Exercises OAF2

Stochastic processes

Exercise 1

a) Consider a stochastic process $X(t) = S(t) + N(t)$.
Component $S(t)$ represents the periodic light curve of a stellar object:

$$S(t) = S_0(t)\left[1 + m\sin\left(\frac{2\pi t}{P} + \phi\right)\right], \tag{1}$$

with P the period and m the so-called modulation index. The stochastic signal $S(t)$ is wide sense stationary with $\overline{S(t)} = \overline{S_0(t)} = S_{av}$ and variance $\overline{(S_0 - S_{av})^2} = \overline{\triangle S_0^2} = \sigma_{S_0}^2$.
Component $N(t)$ represents the sky noise (wide sense stationary), with $\overline{N(t)} = N_{av}$ and $\overline{(N - N_{av})^2} = \overline{\triangle N^2} = \sigma_N^2$.
The periodic signal is submerged in the sky noise and the signal noise: $\sigma_N^2 = 3\sigma_{S_0}^2$ and the modulation index $m = 0.2$.
The periodic component in the stochastic process $X(t)$ can be extracted by using the autocorrelation function of $X(t)$.

Prove that the autocovariance of $X(t)$ is given by:

$$C_X(\tau) = C_N(\tau) + C_{S_0}(\tau)\left[1 + \frac{1}{2}m^2\cos\left(\frac{2\pi\tau}{P}\right)\right] + \frac{1}{2}m^2 S_{av}^2\cos\left(\frac{2\pi\tau}{P}\right) \tag{2}$$

b) Plot the autocovariance function $C_X(\tau)$ for positive values of τ, assuming that $C_N(\tau) \sim \exp(-\tau/\tau_N)$ with $\tau_N^{-1} = 2\pi/(3P)$, and $C_{S_0}(\tau) \sim \exp(-\tau/\tau_{S_0})$ with $\tau_{S_0}^{-1} = 2\pi/P$.

c) Can the periodic component of $X(t)$ be completely reconstructed by the measurement of the autocovariance $C_X(\tau)$. Explain your answer!

Exercise 2

a) If the expected value $\mathbf{E}\{X_T\}$ equals μ for the average number of photons in time period T, the probability p that a photon arrives in a subinterval of T can be equated from $p = \mu/m$ if m equals the number of subintervals within T. The probability that no photon arrives is $1-p$. The measurement can thus be considered as a series of m trials to find a photon, each having a probability of p of succeeding. The probability that in total k photons will be detected is therefore given by the binomial probability function ($k < m$):

$$p_B(k, m, p) = \left(\begin{array}{c} m \\ k \end{array} \right) p^k \, (1-p)^{m-k}. \tag{3}$$

However, if the subinterval is large, there is a finite probability that more than one photon will arrive within this interval, hence the limit should be taken for the number of trials m to go to infinity while maintaining $mp = \mu$ constant. In this limit $m \to \infty$, $p \to 0$, the binomial distribution changes to the Poisson distribution:

$$p_P(k, \mu) = \frac{\mu^k}{k!} \, e^{-\mu}. \tag{4}$$

Derive this distribution transformation and show that the Poisson distribution is normalized.

b) Show that for a *Poisson distributed* stochastic process the autocorrelation of the random variable $X_{\Delta T}$ (number of particles or photons per interval of time ΔT) is given by:

$$R_{X_{\Delta T}} = \mu^2 + \mu \quad \text{for } \tau = 0 \tag{5}$$

$$R_{X_{\Delta T}}(\tau) = \mu^2 + \mu\delta(\tau) \quad \text{in the general case} \tag{6}$$