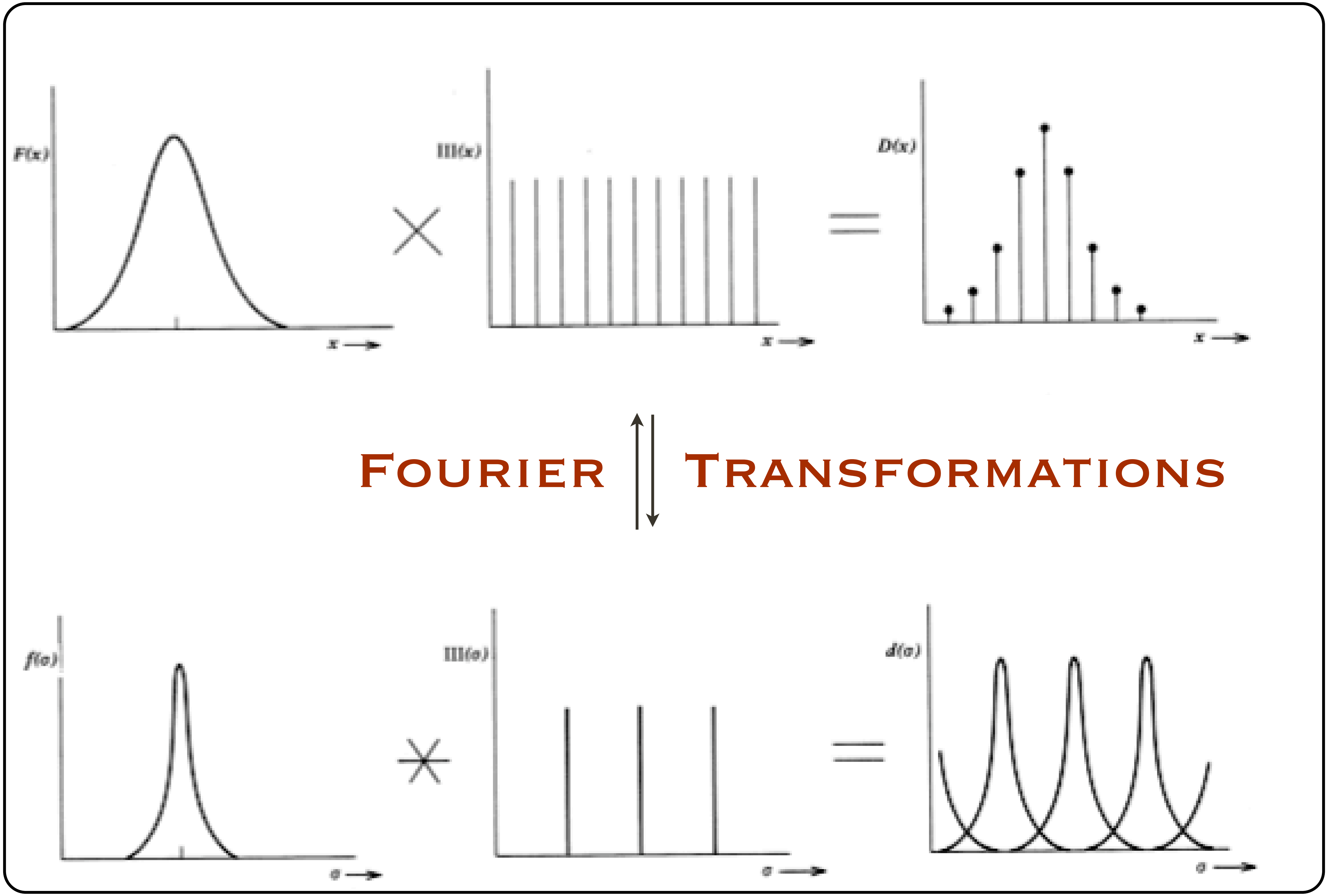


Sampling; Brault & White 1971, A&A, 13, 169 (in list of presentation papers!)

Aliasing


Signal: continuous drawn
h(t)
Δ
t
T
(a)
Signal: samples dots
H(f)
0
f
FT[continuous signal]
(b)
aliased Fourier transform
true Fourier transform
H(f)
−1/2Δ
0
1/2Δ
f
(c)
FT[sampled signal]

Aliasing: cont'd



Convolution with shah function in freq space: replication

Sampling: high frequencies are filtered out

Window: low frequencies are filtered out

→ leads to band limited data

Filtering

→ Frequency Filtering $Y(f) = X(f)H(f)$

$$y(t) = \int_{-\infty}^{\infty} x(t-\theta)h(\theta)d\theta$$

$$y(t) = x(t) * h(t)$$

Filtering of process x with filter h

Sampling: high frequencies are filtered out

Window: low frequencies are filtered out

→

leads to band limited data

Filtering

→ Frequency Filtering $Y(f) = X(f)H(f)$

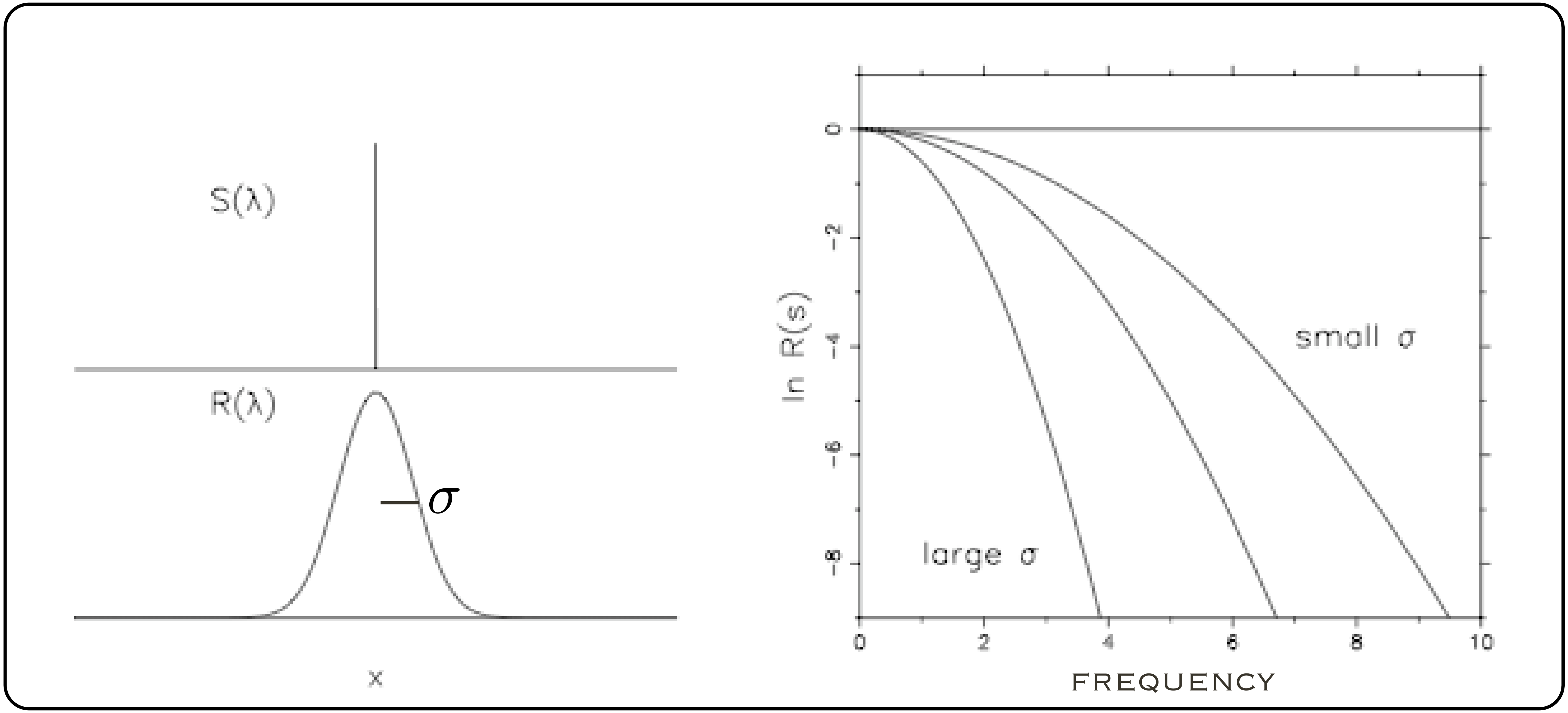
$$y(t) = \int_{-\infty}^{\infty} x(t-\theta)h(\theta)d\theta$$

$$y(t) = x(t) * h(t)$$

Filtering of process x with filter h

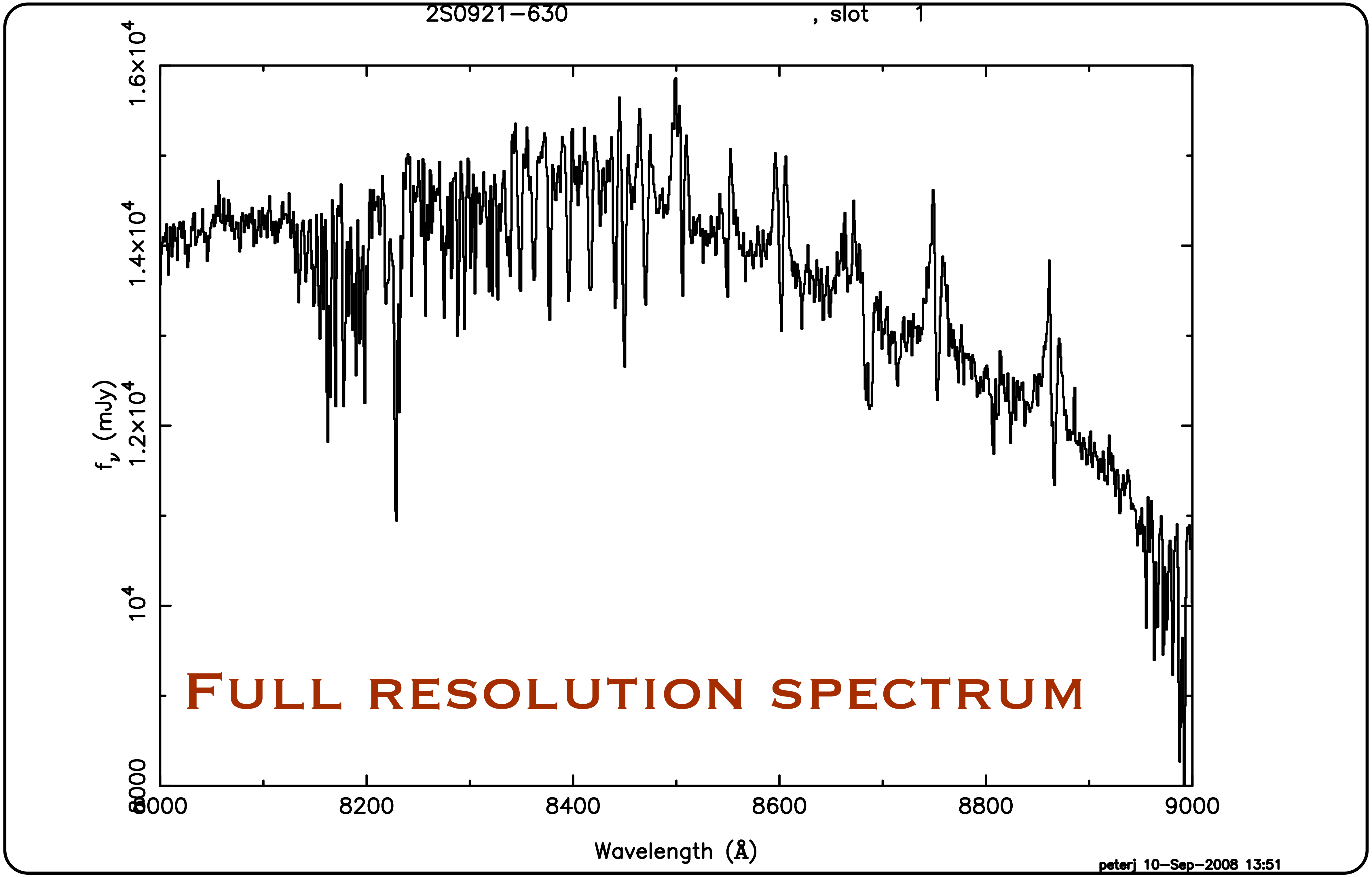
Remember inserting filter in optical imaging

Gaussian response function

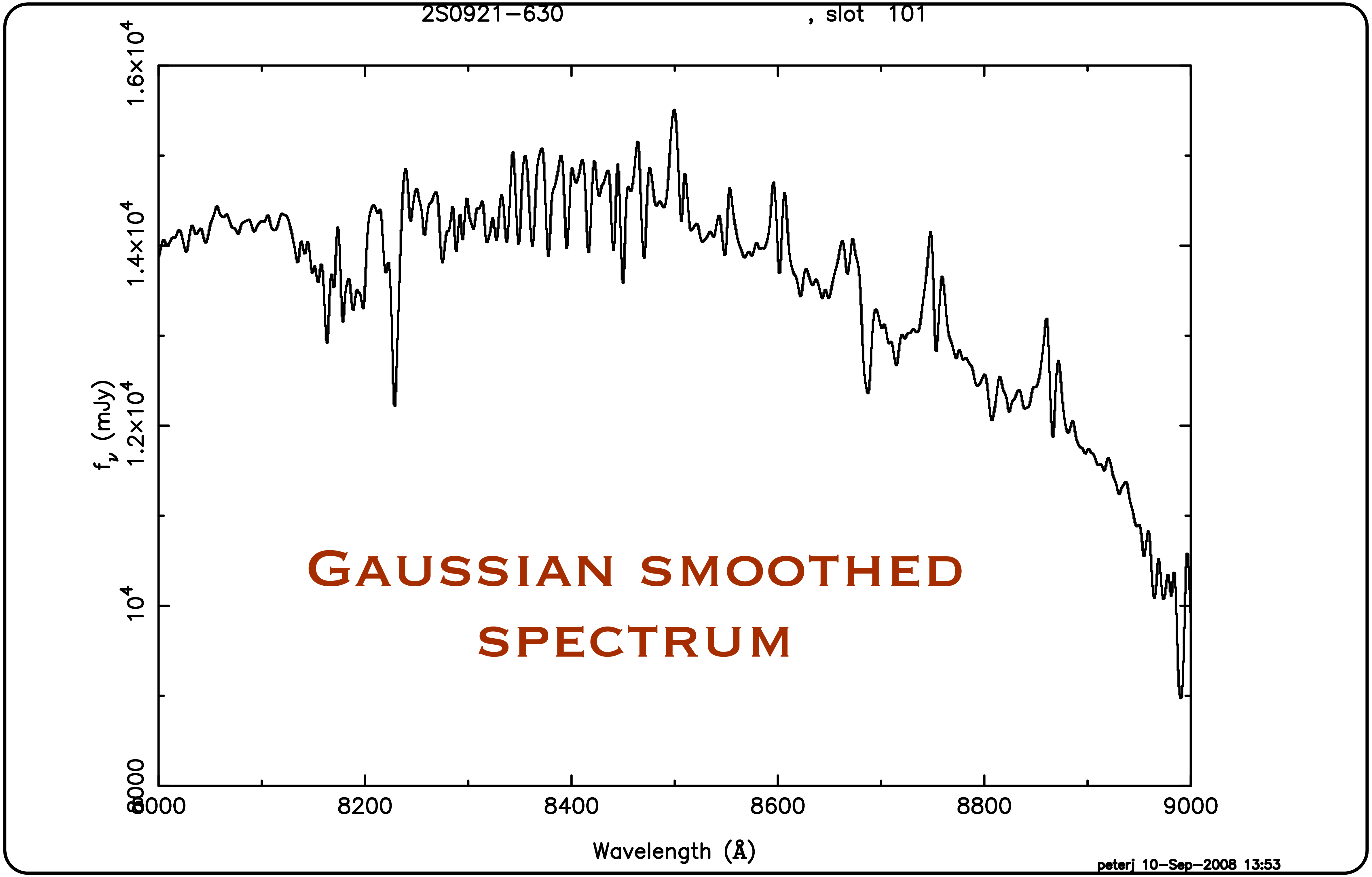


$$R(\lambda)\frac{1}{\sqrt{(2\pi)}\sigma}exp-(\frac{\lambda^2}{2\sigma^2})$$

Gaussian response function



Gaussian response function



Sampling: high frequencies are filtered out

Window: low frequencies are filtered out

→ leads to band limited data

Filtering

→ Time Filtering

measure a process x(t) over interval T assumed zero outside T

$$\equiv y(t) = \Pi(\frac{t}{T})x(t)$$

$$Y(f) = X(f) * Tsinc(Tf)$$

all information about frequencies <1/T is lost!