Lecture 9: Indirect Imaging 2

Outline

Cygnus A at 6 cm

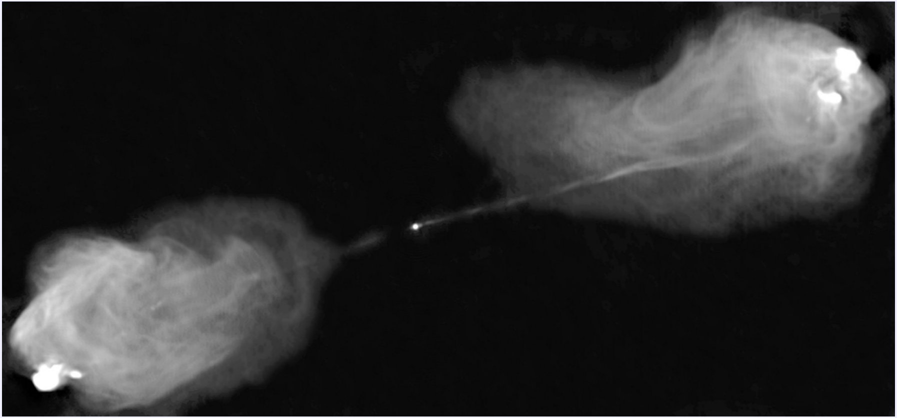


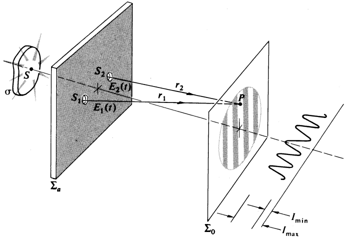
Image courtesy of NRAO/AUI

Very Large Array (VLA), New Mexico, USA

Image courtesy of NRAO/AUI

Two-Element Interferometer

Fringe Pattern



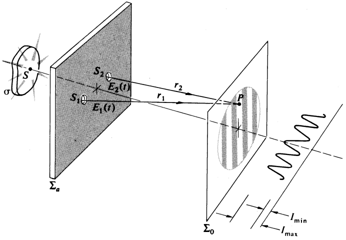
- for $I_1 = I_2 = I_0$

$$I = 2I_0 \left\{1 + |\tilde{\gamma}_{12}(\tau)| \cos\left[\alpha_{12}(\tau) - \phi\right]\right\} \quad V = |\tilde{\gamma}_{12}(\tau)|$$

- source S on central axis, fully coherent waves from two holes

$$I \;=\; 2I_0(1 + \cos\phi) \;=\; 4I_0 \cos^2\frac{\phi}{2}$$

Fringe Pattern (continued)



$$I = 4 I_0 \cos^2 \frac{\phi}{2}$$

$$\phi = \frac{2\pi}{\bar{\lambda}} (r_2 - r_1) = 2\pi \bar{\nu} \tau$$

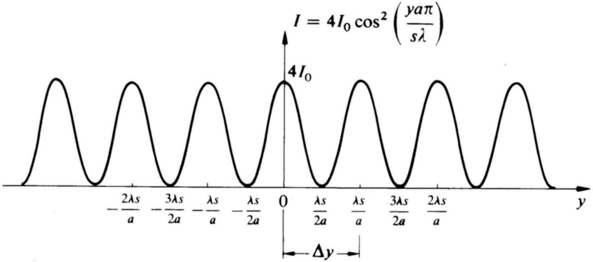
- distance a between pinholes
- distance s to observation plane Σ_O, $s \gg a$
- path difference $(r_2 - r_1)$ in equation for ϕ in good approximation

$$r_2 - r_1 = a\theta = \frac{a}{s} y$$

- and therefore

$$I = 4 I_0 \cos^2 \frac{\pi a y}{s \bar{\lambda}}$$

Interference Fringes from Monochromatic Point Source



- irradiance as a function of the y-coordinate of the fringes in observation plane Σ_O
- irradiance vs. distance distribution is Point-Spread Function (PSF) of ideal two-element interferometer
- non-ideal two-element interferometer with finite apertures using *pupil function* concept (Observational Astrophysics 1)

Finite Apertures

- two circular apertures, diameter d, separated by baseline direction vector $\vec{s}$
- origin of pupil function $P(\vec{r})$ on baseline vector $\vec{s}$ symmetrically positioned between apertures
- pupil function using 2-dimensional circular window function Π

$$P(\vec{r}) = \Pi\left(\frac{\vec{r} - \vec{s}/2}{d}\right) + \Pi\left(\frac{\vec{r} + \vec{s}/2}{d}\right)$$

- with spatial frequency variable $\vec{\zeta} = \vec{r}/\lambda$

$$P(\vec{\zeta}) = \Pi\left(\frac{\vec{\zeta} - \vec{s}/2\lambda}{d/\lambda}\right) + \Pi\left(\frac{\vec{\zeta} + \vec{s}/2\lambda}{d/\lambda}\right)$$

Finite Apertures (continued)

- with $\vec{\zeta} = \vec{r}/\lambda$

$$P(\vec{\zeta}) = \Pi\left(\frac{\vec{\zeta} - \vec{s}/2\lambda}{d/\lambda}\right) + \Pi\left(\frac{\vec{\zeta} + \vec{s}/2\lambda}{d/\lambda}\right)$$

- length of baseline vector $|\vec{s}| = D$
- if $d \ll D$, pupil function can be approximated by

$$P(\vec{\zeta}) = \delta\left(\vec{\zeta} - \vec{s}/2\lambda\right) + \delta\left(\vec{\zeta} + \vec{s}/2\lambda\right)$$

- optical transfer function (OTF) from *self-convolution* of $(\lambda/R)P(\vec{\zeta})$
- symmetrical pupil function: autocorrelation of $(\lambda/R)P(\vec{\zeta})$:

$$OTF = H_\lambda(\vec{\zeta}) = \left(\frac{\lambda}{R}\right)^2 \int\int_{\text{pupil plane}} P(\vec{\zeta}')P(\vec{\zeta}' - \vec{\zeta})d\vec{\zeta}'$$

Finite Apertures (continued)

- pupil function for $d \ll D$

$$P(\vec{\zeta}) = \delta\left(\vec{\zeta} - \vec{s}/2\lambda\right) + \delta\left(\vec{\zeta} + \vec{s}/2\lambda\right)$$

- autocorrelation of $(\lambda/R)P(\vec{\zeta})$:

$$\begin{aligned}
OTF = H_\lambda(\vec{\zeta}) &= \left(\frac{\lambda}{R}\right)^2 \int\int_{\text{pupil plane}} P(\vec{\zeta}')P(\vec{\zeta}' - \vec{\zeta})d\vec{\zeta}' \\
&= \left(\frac{\lambda}{R}\right)^2 \int\int_{\text{pupil plane}} \left[\delta\left(\vec{\zeta}' - \vec{s}/2\lambda\right) + \delta\left(\vec{\zeta}' + \vec{s}/2\lambda\right)\right] \cdot \\
&\quad \cdot \left[\delta\left(\vec{\zeta}' - \vec{\zeta} - \vec{s}/2\lambda\right) + \delta\left(\vec{\zeta}' - \vec{\zeta} + \vec{s}/2\lambda\right)\right] d\vec{\zeta}' \\
&= 2\left(\frac{\lambda}{R}\right)^2 \left[\delta(\vec{\zeta}) + \frac{1}{2}\delta\left(\vec{\zeta} - \vec{s}/\lambda\right) + \frac{1}{2}\delta\left(\vec{\zeta} + \vec{s}/\lambda\right)\right]
\end{aligned}$$

Finite Apertures (continued)

- from before

$$OTF = 2\left(\frac{\lambda}{R}\right)^2\left[\delta(\vec{\zeta}) + \frac{1}{2}\delta\left(\vec{\zeta} - \vec{s}/\lambda\right) + \frac{1}{2}\delta\left(\vec{\zeta} + \vec{s}/\lambda\right)\right]$$

- pair of pinholes transmits three spatial frequencies
  - DC-component $\delta(\vec{0})$
  - two high frequencies related to length of baseline vector $\vec{s}$ at $\pm\vec{s}/\lambda$
- 3 spatial frequencies represent three-point sampling of the **uv-plane** in 2-d spatial frequency space
- complete sampling of uv-plane provides sufficient information completely reconstruct original brightness distribution

Point-Spread Function (PSF)

- Optical Transfer Function (OTF) of 2-element interferometer

$$OTF = 2\left(\frac{\lambda}{R}\right)^2\left[\delta(\vec{\zeta}) + \frac{1}{2}\delta\left(\vec{\zeta} - \vec{s}/\lambda\right) + \frac{1}{2}\delta\left(\vec{\zeta} + \vec{s}/\lambda\right)\right]$$

- PSF is Fourier Transform of $H_\lambda(\vec{\zeta})$

$$\begin{aligned} \delta(\vec{\zeta}) &\Leftrightarrow 1 \\ \delta\left(\vec{\zeta} - \vec{s}/\lambda\right) &\Leftrightarrow e^{i2\pi\vec{\theta}\cdot\vec{s}/\lambda} \\ \delta\left(\vec{\zeta} + \vec{s}/\lambda\right) &\Leftrightarrow e^{-i2\pi\vec{\theta}\cdot\vec{s}/\lambda} \end{aligned}$$

- Point-Spread Function of 2-element interferometer

$$\left(\frac{\lambda}{R}\right)^2\left[2(1 + \cos 2\pi\vec{\theta}\cdot\vec{s}/\lambda)\right] = 4\left(\frac{\lambda}{R}\right)^2\cos^2\pi\vec{\theta}\cdot\vec{s}/\lambda$$

- $\vec{\theta}$: 2-d angular coordinate vector
- attenuation factor $(\lambda/R)^2$ from spherical expansion

Point-Spread Function (continued)

- from before

$$\text{PSF} = 4\left(\frac{\lambda}{R}\right)^2 \cos^2 \pi\vec{\theta}\cdot\vec{s}/\lambda$$

- flux I_0 from each pinhole $\Rightarrow$ brightness distribution for diffracted beam

$$I = 4I_0 \cos^2 \pi\vec{\theta}\cdot\vec{s}/\lambda$$

- compare to previous result

$$I = 4I_0 \cos^2 \frac{\pi a y}{s\lambda}$$

- the same but now in 2-d setting with $\vec{\theta}$ replacing y/s and $\vec{s}$ replacing pinhole distance a

Interpretation of fringe pattern

- flux I_0 from each pinhole $\Rightarrow$ brightness distribution for diffracted beam

$$I = 4 I_0 \cos^2 \pi\vec{\theta}\cdot\vec{s}/\lambda$$

- full constructive interference for

$$I = 4 I_0 \text{ for } \frac{\vec{\theta}\cdot\vec{s}}{\lambda} = n\,(= 0, \pm 1, \pm 2, \ldots) \;\rightarrow\; |\vec{\theta}| = \frac{n\lambda}{|\vec{s}|\cos\phi}$$

- full destructive interference for

$$I = 0 \text{ for } \frac{\vec{\theta}\cdot\vec{s}}{\lambda} = (n + \frac{1}{2})\,(= \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots) \;\rightarrow\; |\vec{\theta}| = \frac{(n+\frac{1}{2})\lambda}{|\vec{s}|\cos\phi}$$

- $\cos\phi$ is angle between $\vec{\theta}$ and baseline vector $\vec{s}$

Interpretation of fringe pattern (continued)

- PSF represents *corrugated sheet* with modulation along direction of baseline vector $\vec{s}$ and periodicity $(\Delta\theta)_s = \lambda/|\vec{s}|$
- PSF is pattern of alternating bright and dark *stripes* orthogonal to direction of baseline vector $\vec{s}$
- aperture have finite size, diffracted light from apertures is localized, i.e. $d < S$, but approximation by δ-functions no longer holds

Finite Apertures: PSF

- PSF (amplitude squared of diffracted field) from Fourier transformation of pupil function

$$\tilde{a}(\vec{\theta}) \Leftrightarrow \frac{\lambda}{R}\left[\Pi\left\{\frac{\lambda}{d}\left(\vec{\zeta}-\frac{\vec{s}}{2\lambda}\right)\right\}+\Pi\left\{\frac{\lambda}{d}\left(\vec{\zeta}+\frac{\vec{s}}{2\lambda}\right)\right\}\right]$$

- shift and scale theorems of Fourier theory: if $f(x) \Leftrightarrow F(s)$, then $f[a(x-b)] \Leftrightarrow \left(e^{-2\pi ibs}/a\right) F(s/a)$
- amplitude of diffracted field

$$\tilde{a}(|\vec{\theta}|) = \left(\frac{\lambda}{R}\right)\left[\frac{1}{4}\pi\,(d/\lambda)^2\right]\left[\frac{2J_1(\pi|\vec{\theta}|d/\lambda)}{\pi|\vec{\theta}|d/\lambda}\right]\left(e^{-2\pi i\vec{\theta}\cdot\vec{s}/2\lambda}+e^{2\pi i\vec{\theta}\cdot\vec{s}/2\lambda}\right)$$

$$= 2\left(\frac{\lambda}{R}\right)\left[\frac{1}{4}\pi\,(d/\lambda)^2\right]\left[\frac{2J_1(|\vec{u}|)}{|\vec{u}|}\right]\cos\vec{u}\cdot\vec{s}/d \qquad \vec{u}=\pi\,\vec{\theta}\,d/\lambda$$

$$\mathrm{PSF} = |\tilde{a}(|\vec{\theta}|)|^2 = 4\left(\frac{\lambda}{R}\right)^2\left[\frac{1}{4}\pi\,(d/\lambda)^2\right]^2\left[\frac{2J_1(|\vec{u}|)}{|\vec{u}|}\right]^2\cos^2\vec{u}\cdot\vec{s}/d$$

Finite Apertures: PSF Interpretation

- from before

$$\text{PSF} = |\tilde{a}(|\vec{\theta}|)|^2 = 4\left(\frac{\lambda}{R}\right)^2 \left[\frac{1}{4}\pi\,(d/\lambda)^2\right]^2 \left[\frac{2J_1(|\vec{u}\,|)}{|\vec{u}\,|}\right]^2 \cos^2 \vec{u}\cdot\vec{s}/d$$

- full constructive interference for

$$\frac{\vec{\theta}\cdot\vec{s}}{\lambda} = n\,(= 0, \pm 1, \pm 2, \ldots) \;\rightarrow\; |\vec{\theta}| = \frac{n\lambda}{|\vec{s}|\cos\phi}$$

- full destructive interference for

$$\frac{\vec{\theta}\cdot\vec{s}}{\lambda} = (n+\frac{1}{2})\,(= \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots) \;\rightarrow\; |\vec{\theta}| = \frac{(n+\frac{1}{2})\lambda}{|\vec{s}|\cos\phi}$$

- $\cos\phi$ is angle between $\vec{\theta}$ and baseline vector $\vec{s}$

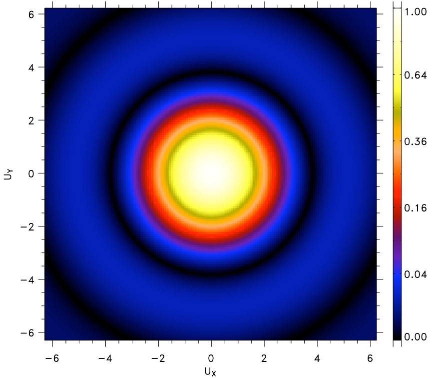
PSF Interpretation (continued)

- from before

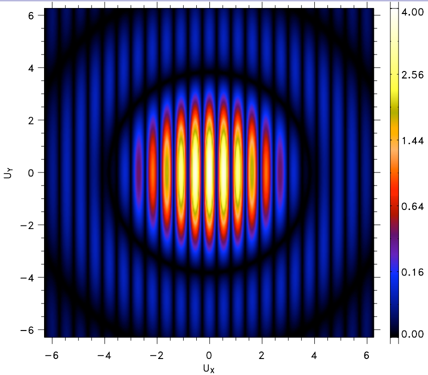
$$\text{PSF} = |\tilde{a}(|\vec{\theta}|)|^2 = 4\left(\frac{\lambda}{R}\right)^2 \left[\frac{1}{4}\pi\,(d/\lambda)^2\right]^2 \left[\frac{2J_1(|\vec{u}\,|)}{|\vec{u}\,|}\right]^2 \cos^2 \vec{u}\cdot\vec{s}/d$$

- first two terms give normalisation for $|\vec{\theta}\,| = 0$
- other terms represent corrugated 2-d Airy brightness distribution, intensity-modulated along the direction of the baseline vector $\vec{s}$ with periodicity $(\Delta\theta)_s = \lambda/|\vec{s}|$
- pattern of alternating bright and dark annuli
- separation determined by $(\Delta\theta)_d = 1.220\lambda/d,\ 2.233\lambda/d,\ 3.238\lambda/d, \ldots$
- superimposed by fringes orthogonal to direction of baseline vector $\vec{s}$ at distance $(\Delta\theta)_s = \lambda/|\vec{s}\,|$

2-d Brightness Distribution



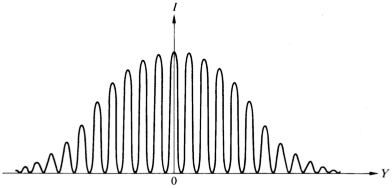
PSF of single circular aperture



PSF of two-element interferometer, aperture diameter $d = 25$ m, length of baseline vector $|\vec{s}| = 144$ m

- double beam interference fringes showing modulation effect of diffraction by aperture of a single pinhole

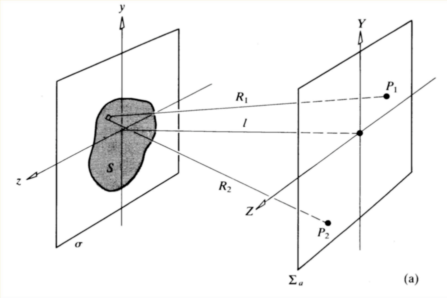
Modulation Effect of Aperture



- typical one-dimensional cross-section along $u_y = 0$ of the central part of the interferogram
- visibilities are equal to one, because $I_{min} = 0$
- $|\tilde{\gamma}_{12}(\tau)| = 1$ for all values of τ and any pair of spatial points, if and only if the radiation field is *strictly monochromatic*
- a non-zero radiation field for which $|\tilde{\gamma}_{12}(\tau)| = 0$ for all values of τ and any pair of spatial points cannot exist in free space

Van Cittert-Zernike Theorem

Basic Approach



- relates brightness distribution of extended source and phase correlation between two points in radiation field
- extended source S incoherent, quasi-monochromatic
- positions P_1 and P_2 in observers plane Σ

$$\tilde{E}_1(t)\tilde{E}_2^*(t) = \mathbf{E}\{\tilde{E}_1(t)\tilde{E}_2^*(t)\} = \tilde{\Gamma}_{12}(0)$$

Basic Approach (continued)

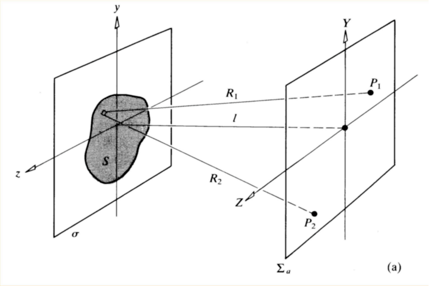
- in observers plane Σ with subscripts 1 and 2 referring to positions P_1 and P_2

$$\tilde{E}_1(t)\tilde{E}_2^*(t) = \mathbf{E}\{\tilde{E}_1(t)\tilde{E}_2^*(t)\} = \tilde{\Gamma}_{12}(0)$$

- if $\tilde{E}_1$ and $\tilde{E}_2$ are uncorrelated, then $|\tilde{\Gamma}_{12}(0)| = 0$
- full correlation: $|\tilde{\gamma}_{12}| \left(= \frac{|\tilde{\Gamma}_{12}(0)|}{\sqrt{I_1 I_2}} \right) = 1$
- partial correlation: $0 < |\tilde{\gamma}_{12}(0)| < 1$
- extended source is collection of non-coherent infinitesimal sources
- reduction in fringe contrast described by *Visibility function V*:

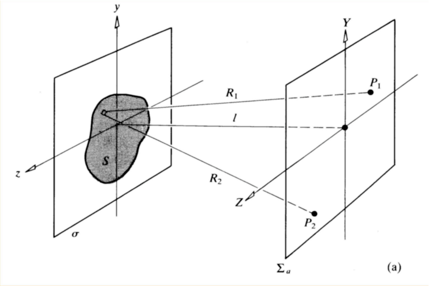
$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = |\tilde{\gamma}_{12}(0)|$$

Overview without Equations



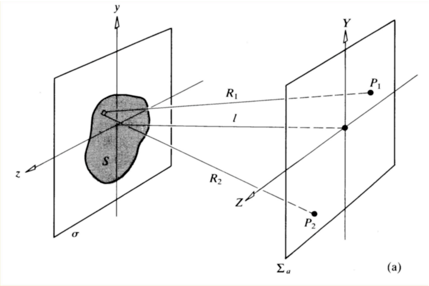
- relate $\gamma_{12}(0)$ to brightness distribution of extended source S
- source S is quasi-monochromatic, incoherent source in plane σ, intensity distribution $I(y, z)$
- observation plane Σ parallel to σ
- l is perpendicular to both planes (coincident with the X-axis) connecting center of extended source $(y = 0, z = 0)$ to zero reference in Σ $(Y = 0, Z = 0)$.

Overview without Equations (continued)



- two positions P_1, P_2
- describe coherence $\gamma_{12}(0)$ in plane Σ
- infinitesimal radiation element *dS* in source at distances R_1 and R_2 from P_1 and P_2
- assume *S not a source* but *an aperture* of identical size and shape
- assume that $I(y, z)$ is not description of intensity distribution but its functional form corresponds to the *field distribution* across that aperture

Overview without Equations (continued)



- imagine a transparancy at aperture with amplitude transmission characteristics that correspond functionally to the irradiance distribution $I(y, z)$
- imagine that aperture is illuminated by spherical wave converging towards fixed point P_2 so that diffraction pattern is centered at P_2
- this diffracted field distribution, normalised to unity at P_2, is everywhere (e.g. at P_1) equal to the value of $\gamma_{12}(0)$ at that point

Overview with Equations

- in limit R_1, R_2 much larger than source diameter and relevant part of Σ-plane, equivalent of Fraunhofer diffraction
- almost always satisfied for astronomical observations
- van Cittert-Zernike theorem

$$\tilde{\Gamma}(\vec{r}) = \int\int_{\text{source}} I(\vec{\Omega}) e^{\frac{2\pi i \vec{\Omega}.\vec{r}}{\lambda}} d\vec{\Omega}$$

$$I(\vec{\Omega}) = \lambda^{-2} \int\int_{\Sigma\text{-plane}} \tilde{\Gamma}(\vec{r}) e^{-\frac{2\pi i \vec{\Omega}.\vec{r}}{\lambda}} d\vec{r}$$

- $I(\vec{\Omega})$ is intensity distribution of extended source as function of unit direction vector $\vec{\Omega}$ as seen from observation plane Σ
- $\tilde{\Gamma}(\vec{r})$ is coherence function in Σ-plane
- center of extended source S is origin of $\vec{\Omega}$ (coincident with central axis l), small angular extent of source: $I(\vec{\Omega}) = I(\theta_y, \theta_z)$ and $d\vec{\Omega} = d\theta_y d\theta_z$, where θ_y, θ_z two orthogonal angular coordinate axes across the source

Overview with Equations (continued)

- van Cittert-Zernike theorem

$$\tilde{\Gamma}(\vec{r}) = \int\int_{\text{source}} I(\vec{\Omega}) e^{\frac{2\pi i \vec{\Omega}.\vec{r}}{\lambda}} d\vec{\Omega}$$

$$I(\vec{\Omega}) = \lambda^{-2} \int\int_{\Sigma\text{-plane}} \tilde{\Gamma}(\vec{r}) e^{-\frac{2\pi i \vec{\Omega}.\vec{r}}{\lambda}} d\vec{r}$$

- vector $\vec{r}$ represents arbitrary baseline $\vec{r}(X, Y)$ in plane with $d\vec{r} = dYdZ$ (e.g. $\overline{P_1 P_2} = \vec{r}_{P_1} - \vec{r}_{P_2}$)
- $\tilde{\Gamma}(\vec{r})$ and $I(\vec{\Omega})$ are linked through Fourier transform, except for scaling with wavelength λ
- scaling might be perceived as "true" Fourier transform with *conjugate variables* $\vec{\Omega}$ and $\vec{r}/\lambda$,
- van Cittert-Zernike theorem as Fourier pair

$$I(\vec{\Omega}) \quad \Leftrightarrow \quad \tilde{\Gamma}(\vec{r}/\lambda)$$

Overview with Equations (continued)

- van Cittert-Zernike theorem

$$\tilde{\Gamma}(\vec{r}) = \int\int_{\text{source}} I(\vec{\Omega}) e^{\frac{2\pi i \vec{\Omega}.\vec{r}}{\lambda}} d\vec{\Omega}$$

$$I(\vec{\Omega}) = \lambda^{-2} \int\int_{\Sigma\text{-plane}} \tilde{\Gamma}(\vec{r}) e^{-\frac{2\pi i \vec{\Omega}.\vec{r}}{\lambda}} d\vec{r}$$

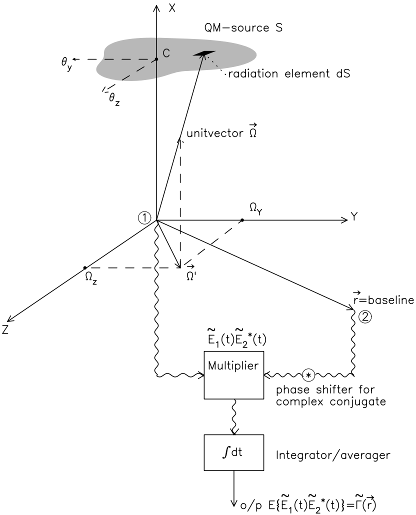
- complex *spatial* degree of coherence

$$\tilde{\gamma}(\vec{r}) = \frac{\tilde{\Gamma}(\vec{r})}{\int\int_{\text{source}} I(\vec{\Omega}) d\vec{\Omega}}$$

  i.e. normalising by total source intensity

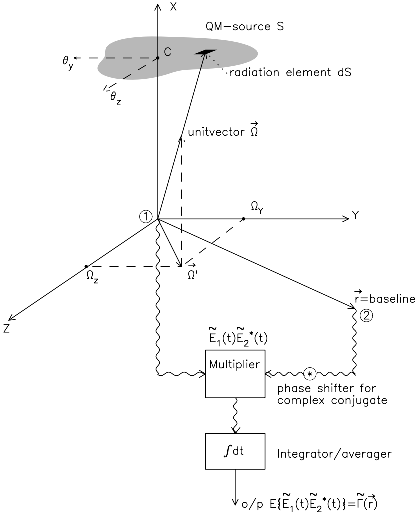
- extended source S is spatially incoherent, but partially correlated radiation field at e.g. positions P_1 and P_2 exists since all individual source elements contribute to a specific location P in Σ-plane

Derivation of Van Cittert-Zernike Theorem



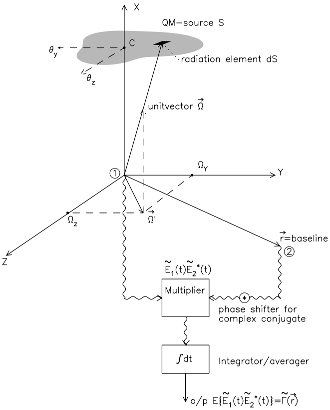
- observation plane Σ contains baseline vector $\vec{r}(Y, Z)$ and is perpendicular to vector pointing at centre of radiation source
- angular coordinates θ_Y and θ_Z across the source correspond to linear coordinates of unit direction vector $\vec{\Omega}(\Omega_X, \Omega_Y, \Omega_Z)$, i.e. direction cosines of $\vec{\Omega}$ relative to X, Y, Z coordinate system ($\Omega_X^2 + \Omega_Y^2 + \Omega_Z^2 = 1$).

Derivation of Van Cittert-Zernike Theorem (continued)



- spatial coherence of *EM*-field between two positions 1 and 2 is outcome of correlator producing $\mathbf{E}\left\{\tilde{E}_1(t)\tilde{E}_2^*(t)\right\}$
- positions 1, 2 not point like, represent *radio antennae* or *optical telescopes*
- if $I(\vec{\Omega}) = I_0\delta(\vec{\Omega})$, i.e. a point source on the *X*-axis, the Van Cittert-Zernike relation yields $|\tilde{\Gamma}(\vec{r})| = I_0$ and $|\tilde{\gamma}(\vec{r})| = 1$: a plane wave simultaneously hits entire *YZ*-plane, full coherence is preserved

Derivation of Van Cittert-Zernike Theorem (continued)



- infinitesimal source element in direction $\vec{\Omega}_0 \Longrightarrow I_0\delta(\vec{\Omega} - \vec{\Omega}_0)$ projection of $\vec{\Omega}_0$ on the $\Sigma-$plane is $\vec{\Omega}'_0(\Omega_Y, \Omega_Z)$
- difference in path length between positions 1 and 2 given by projection of $\vec{r}$ on $\vec{\Omega}_0$, i.e. $\vec{r}.\vec{\Omega}'_0 = \Omega_Y Y + \Omega_Z Z$
- therefore

$$\tilde{E}_1(t) = \tilde{E}_0(t) e^{2\pi i \nu_0 \left(t + \frac{\vec{\Omega}_0 . \vec{r}}{c}\right)}$$

$$= \tilde{E}_0(t) e^{\left(2\pi i \nu_0 t + \frac{2\pi i \vec{\Omega}_0 . \vec{r}}{\lambda}\right)}$$

$$\tilde{E}_2^*(t) = \tilde{E}_0(t) e^{-2\pi i \nu_0 t}$$

Derivation of Van Cittert-Zernike Theorem (continued)

$$\begin{aligned}
\tilde{E}_1(t) &= \tilde{E}_0(t) e^{2\pi i \nu_0 \left(t + \frac{\vec{\Omega}_0 . \vec{r}}{c}\right)} \\
&= \tilde{E}_0(t) e^{\left(2\pi i \nu_0 t + \frac{2\pi i \vec{\Omega}_0 . \vec{r}}{\lambda}\right)} \\
\tilde{E}_2^*(t) &= \tilde{E}_0(t) e^{-2\pi i \nu_0 t}
\end{aligned}$$

- $\mathbf{E}\left\{\tilde{E}_1(t)\tilde{E}_2^*(t)\right\} = \mathbf{E}\left\{|\tilde{E}_0(t)|^2\right\} e^{\frac{2\pi i \vec{\Omega}_0 . \vec{r}}{\lambda}} = I_0(\vec{\Omega}_0) e^{\frac{2\pi i \vec{\Omega}_0 . \vec{r}}{\lambda}}$
- integration over full source extent (all source elements are spatially uncorrelated)

$$\tilde{\Gamma}(\vec{r}) = \int\int_{\text{source}} I_0(\vec{\Omega}) e^{2\pi i \vec{\Omega} . \vec{r}/\lambda} d\vec{\Omega}$$

$$\tilde{\gamma}(\vec{r}) = \frac{\int\int_{\text{source}} I_0(\vec{\Omega}) e^{2\pi i \vec{\Omega} . \vec{r}/\lambda} d\vec{\Omega}}{\int\int_{\text{source}} I_0(\vec{\Omega}) d\vec{\Omega}}$$

Derivation of Van Cittert-Zernike Theorem (continued)

- coherence

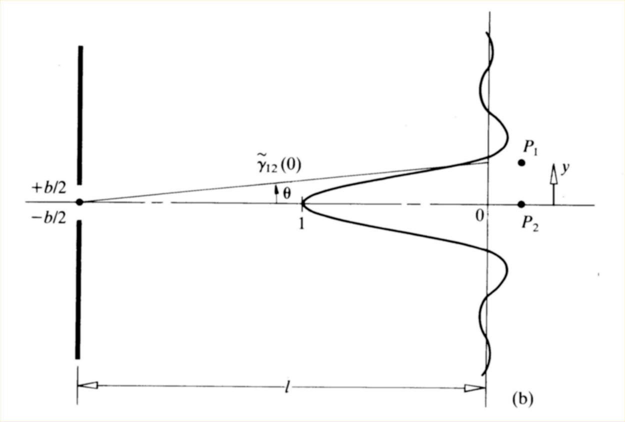
$$\tilde{\Gamma}(\vec{r}) = \int\int_{\text{source}} I_0(\vec{\Omega}) e^{2\pi i \vec{\Omega}.\vec{r}/\lambda} d\vec{\Omega}$$

- $\tilde{\Gamma}(\vec{r})$ at a certain point represents a *single* Fourier component (with baseline $\vec{r}$) of the intensity distribution of the source with strength $\tilde{\Gamma}(\vec{r})d\vec{r}$
- short baseline (small $|\vec{r}|$) corresponds to low spatial frequency component in brightness distribution $I(\theta_y, \theta_z)$, i.e. *coarse* structure
- large values of $|\vec{r}|$ correspond to *fine* structure in $I(\theta_y, \theta_z)$
- *diffraction limited* resolution in *aperture synthesis* is

$$|\vec{r}_{max}| = L_{max} \Longrightarrow \theta_{min} = \frac{\lambda}{2L_{max}}$$

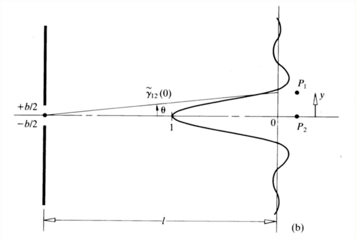
- factor 2 in denominator of expression for θ_{min} follows from rotation symmetry in aperture synthesis

Example: Uniform Slit



- coherence function $\tilde{\gamma}_{12}(0)$ for uniform slit source
- slit width *b*, running coordinate ξ, observation plane Σ, running coordinate *y*, located at large distance *l* from slit source (Fraunhofer limit)

Example: Uniform Slit (continued)



- source function is window function $\Pi\left(\frac{\xi}{b}\right)$
- in angular equivalent $\Pi\left(\frac{\beta}{\beta_0}\right)$, with $\beta_0 = b/l$
- application of Van Cittert-Zernike theorem $I(\vec{\Omega}) \Leftrightarrow \tilde{\Gamma}(\vec{r}/\lambda)$:

$$\Pi\left(\frac{\beta}{\beta_0}\right) \Leftrightarrow \beta_0 \text{sinc}\left(\frac{y\beta_0}{\lambda}\right) = \beta_0 \text{sinc}\left(\frac{yb}{\lambda l}\right)$$

$\text{sinc}(x) = \left(\frac{\sin \pi x}{\pi x}\right)$

Example: Uniform Slit (continued)

- from Van Cittert-Zernike theorem $I(\vec{\Omega}) \Leftrightarrow \tilde{\Gamma}(\vec{r}/\lambda)$:

$$\Pi\left(\frac{\beta}{\beta_0}\right) \Leftrightarrow \beta_0 \mathrm{sinc}\left(\frac{y\beta_0}{\lambda}\right) = \beta_0 \mathrm{sinc}\left(\frac{yb}{\lambda l}\right)$$

- modulus of normalised complex coherence function

$$|\tilde{\gamma}(y)| = \left|\frac{\beta_0 \mathrm{sinc}\frac{yb}{\lambda l}}{\beta_0}\right| = \left|\mathrm{sinc}\frac{yb}{\lambda l}\right| = V \Rightarrow \text{Visibility}$$

- enlarging b with a factor shrinks coherence function by same factor
- if brightness structure of radiation source covers wide range of angular scales from Δ down to δ, spatial coherence function shows finest detail of λ/Δ over maximum extent of $\approx \lambda/\delta$