# Lecture 8: Indirect Imaging 1

## Outline

## Motivation

- understand electromagnetic radiation and its propagation from astronomical sources to astronomical detectors
- understand how to analyse data from telescopes at all wavelengths
- different detection techniques at different wavelengths

### Electromagnetic Waves in Matter

- electromagnetic waves are a direct consequence of *Maxwell's equations*
- optics: interaction of electromagnetic waves with matter as described by *material equations*

## Maxwell's Equations in Matter

$$\nabla \cdot \vec{D} = 4\pi\rho$$

$$\nabla \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c}\vec{j}$$

$$\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{B} = 0$$

## Symbols

- $\vec{D}$ *electric displacement*
- $\rho$ *electric charge density*
- $\vec{H}$ *magnetic field vector*
- $c$ *speed of light in vacuum*
- $\vec{j}$ *electric current density*
- $\vec{E}$ *electric field vector*
- $\vec{B}$ *magnetic induction*
- $t$ time

## Linear Material Equations

$$\vec{D} = \epsilon\vec{E}$$

$$\vec{B} = \mu\vec{H}$$

$$\vec{j} = \sigma\vec{E}$$

## Symbols

- $\epsilon$ *dielectric constant*
- $\mu$ *magnetic permeability*
- $\sigma$ *electrical conductivity*

## Wave Equation in Matter

- static, homogeneous medium with no net charges ($\rho = 0$)
- for most materials $\mu = 1$
- combination of Maxwell's and material equations leads to differential equations for a damped (vector) wave

$$\nabla^2 \vec{E} - \frac{\mu\epsilon}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{4\pi\mu\sigma}{c^2}\frac{\partial \vec{E}}{\partial t} = 0$$

$$\nabla^2 \vec{H} - \frac{\mu\epsilon}{c^2}\frac{\partial^2 \vec{H}}{\partial t^2} - \frac{4\pi\mu\sigma}{c^2}\frac{\partial \vec{H}}{\partial t} = 0$$

- $\vec{E}$ and $\vec{H}$ are equivalent
- interaction with matter almost always through $\vec{E}$
- but: at interfaces, boundary conditions for $\vec{H}$ are crucial
- damping controlled by conductivity $\sigma$

## Plane-Wave Solutions

Plane Vector Wave ansatz

$$\vec{E} = \vec{E}_0 e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

- $\vec{k}$ spatially and temporally constant *wave vector*
- $\vec{k}$ normal to surfaces of constant phase
- $|\vec{k}|$ *wave number*
- $\vec{x}$ spatial location
- $\omega$ *angular frequency* ($2\pi\times$ frequency)
- $t$ time
- $\vec{E}_0$ a (generally complex) vector independent of time and space
  - damping if $\vec{k}$ is complex
  - real electric field vector given by real part of $\vec{E}$

## Complex index of refraction

- after doing temporal derivatives $\Rightarrow$ Helmholtz-equation

$$\nabla^2 \vec{E} + \frac{\omega^2 \mu}{c^2} \left( \epsilon + i \frac{4\pi\sigma}{\omega} \right) \vec{E} = 0,$$

- *dispersion relation* between $\vec{k}$ and $\omega$

$$\vec{k} \cdot \vec{k} = \frac{\omega^2 \mu}{c^2} \left( \epsilon + i \frac{4\pi\sigma}{\omega} \right)$$

- *complex index of refraction*

$$\tilde{n}^2 = \mu \left( \epsilon + i \frac{4\pi\sigma}{\omega} \right), \quad \vec{k} \cdot \vec{k} = \frac{\omega^2}{c^2} \tilde{n}^2$$

## Transverse Waves

- plane-wave solution must also fulfill Maxwell's equations

$$\vec{E}_0 \cdot \vec{k} = 0, \quad \vec{H}_0 \cdot \vec{k} = 0$$

$$\vec{H}_0 = \frac{\tilde{n}}{\mu} \frac{\vec{k}}{|\vec{k}|} \times \vec{E}_0$$

- isotropic media: electric, magnetic field vectors normal to wave vector $\Rightarrow$ transverse waves
- $\vec{E}_0$, $\vec{H}_0$, and $\vec{k}$ orthogonal to each other, right-handed vector-triple
- conductive medium $\Rightarrow$ complex $\tilde{n}$, $\vec{E}_0$ and $\vec{H}_0$ out of phase
- $\vec{E}_0$ and $\vec{H}_0$ have constant relationship $\Rightarrow$ consider only one of two fields

## Polarization

- spatially, temporally constant vector $\vec{E}_0$ lays in plane perpendicular to propagation direction $\vec{k}$
- represent $\vec{E}_0$ in 2-D basis, unit vectors $\vec{e}_1$ and $\vec{e}_2$, both perpendicular to $\vec{k}$

$$\vec{E}_0 = E_1 \vec{e}_1 + E_2 \vec{e}_2.$$

  $E_1$, $E_2$: arbitrary complex scalars
- damped plane-wave solution with given $\omega$, $\vec{k}$ has 4 degrees of freedom (two complex scalars)
- additional property is called *polarization*
- many ways to represent these four quantities

## Scalar Wave

- electric vector of wave field at position $\vec{r}$ at time $t$ is $\tilde{E}(\vec{r}, t)$
- complex notation to easily express amplitude and phase
- real part of complex quantity is the physical quantity

## Quasi-Monochromatic Light

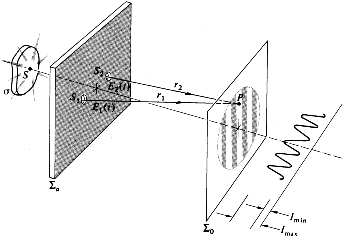
- monochromatic light: purely theoretical concept
- monochromatic light wave always fully polarized
- real life: light includes range of wavelengths $\Rightarrow$ *quasi-monochromatic light*
- quasi-monochromatic: superposition of mutually incoherent monochromatic light beams whose wavelengths vary in narrow range $\delta\lambda$ around central wavelength $\lambda_0$

$$\frac{\delta\lambda}{\lambda} \ll 1$$

- measurement of quasi-monochromatic light: integral over measurement time $t_m$
- amplitude, phase (slow) functions of time for given spatial location
- *slow*: variations occur on time scales much longer than the mean period of the wave

# Interference

## Young's Double Slit Experiment



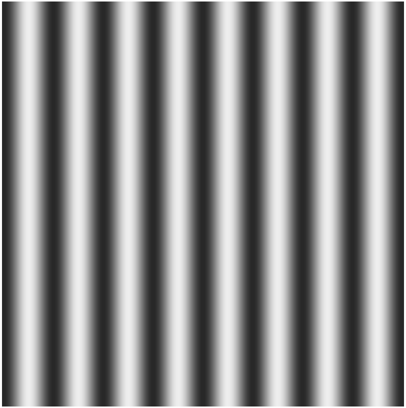

- monochromatic wave
- infinitely small holes (pinholes)
- source $S$ generates fields $\tilde{E}(\vec{r}_1, t) \equiv \tilde{E}_1(t)$ at $S_1$ and $\tilde{E}(\vec{r}_2, t) \equiv \tilde{E}_2(t)$ at $S_2$
- two spherical waves from pinholes interfere on screen
- electrical field at $P$

$$\tilde{E}_P(t) = \tilde{C}_1 \tilde{E}_1(t - t_1) + \tilde{C}_2 \tilde{E}_2(t - t_2)$$

- $t_1 = r_1/c$, $t_2 = r_2/c$
- $r_1$, $r_2$: path lengths to $P$ from $S_1$, $S_2$
- propagators $\tilde{C}_{1,2} = \frac{i}{\lambda}$

## no tilt

## tilt by 0.5 $\lambda/d$



## Change in Angle of Incoming Wave

- phase of fringe pattern changes, but not fringe spacing
- tilt of $\lambda/d$ produces identical fringe pattern

### long wavelength

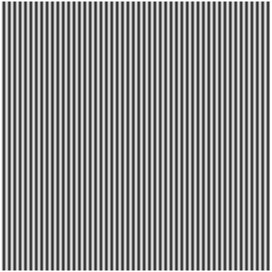
### short wavelength

### wavelength average

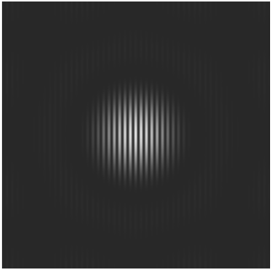
### Change in Wavelength

- fringe spacing changes, central fringe broadens
- integral over 0.8 to 1.2 of central wavelength
- intergral over wavelength makes fringe envelope

## 2 pinholes



## 2 small holes



## 2 large holes

## Finite Hole Diameter

- fringe spacing only depends on separation of holes and wavelength
- the smaller the hole, the larger the ’illuminated’ area
- fringe envelope is an Aity pattern (diffraction pattern of a single hole)

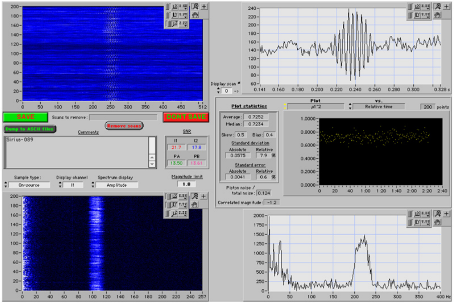
## Visibility

- "quality" of fringes described by **Visibility function**

$$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$$

- $I_{\max}, I_{\min}$ are maximum and adjacent minimum in fringe pattern

## First Fringes from VLT Interferometer




'First Fringes' from Sirius with VLTI

ESO PR Photo 10a/01 (18 March 2001 )

## Mutual Coherence

- total field in point $P$

$$\tilde{E}_P(t) = \tilde{C}_1\tilde{E}_1(t-t_1) + \tilde{C}_2\tilde{E}_2(t-t_2)$$

- irradiance at $P$, averaged over time

$$I = \mathbf{E}|\tilde{E}_P(t)|^2 = \mathbf{E}\left\{\tilde{E}_P(t)\tilde{E}_P^*(t)\right\}$$

- writing out all the terms

$$\begin{aligned} I = \quad & \tilde{C}_1\tilde{C}_1^*\mathbf{E}\left\{\tilde{E}_1(t-t_1)\tilde{E}_1^*(t-t_1)\right\} \\ & +\tilde{C}_2\tilde{C}_2^*\mathbf{E}\left\{\tilde{E}_2(t-t_2)\tilde{E}_2^*(t-t_2)\right\} \\ & +\tilde{C}_1\tilde{C}_2^*\mathbf{E}\left\{\tilde{E}_1(t-t_1)\tilde{E}_2^*(t-t_2)\right\} \\ & +\tilde{C}_1^*\tilde{C}_2\mathbf{E}\left\{\tilde{E}_1^*(t-t_1)\tilde{E}_2(t-t_2)\right\} \end{aligned}$$

## Mutual Coherence (continued)

- as before

$$\begin{aligned} I = \quad & \tilde{C}_1\tilde{C}_1^*\mathbf{E}\left\{\tilde{E}_1(t-t_1)\tilde{E}_1^*(t-t_1)\right\} \\ & +\tilde{C}_2\tilde{C}_2^*\mathbf{E}\left\{\tilde{E}_2(t-t_2)\tilde{E}_2^*(t-t_2)\right\} \\ & +\tilde{C}_1\tilde{C}_2^*\mathbf{E}\left\{\tilde{E}_1(t-t_1)\tilde{E}_2^*(t-t_2)\right\} \\ & +\tilde{C}_1^*\tilde{C}_2\mathbf{E}\left\{\tilde{E}_1^*(t-t_1)\tilde{E}_2(t-t_2)\right\} \end{aligned}$$

- *stationary* wave field, time average independent of absolute time

$$I_{S_1} = \mathbf{E}\left\{\tilde{E}_1(t)\tilde{E}_1^*(t)\right\} \text{ and } I_{S_2} = \mathbf{E}\left\{\tilde{E}_2(t)\tilde{E}_2^*(t)\right\}$$

- irradiance at $P$ is now

$$\begin{aligned} I = \quad & \tilde{C}_1\tilde{C}_1^* I_{S_1} + \tilde{C}_2\tilde{C}_2^* I_{S_2} + \tilde{C}_1\tilde{C}_2^*\mathbf{E}\left\{\tilde{E}_1(t-t_1)\tilde{E}_2^*(t-t_2)\right\} \\ & +\tilde{C}_1^*\tilde{C}_2\mathbf{E}\left\{\tilde{E}_1^*(t-t_1)\tilde{E}_2(t-t_2)\right\} \end{aligned}$$

## Mutual Coherence (continued)

- as before

$$I = \tilde{C}_1 \tilde{C}_1^* I_{S_1} + \tilde{C}_2 \tilde{C}_2^* I_{S_2} + \tilde{C}_1 \tilde{C}_2^* \mathbf{E}\left\{\tilde{E}_1(t-t_1)\tilde{E}_2^*(t-t_2)\right\} + \tilde{C}_1^* \tilde{C}_2 \mathbf{E}\left\{\tilde{E}_1^*(t-t_1)\tilde{E}_2(t-t_2)\right\}$$

- with time difference $\tau = t_2 - t_1$ last two terms become

$$\tilde{C}_1 \tilde{C}_2^* \mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_2^*(t)\right\} + \tilde{C}_1^* \tilde{C}_2 \mathbf{E}\left\{\tilde{E}_1^*(t+\tau)\tilde{E}_2(t)\right\}$$

- this is equivalent to

$$2\ Re\left[\tilde{C}_1 \tilde{C}_2^* \mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_2^*(t)\right\}\right]$$

- propagators $\tilde{C}$ purely imaginary: $\tilde{C}_1 \tilde{C}_2^* = \tilde{C}_1^* \tilde{C}_2 = |\tilde{C}_1||\tilde{C}_2|$
- cross-term becomes

$$2|\tilde{C}_1||\tilde{C}_2| Re\left[\mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_2^*(t)\right\}\right]$$

## Mutual Coherence (continued)

- irradiance at point $P$ is now

$$I = \tilde{C}_1 \tilde{C}_1^* I_{S_1} + \tilde{C}_2 \tilde{C}_2^* I_{S_2} + 2|\tilde{C}_1||\tilde{C}_2| Re\left[\mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_2^*(t)\right\}\right]$$

- **mutual coherence function** of wave field at $S_1$ and $S_2$

$$\tilde{\Gamma}_{12}(\tau) = \mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_2^*(t)\right\}$$

- irradiance at point $P$ becomes

$$I = |\tilde{C}_1|^2 I_{S_1} + |\tilde{C}_2|^2 I_{S_2} + 2|\tilde{C}_1||\tilde{C}_2|\ Re\,\tilde{\Gamma}_{12}(\tau)$$

- $I_1 = |\tilde{C}_1|^2 I_{S_1}$, $I_2 = |\tilde{C}_2|^2 I_{S_2}$ are irradiances at $P$ from a single aperture

$$I = I_1 + I_2 + 2|\tilde{C}_1||\tilde{C}_2|\ Re\,\tilde{\Gamma}_{12}(\tau)$$

## Self-Coherence

- if $S_1 = S_2$, mutual coherence function becomes autocorrelation function

$$\tilde{\Gamma}_{11}(\tau) = \tilde{R}_1(\tau) = \mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_1^*(t)\right\}$$
$$\tilde{\Gamma}_{22}(\tau) = \tilde{R}_2(\tau) = \mathbf{E}\left\{\tilde{E}_2(t+\tau)\tilde{E}_2^*(t)\right\}$$

- autocorrelation functions are also called *self-coherence functions*
- for $\tau = 0$

$$I_{S_1} = \mathbf{E}\left\{\tilde{E}_1(t)\tilde{E}_1^*(t)\right\} = \Gamma_{11}(0) = \mathbf{E}\left\{|\tilde{E}_1(t)|^2\right\}$$
$$I_{S_2} = \mathbf{E}\left\{\tilde{E}_2(t)\tilde{E}_2^*(t)\right\} = \Gamma_{22}(0) = \mathbf{E}\left\{|\tilde{E}_2(t)|^2\right\}$$

- autocorrelation function with zero lag ($\tau = 0$) represent (average) irradiance (power) of wave field at $S_1$, $S_2$

## Complex Degree of Coherence

- using selfcoherence functions

$$|\tilde{C}_1||\tilde{C}_2| = \frac{\sqrt{I_1}\sqrt{I_2}}{\sqrt{\Gamma_{11}(0)}\sqrt{\Gamma_{22}(0)}}$$

- normalized mutual coherence defines the **complex degree of coherence**

$$\tilde{\gamma}_{12}(\tau) \equiv \frac{\tilde{\Gamma}_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}} = \frac{\mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_2^*(t)\right\}}{\sqrt{\mathbf{E}\left\{|\tilde{E}_1(t)|^2\right\}\mathbf{E}\left\{|\tilde{E}_2(t)|^2\right\}}}$$

- irradiance in point *P* as *general interference law for a partially coherent radiation field*

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\, Re\, \tilde{\gamma}_{12}(\tau)$$

## Spatial and Temporal Coherence

- complex degree of coherence

$$\tilde{\gamma}_{12}(\tau) \equiv \frac{\tilde{\Gamma}_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}} = \frac{\mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_2^*(t)\right\}}{\sqrt{\mathbf{E}\left\{|\tilde{E}_1(t)|^2\right\}\mathbf{E}\left\{|\tilde{E}_2(t)|^2\right\}}}$$

- measures both
  - *spatial coherence* at $S_1$ and $S_2$
  - *temporal coherence* through time lag $\tau$
- $\tilde{\gamma}_{12}(\tau)$ is a complex variable and can be written as:

$$\tilde{\gamma}_{12}(\tau) = |\tilde{\gamma}_{12}(\tau)|e^{i\psi_{12}(\tau)}$$

- $0 \leq |\tilde{\gamma}_{12}(\tau)| \leq 1$
- phase angle $\psi_{12}(\tau)$ relates to
  - phase angle between fields at $S_1$ and $S_2$
  - phase angle difference in $P$ resulting in time lag $\tau$

## Coherence of Quasi-Monochromatic Light

- quasi-monochromatic light, mean wavelength $\overline{\lambda}$, frequency $\overline{\nu}$, phase difference $\phi$ due to optical path difference:

$$\phi = \frac{2\pi}{\overline{\lambda}}(r_2 - r_1) = \frac{2\pi}{\overline{\lambda}}c(t_2 - t_1) = 2\pi\overline{\nu}\tau$$

- with phase angle $\alpha_{12}(\tau)$ between fields at pinholes $S_1$, $S_2$

$$\psi_{12}(\tau) = \alpha_{12}(\tau) - \phi$$

- and

$$Re\,\tilde{\gamma}_{12}(\tau) = |\tilde{\gamma}_{12}(\tau)| \cos\left[\alpha_{12}(\tau) - \phi\right]$$

- intensity in $P$ becomes

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\,|\tilde{\gamma}_{12}(\tau)| \cos\left[\alpha_{12}(\tau) - \phi\right]$$

## Visibility of Quasi-Monochromatic, Partially Coherent Light

- intensity in $P$

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\,|\tilde{\gamma}_{12}(\tau)|\cos\left[\alpha_{12}(\tau) - \phi\right]$$

- maximum, minimum $I$ for $\cos(...) = \pm 1$
- visibility $V$ at position $P$

$$V = \frac{2\sqrt{I_1}\sqrt{I_2}}{I_1 + I_2}|\tilde{\gamma}_{12}(\tau)|$$

- for $I_1 = I_2 = I_0$

$$\begin{aligned} I &= 2I_0\left\{1 + |\tilde{\gamma}_{12}(\tau)|\cos\left[\alpha_{12}(\tau) - \phi\right]\right\} \\ V &= |\tilde{\gamma}_{12}(\tau)| \end{aligned}$$

## Interpretation of Visibility

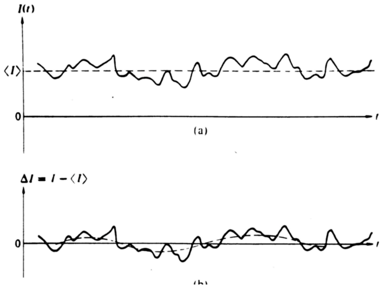
- for $I_1 = I_2 = I_0$

$$\begin{aligned} I &= 2I_0 \left\{1 + |\tilde{\gamma}_{12}(\tau)| \cos\left[\alpha_{12}(\tau) - \phi\right]\right\} \\ V &= |\tilde{\gamma}_{12}(\tau)| \end{aligned}$$

- *modulus of complex degree of coherence = visibility of fringes*
- modulus can therefore be measured
- shift in location of central fringe (no optical path length difference, $\phi = 0$) is measure of $\alpha_{12}(\tau)$
- measurements of visibility and fringe position yield amplitude and phase of complex degree of coherence

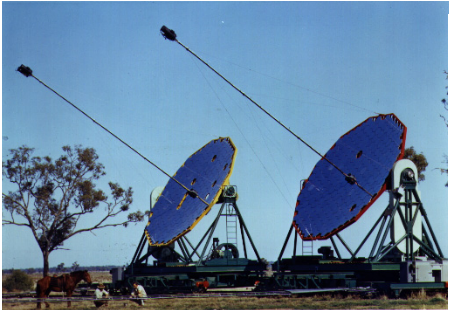
# Intensity Interferometry

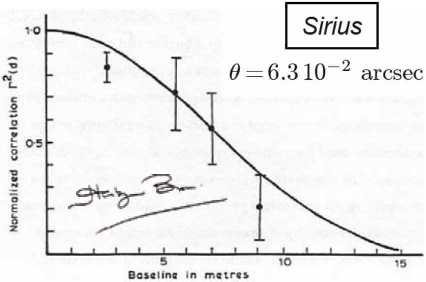
## Basic Idea




- intensity fluctuations are also partially coherent
- random superposition of wave packets results in Gaussian distribution of amplitude fluctuations
- intensity cross-correlation $\mathbf{E}\{\triangle I_1(t+\tau)\triangle I_2(t)\}$ between two different parts of incoming wave is not zero
- implies correlation between photons from thermal sources

# Hanbury Brown and Twiss






- intensity interferometry can measure stellar diameters
- diameter of Sirius in 1956 by Hanbury Brown and Twiss
- only light buckets are needed to collect photons
- basic principle was doubted by many
- classical explanation (see exercises)
- quantum-mechanical explanation: “photon bunching”, Bose-Einstein statistics makes photons to be bunched together