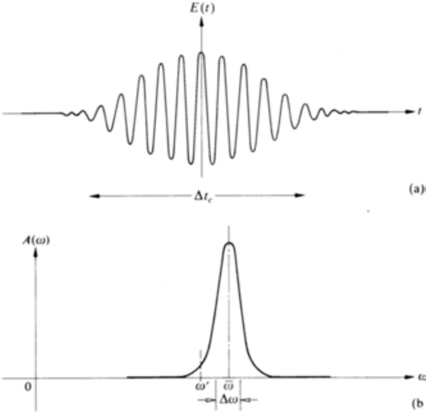
Lecture 10: Aperture Synthesis Imaging

Outline

1. Spatial and Temporal Coherence
2. Etendue of Coherence
3. Aperture Synthesis
4. Earth-Rotation Aperture Synthesis

Temporal and Spatial Coherence

Temporal Coherence



Gaussian shaped line profile of quasi-monochromatic source and shape of associated wave packet

- temporal coherence characterised by coherence time τ_c
- τ_c due to finite bandwidth of source
- quasi-monochromatic source

$$\tau_c \approx \frac{1}{\triangle\nu}$$

$\triangle\nu$: frequency band width

Temporal Coherence (continued)

- Wiener-Khinchin theorem relates power spectrum $S(\nu)$ and autocorrelation $R(\tau)$:

$$S(\nu) = \int_{-\infty}^{+\infty} R(\tau) e^{-2\pi i \nu \tau} d\tau$$

$$R(\tau) = \int_{-\infty}^{+\infty} S(\nu) e^{2\pi i \nu \tau} d\nu$$

- example: Gaussian-shaped spectral profile

$$S(\nu) \sim e^{-\left(\frac{\nu}{\triangle\nu}\right)^2} \Longleftrightarrow R(\tau) \sim e^{-\left(\frac{\tau}{\tau_c}\right)^2}$$

- corresponding wave packet has Gaussian autocorrelation function with characteristic width τ_c
- (autocorrelation $R(\tau)$ equals the autocovariance $C(\tau)$)

Temporal Coherence (continued)

- useful relations:
  - first order system shows exponential autocorrelation function $R(\tau)$
  - Gaussian spectral frequency domain shows amplitude-modulated wave train with Gaussian envelope in time domain
  - Lorentz line profile in frequency domain shows exponentially damped oscillator profile in time domain
- infrared and shorter wavelengths, disperse incoming radiation with wavelength-dispersive device
- spectroscopy at radio wavelengths employs indirect method
- incoming wave signal is fed into *correlator* that produces temporal coherence function $R(\tau)$
- subsequent Fourier transform yields spectral distribution $S(\nu)$

Coherence Length

- *coherence length*

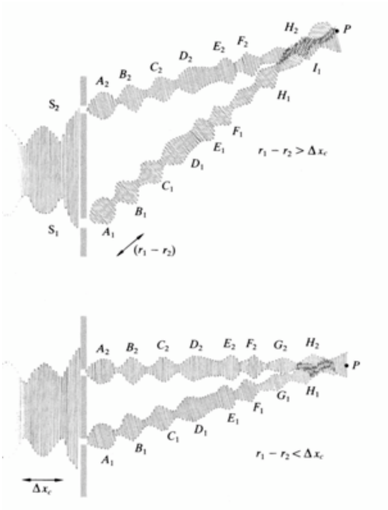
$$l_c = c\tau_c$$

- coherence length in wavelength domain

$$l_c = \frac{\lambda^2}{\triangle\lambda}$$

- quasi-monochromatic wave propagating along a line
  - two positions P_1 and P_2 on this line of propagation at distance R_{12}
  - if $R_{12} \ll l_c$, there will be strong correlation between the EM-fields at P_1 and P_2, interference effects will be possible
  - if $R_{12} \gg l_c$, no interference effects are possible
- also called *longitudinal correlation* or *longitudinal spatial coherence*

Coherence Length (continued)



influence of coherence length on interference pattern of two diffracted coherent thermal sources S_1, S_2

- waves in Young's interference experiment
- diffracted beams from coherent sources S_1 and S_2 cause interference pattern
- large path differences $\Rightarrow$ interference contrast reduced

Spatial Coherence

- spatial coherence relates to spatial extent of source
- for $\tau \ll \tau_c$

$$\tilde{\gamma}_{12}(\tau) = \tilde{\gamma}_{12}(0) e^{2\pi i \nu_0 \tau}$$

  - $|\tilde{\gamma}_{12}(\tau)| = |\tilde{\gamma}_{12}(0)|$
  - fixed phase difference $\alpha_{12}(\tau) = 2\pi\nu_0\tau$
  - ν_0: average frequency of wave
- frequency bandwidth of radiation source suffiently narrow: comparison between two points with respect to spatial coherence occurs at times differing by $\triangle t \ll \tau_c$

Etendue of Coherence

- circular source of uniform intensity with angular diameter θ_s
- source brightness distribution described as circular two-dimensional window function

$$I(\vec{\Omega}) = \Pi\left(\frac{\theta}{\theta_s}\right)$$

- complex degree of coherence in observation plane Σ at two positions: position 1 at origin, position 2 at distance ρ from origin
- applying van Cittert-Zernike theorem

$$\Pi\left(\frac{\theta}{\theta_s}\right) \Leftrightarrow \tilde{\Gamma}(\rho/\lambda) = \frac{(\theta_s/2)J_1(\pi\theta_s\rho/\lambda)}{\rho/\lambda}$$

- J_1: Bessel function of first kind

Etendue of Coherence (continued)

- normalisation to source brightness $(\pi\theta_s^2)/4$

$$\tilde{\gamma}(\rho) = \frac{2J_1(\pi\theta_s\rho/\lambda)}{\pi\theta_s\rho/\lambda}$$

- modulus of complex degree of coherence

$$|\tilde{\gamma}(\rho)| = \left|\frac{2J_1(u)}{u}\right|$$

  with $u = \pi\theta_s\rho/\lambda$
- defines extent of coherence in observation plane Σ
- for $u = 2$, $|\tilde{\gamma}(\rho)| = J_1(2) = 0.577$
- coherence remains significant for $u \leq 2$, or

$$\rho \leq 2\lambda/(\pi\theta_s)$$

Etendue of Coherence (continued)

- area S in Σ over which coherence remains significant

$$\pi\rho^2 = 4\lambda^2/(\pi\theta_s^2)$$

- $\pi\theta_s^2/4$ equals solid angle Ω_{source} of source
- significant coherence if

$$\epsilon = S\Omega_{\text{source}} \leq \lambda^2$$

- condition $\epsilon = S\Omega_{\text{source}} = \lambda^2$ is called the *Etendue of Coherence*
- needs to be fulfilled if coherent detection is required

Etendue of Coherence: Examples

- red giant, radius $r_0 = 1.5 \text{ x}10^{11}$ meter at 10 parsec distance, $\theta_s = 10^{-6}$ radians
- at $\lambda = 0.5\mu$m, coherence radius ρ, on earth, on screen normal to incident beam is $\rho = 2\lambda/(\pi\theta_s) = 32$ cm
- at $\lambda = 25\mu$m, radius ρ is increased fifty fold to ≈ 15 m
- in radio domain at$\lambda = 6$ cm, $\rho \approx 35$ km

Good Coherence

- good coherence means visibility of 0.88 or better
- uniform circular source: occurs for $u = 1$, that is when $\rho = 0.32\lambda/\theta$
- narrow-bandwidth uniform radiation source at distance R

$$\rho = 0.32(\lambda R)/D$$

- example: red filter over 1-mm-diameter, disk-shaped flashlight at 20 m away: $\rho = 3.8$ mm
- set of apertures spaced at about 4 mm or less should produce clear fringes
- we always assume that comparison between two points occurs at times differing by a $\triangle t \ll \tau_c$
- if necessary, additional frequency filtering required to reduce spectral bandwidth of source signal

Aperture Synthesis

Overview

- positions 1, 2 in observation plane Σ not pointlike, but finite aperture with diameter D
- single aperture has diffraction-sized beam of λ/D
- Van Cittert-Zernike relation needs to be "weighted" with telescope element (single dish) transfer function $H(\vec{\Omega})$
- circular dish antenna: $H(\vec{\Omega})$ is *Airy brightness function*
- The Van Cittert-Zernike relations now become:

$$\tilde{\Gamma}'(\vec{r}) = \int\int_{\text{source}} I(\vec{\Omega})H(\vec{\Omega})e^{\frac{2\pi i\vec{\Omega}.\vec{r}}{\lambda}}d\vec{\Omega}$$

$$I(\vec{\Omega})H(\vec{\Omega}) = \lambda^{-2}\int\int_{\Sigma\text{-plane}} \tilde{\Gamma}'(\vec{r})e^{-\frac{2\pi i\vec{\Omega}.\vec{r}}{\lambda}}d\vec{r}$$

- field of view scales with λ/D, e.g. if λ decreases, the synthesis resolution improves but the field of view reduces proportionally!

Overview (continued)

- aperture synthesis: incoming beams from antenna dish 1 and antenna dish 2 are fed into a *correlator (multiplier)* producing as output $\tilde{E}_1(t)\tilde{E}_2^*(t)$
- output subsequently fed into *integrator/averager* producing

$$\mathbf{E}\left\{\tilde{E}_1(t)\tilde{E}_2^*(t)\right\} = \tilde{\Gamma}'(\vec{r})$$

- applying Fourier transform and correcting for beam profile of single dish $H(\vec{\Omega})$, source brightness distribution $I(\vec{\Omega})$ can be reconstructed
- Indirect imaging with aperture synthesis system is limited to measuring image details within the *single pixel* defined by the beam profile of an individual telescope element, i.e. a single dish!

Pupil Function

- pupil function of linear array comprising N circular apertures with diameter d, aligned along baseline unit vector $\vec{b}$, equally spaced at distance $|\vec{s}| = \vec{b} \cdot \vec{s}$

$$P(\vec{\zeta}) = \left[\Pi\left(\frac{\lambda\vec{\zeta}}{d}\right) + \Pi\left\{\frac{\lambda}{d}\left(\vec{\zeta} - \frac{\vec{s}}{\lambda}\right)\right\} + \Pi\left\{\frac{\lambda}{d}\left(\vec{\zeta} - 2 \cdot \frac{\vec{s}}{\lambda}\right)\right\} + \ldots\right]$$
$$= \sum_{n=0}^{N-1} \Pi\left\{\frac{\lambda}{d}\left(\vec{\zeta} - n \cdot \frac{\vec{s}}{\lambda}\right)\right\}$$

- amplitude of diffracted field using Fourier shift and scale theorems

$$\tilde{a}(|\vec{\theta}|) = \left(\frac{\lambda}{R}\right)\left[\frac{1}{4}\pi(d/\lambda)^2\right]\left[\frac{2J_1(\pi|\vec{\theta}|d/\lambda)}{\pi|\vec{\theta}|d/\lambda}\right]\sum_{n=0}^{N-1}\left(e^{-i(2\pi\vec{\theta}\cdot\vec{s}/\lambda)}\right)^n$$

Point-Spread Function

- sum of geometric series of N complex exponentials

$$\sum_{n=0}^{N-1}\left(e^{-i(2\pi\vec{\theta}\cdot\vec{s}/\lambda)}\right)^n = \frac{e^{-iN(2\pi\vec{\theta}\cdot\vec{s}/\lambda)}-1}{e^{-i(2\pi\vec{\theta}\cdot\vec{s}/\lambda)}-1}$$

$$= \frac{e^{-iN(2\pi\vec{\theta}\cdot\vec{s}/2\lambda)}}{e^{-i(2\pi\vec{\theta}\cdot\vec{s}/2\lambda)}}\frac{\left(e^{-iN(2\pi\vec{\theta}\cdot\vec{s}/2\lambda)}-e^{iN(2\pi\vec{\theta}\cdot\vec{s}/2\lambda)}\right)}{\left(e^{-i(2\pi\vec{\theta}\cdot\vec{s}/2\lambda)}-e^{i(2\pi\vec{\theta}\cdot\vec{s}/2\lambda)}\right)}$$

$$= e^{-i(N-1)\pi\vec{\theta}\cdot\vec{s}/\lambda}\left[\frac{\sin N(\pi\vec{\theta}\cdot\vec{s}/\lambda)}{\sin(\pi\vec{\theta}\cdot\vec{s}/\lambda)}\right]$$

- PSF from PSF $= \tilde{a}(|\vec{\theta}|)\cdot\tilde{a}^*(|\vec{\theta}|)$:

$$\mathrm{PSF} = \left(\frac{\lambda}{R}\right)^2\left[\frac{1}{4}\pi(d/\lambda)^2\right]^2\left[\frac{2J_1(|\vec{u}\,|)}{|\vec{u}\,|}\right]^2\frac{\sin^2 N(\vec{u}\cdot\vec{s}/d)}{\sin^2(\vec{u}\cdot\vec{s}/d)}$$

$$\vec{u} = \pi\vec{\theta}\,d/\lambda$$

PSF

- PSF from before

$$\mathrm{PSF} \;=\; \left(\frac{\lambda}{R}\right)^2 \left[\frac{1}{4}\pi\,(d/\lambda)^2\right]^2 \left[\frac{2J_1(|\vec{u}\,|)}{|\vec{u}\,|}\right]^2 \frac{\sin^2 N(\vec{u}\cdot\vec{s}/d)}{\sin^2(\vec{u}\cdot\vec{s}/d)}$$

- $N = 1$: Airy brightness function for single circular aperture
- $N = 2$: Michelson:

$$\sin^2 N(\vec{u}\cdot\vec{s}/d)/\sin^2(\vec{u}\cdot\vec{s}/d) \;=\; \left[2\sin(\vec{u}\cdot\vec{s}/d)\cos(\vec{u}\cdot\vec{s}/d)\right]^2/\sin^2(\vec{u}\cdot$$

- N apertures: **maximum** constructive interference occurs for

$$\sin N(\pi\vec{\theta}\cdot\vec{s}/\lambda)/\sin(\pi\vec{\theta}\cdot\vec{s}/\lambda) \;=\; N$$

$$\frac{\vec{\theta}\cdot\vec{s}}{\lambda} \;=\; n\,(=\;0,\pm1,\pm2,\ldots) \;\;\rightarrow\;\; |\vec{\theta}| \;=\; \frac{n\lambda}{|\vec{s}|\cos\phi}$$

- *principal maxima* found at same $|\vec{\theta}|$-locations, regardless of value of $N \geq 2$

PSF (continued)

- PSF from before

$$\mathrm{PSF} = \left(\frac{\lambda}{R}\right)^2 \left[\frac{1}{4}\pi\,(d/\lambda)^2\right]^2 \left[\frac{2J_1(|\vec{u}\,|)}{|\vec{u}\,|}\right]^2 \frac{\sin^2 N(\vec{u}\cdot\vec{s}/d)}{\sin^2(\vec{u}\cdot\vec{s}/d)}$$

- **Minima**, of **zero** flux density, exist whenever

$$\sin N(\pi\vec{\theta}\cdot\vec{s}/\lambda)/\sin(\pi\vec{\theta}\cdot$$

$$\frac{\vec{\theta}\cdot\vec{s}}{\lambda} = \pm\frac{1}{N}, \pm\frac{2}{N}, \pm\frac{3}{N}, \ldots, \pm\frac{N-1}{N}, \pm\frac{N+1}{N}, \ldots$$

$$\Rightarrow \quad |\vec{\theta}| = \frac{n\lambda}{N|\vec{s}|\cos\phi}, \quad \text{for} \quad n = \pm 1, \pm 2, \ldots \quad \text{but} \quad n \neq kN \; (k = 0, \pm$$

- $\cos\phi$: angle between $\vec{\theta}$ and baseline vector $\vec{s}$
- between consecutive principal maxima there will therefore be **N-1 minima**
- each pair of minima there will have to be a **subsidiary maximum**, i.e. a total of **N-2 subsidiary maxima** between

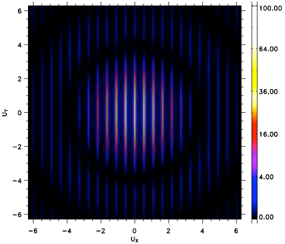
PSF Interpretation

- PSF from before

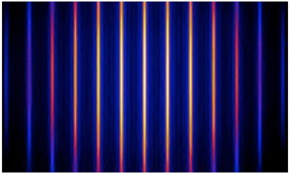
$$\mathrm{PSF} = \left(\frac{\lambda}{R}\right)^2 \left[\frac{1}{4}\pi\,(d/\lambda)^2\right]^2 \left[\frac{2J_1(|\vec{u}\,|)}{|\vec{u}\,|}\right]^2 \frac{\sin^2 N(\vec{u}\cdot\vec{s}/d)}{\sin^2(\vec{u}\cdot\vec{s}/d)}$$

- first two terms give normalisation for $|\vec{\theta}\,| = 0$
- other terms represent 2-d Airy distribution, intensity-modulated along direction of baseline vector $\vec{s}$ with periodicity $(\Delta\theta)_s = \lambda/|\vec{s}|$ of narrow bright principal maxima and with a periodicity $(\Delta\theta)_{Ns} = \lambda/(N|\vec{s}|)$ of narrow weak subsidiary maxima, interleaved with zero-intensity minima

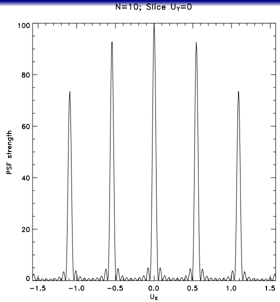
Multi-Aperture Array PSF



PSF of 10-element interferometer with circular apertures



Magnification of central part



Cross-section of central part of PSF for 10-element interferometer

Optical Transer Function (OTF)

- OTF for N circular apertures from autocorrelation of pupil function

$$H_\lambda(\vec{\zeta}, N\vec{s}/\lambda) = \left(\frac{\lambda}{R}\right)^2 \left[\sum_{n=0}^{N-1} \Pi\left\{\frac{\lambda}{d}\left(\vec{\zeta} - n\cdot\frac{\vec{s}}{\lambda}\right)\right\}\right] * \left[\sum_{m=0}^{N-1} \Pi\left\{\frac{\lambda}{d}\left(\vec{\zeta} - m\cdot\frac{\vec{s}}{\lambda}\right)\right\}\right]$$

$$= \left(\frac{\lambda}{R}\right)^2 \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} A_{nm}(\vec{\zeta}, \vec{s}/\lambda)$$

- A_{nm} represents element of $N \times N$ autocorrelation matrix $\mathbf{A}$

$$A_{nm}(\vec{\zeta}, \vec{s}/\lambda) = \int\int_{pupil\ plane} \Pi\left\{\frac{\lambda}{d}\left(\vec{\zeta}' - n\cdot\frac{\vec{s}}{\lambda}\right)\right\}\Pi\left\{\frac{\lambda}{d}\left(\vec{\zeta}' - \vec{\zeta} - m\cdot\frac{\vec{s}}{\lambda}\right)\right\} d\vec{\zeta}'$$

OTF (continued)

- values $A_{nm} \neq 0$ are Chinese-hat functions for single circular aperture
- multi-aperture case: series of **principal maxima** in $H_\lambda(\vec{\zeta}, \vec{s}/\lambda)$ plane
- this is the uv-plane representing 2-d spatial frequency space
- principal maxima at

$$\vec{\zeta}_{max} = \vec{\zeta} - k \cdot \frac{\vec{s}}{\lambda} \text{ with } k = n - m = 0, \pm 1, \pm 2, \ldots, \pm(N-2), \pm(N-1)$$

- replace A_{nm} by A_k, where k refers to diagonals of autocorrelation matrix **A**
  - $k = 0$: main diagonal
  - $k = \pm 1$: two diagonals contiguous to main diagonal
  - ...

OTF (continued)

- diagonal terms A_k computed in same way as for single circular aperture with vector notation

$$\cdot \left[\arccos\left(\frac{\lambda}{d}\left|\vec{\zeta} - k\cdot\frac{\vec{s}}{\lambda}\right|\right) - \left(\frac{\lambda}{d}\left|\vec{\zeta} - k\cdot\frac{\vec{s}}{\lambda}\right|\right)\left(1 - \left(\frac{\lambda}{d}\left|\vec{\zeta} - k\cdot\frac{\vec{s}}{\lambda}\right|\right)^2\right.\right.$$

- with Chinese-hat functions $\hat{C}_k(\vec{\zeta} - k\cdot\vec{s}/\lambda)$ normalised to unit aperture area

$$A_k = \frac{1}{4}\pi\left(\frac{d}{\lambda}\right)^2 \hat{C}_k(\vec{\zeta} - k\cdot\vec{s}/\lambda)$$

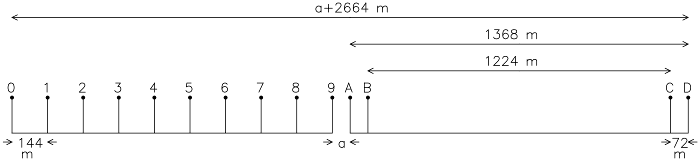
- sum over all elements of matrix **A**

$$\sum^{N-1}\sum^{N-1} A_{nm} \equiv A_{sa} \sum^{N-1} (N-|k|)\hat{C}_k(\vec{\zeta} - k\cdot\vec{s}/\lambda)$$

Introduction

- due to rotation of Earth, baseline vectors $k \cdot \vec{s}/\lambda$ of N-element array scan the YZ-plane if X-axis is lined up with North polar axis
- principal maxima or 'grating lobes' in PSF are concentric annuli around central source peak at angular distances $k \cdot \lambda / |\vec{s}\,|$
- if circular scans in YZ-plane are too widely spaced ($|\vec{s}\,|$ is larger than single dish diameter), the Nyquist criterion is not respected and undersampling of spatial frequency uv-plane (=YZ-plane) occurs
- consequently, grating lobes will show up within the field of view defined by the single-dish beam profile
- can be avoided by decreasing sampling distance $|\vec{s}\,|$

Westerbork Radio Synthesis Telescope (WSRT)

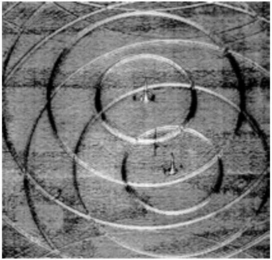


- 14 parabolic antennae, diameters $D = 25$ m
- lined up along East-West direction over ≈ 2750 m
- 10 antennae have fixed mutual distance of 144 m
- 4 antennae can be moved collectively with respect to fixed array, without changing their mutual distance
- 14 antennae comprise 40 simultaneously operating interferometers
- array is rotated in plane containing Westerbork perpendicular to Earth's rotation axis
- limited to sources near the North polar axis
- standard distance between 9 and A equals 72 meters

WSRT (continued)

- after 12 hours, 38 concentric semi-circles with radii ranging from $L_{min} = 72$ meters to $L_{max} = 2736$ meters in increments of $\triangle L = 72$ meters
- correlators integrate over 10 s, sampling of semi-circles every 1/24 degrees
- other half can be found by mirroring the first half since $I(\vec{\Omega})$ is a real function

Imaging



- brightness distribution $I(\vec{\Omega})$ by Fourier inversion
- only discrete samples of spatial coherence function $\tilde{\Gamma}(\vec{r})$, integral replaced by sum
- weighting function to get considerable reduction of side lobes at expense of ultimate angular resolution
- reconstructed $\hat{I}(\vec{\Omega})$ needs to be corrected for single dish response function $H(\vec{\Omega})$

Point-Spread Function

- spatial frequency response function of rotated array in uv-plane from geometry of concentric scans
- scalar function due to circular symmetry

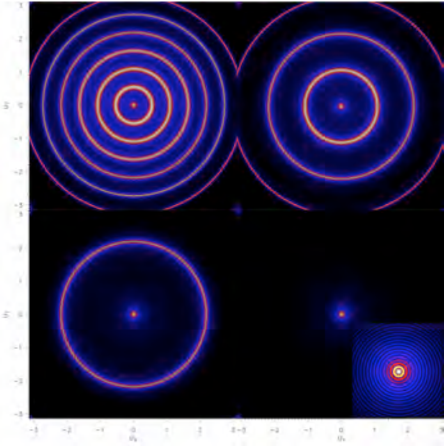
$$PSF_{ERAS} = \left(\frac{\lambda}{R}\right)^2 \left[\frac{1}{4}\pi\left(d/\lambda\right)^2\right]^2 \left[\frac{2J_1(u)}{u}\right]^2 \frac{\sin^2 N(u\triangle L/D)}{\sin^2(u\triangle L/D)}$$

  with $u = \pi\theta D/\lambda$ and θ, the radially symmetric, diffraction angle
- central peak: similar to Airy function with spatial resolution

$$\triangle\theta = \frac{\lambda}{2L_{max}} radians$$

  with $2L_{max}$ the maximum diameter of the array in the YZ-plane
- concentric grating lobes: angular distances of annuli from central peak follow from the location of principal maxima given by modulation term $\sin^2 N(u\triangle L/D)/\sin^2(u\triangle L/D)$

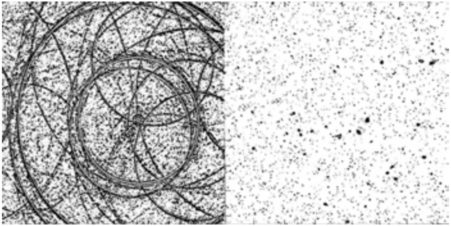
PSF (continued)



- for an N-element array with increment $\triangle L$, these angular positions are given by:

$$\theta_{grating} = \frac{\lambda}{\triangle L}, 2\frac{\lambda}{\triangle L}, ..., (N-1)\frac{\lambda}{\triangle L}$$

CLEAN



- undersampling of YZ-plane since grating lobes are well within field of view
- can decrease distance between antennae 9 and A during second half rotation for 36 meter increment coverage
- four half rotations in 48 hours can increase coverage to 18 meter increments $\Rightarrow$ complete uv coverage
- incomplete coverage of YZ-plane means that values of coherence function $\tilde{\Gamma}(\vec{r})$ are set to zero in empty spaces, which will certainly give an erroneous result
- apply CLEAN method for improving dirty radio maps

Bandwidth Restrictions

- coherence length of source needs to be larger than maximum path length difference at longest baseline
- imposes maximum frequency bandwidth for observations
- largest angle of incidence equals half the field of view, i.e. $\lambda/2D$
- coherence length compliant with largest baseline $L_{coh} \gg \frac{\lambda}{2D} L_{max}$
- frequency bandwidth requirement

$$\frac{\triangle\nu}{\nu_0} \ll \frac{2D}{L_{max}}$$

- WRST: $2D/L_{max} \approx 1/50$, at 21 cm ($\approx$ 1400 MHz), $\triangle\nu \ll 28$ MHz, coherence length $>$ 10 m
- in practise: $\triangle\nu \approx 10$ MHz
- increase bandwidth by division into frequency subbands
- subband maps scaled with λ and added

General Case

- extended source in arbitrary direction
- during Earth's rotation, antenna beams kept pointed at source
- tip of baseline vector describes a trajectory
- maintain maximum coherence by delaying one antenna signal with respect to the other antenna within fraction of a wavelength
- source at angle ϕ_0 to Earth's rotation axis
- circles in uv-plane change into ellipses and coherence function is sampled on ellipses rather than on circles
- major axes of these ellipses remain equal to the physical length of the WSRT baselines, minor axes are shortened by $\cos\phi_0$
- PSF becomes elliptical

$$PSF = \frac{\alpha\lambda}{2L_{max}\cos\phi_0}$$

- source in equatorial plane: no resolution in one direction
- baselines need North-South components (e.g. VLA)