Recap lecture 4 + Frits Paerels

thermal limit of stochastic radiation processes

$h\nu << kT$ Thermal limit $\overline{\Delta P^2}(\nu) = \bar{P}^2(\nu)$

Paerels showed that one doesn't always get the exact Poisson quantum noise at $h\nu >> kT$

Use incomplete gamma function to calculate Poisson and Gauss cumulative distribution function

Propagation of errors under the assumption of independent variables

$$\bar{f} = f(\bar{u}, \bar{v}, ..)$$

$$\sigma_f^2 = \sigma_u^2 (\frac{\partial f}{\partial u})^2 + \sigma_v^2 (\frac{\partial f}{\partial v})^2 + ...$$

Today:

Comparing data with a model:
Least-squares fitting, maximum likelihood method: Gaussian data

"real" maximum likelihood method: Poissonian data

OAF2 chapter 5.3+5.4
see also Num Res Chapter 15

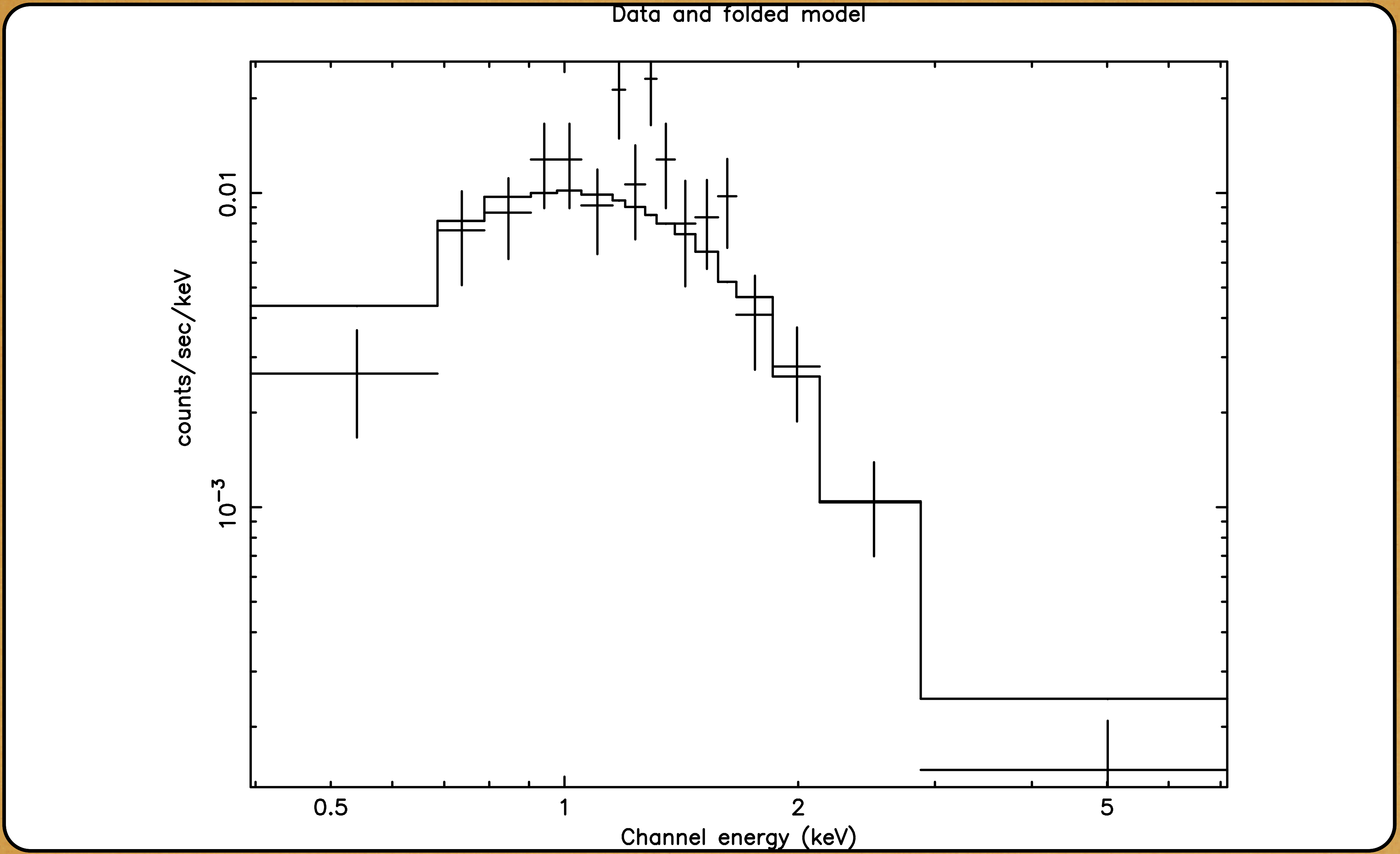
Compare data with a model

describe data in terms of a continuous function

Compare observations (data) with theoretical model prediction

Describe the data in a few parameters

Example: continuous model through discrete data & model prediction



X-ray binary in quiescence neutron star atmosphere model

Maximum likelihood: most likely outcome is assumed to be the 'correct' one

Method of least squares

$$dQ_i = P_i dx$$

Probability density function $\frac{dQ_i}{dx} = P_i$

$\hookrightarrow$ Gauss, Poisson

$$P(y_i)\Delta y = \frac{1}{\sqrt{2\pi}\sigma_i} exp(-\frac{1}{2}\frac{(y_i - y_m)^2}{\sigma_i^2})\Delta y$$

note: y_m = model value not mean here!

Method of least squares

$$P \propto \prod_{i=1}^{N} \{exp[-\frac{1}{2}(\frac{y_i - y_m}{\sigma_i})^2]\}$$

$$\propto exp[-\frac{1}{2}\sum_{i=1}^{N}(\frac{y_i - y_m}{\sigma_i})^2]\}$$

MINIMISE: $\chi^2 \equiv \sum_{i=1}^{N}(\frac{y_i - y_m}{\sigma_i})^2$

MINIMISATION: ROOT FINDING PROBLEM

1D: $\frac{\partial}{\partial y_i}\sum_{i=1}^{N}(\frac{y_i - y_m}{\sigma_i})^2 = 0$

more about χ^2

drawn from normal distribution

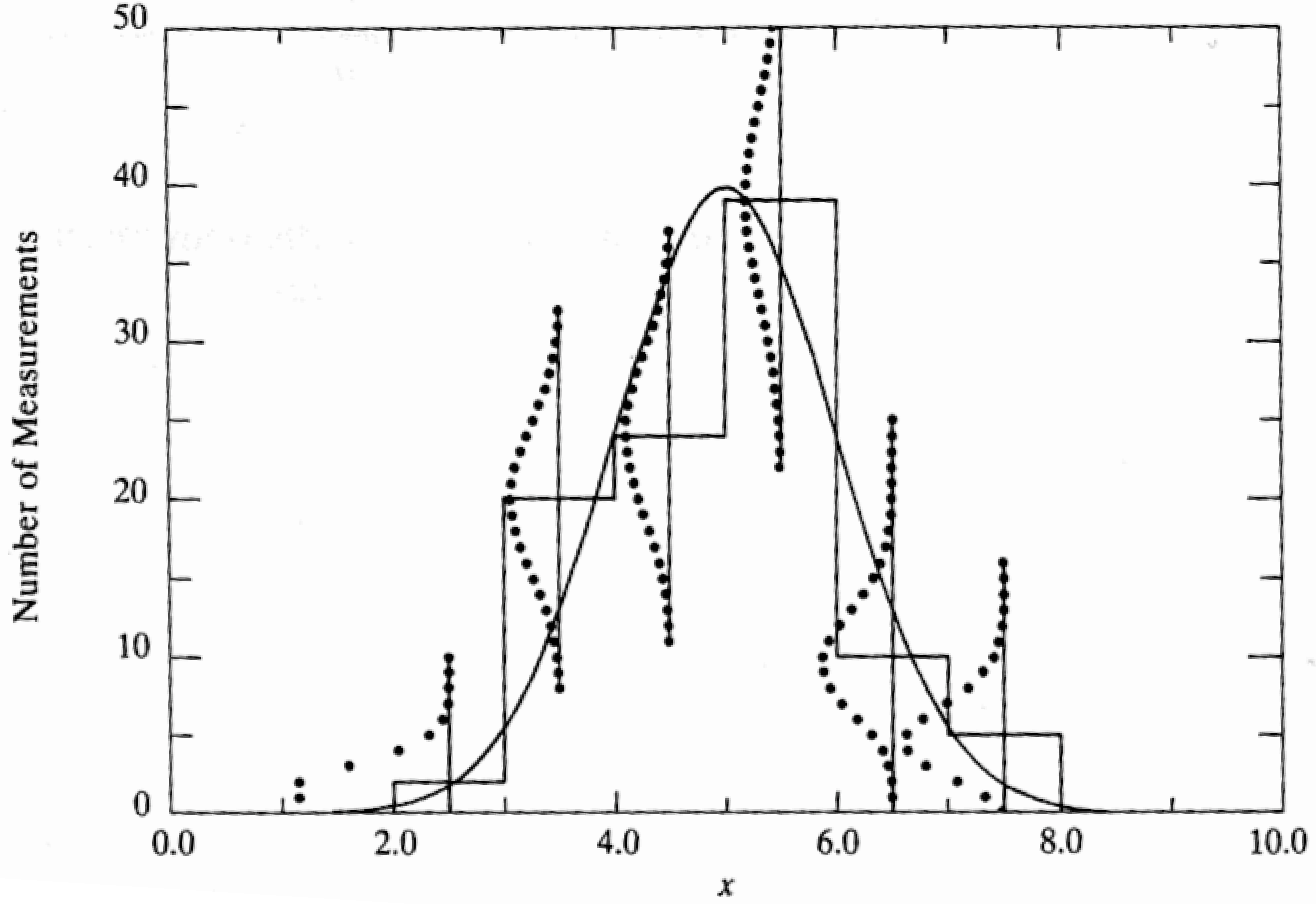
distribution of χ_i^2 is a χ^2 distribution

for N measurements described by M variables, there are N-M Degrees of Freedom (d.o.f.)

Probability of obtaining a certain χ^2

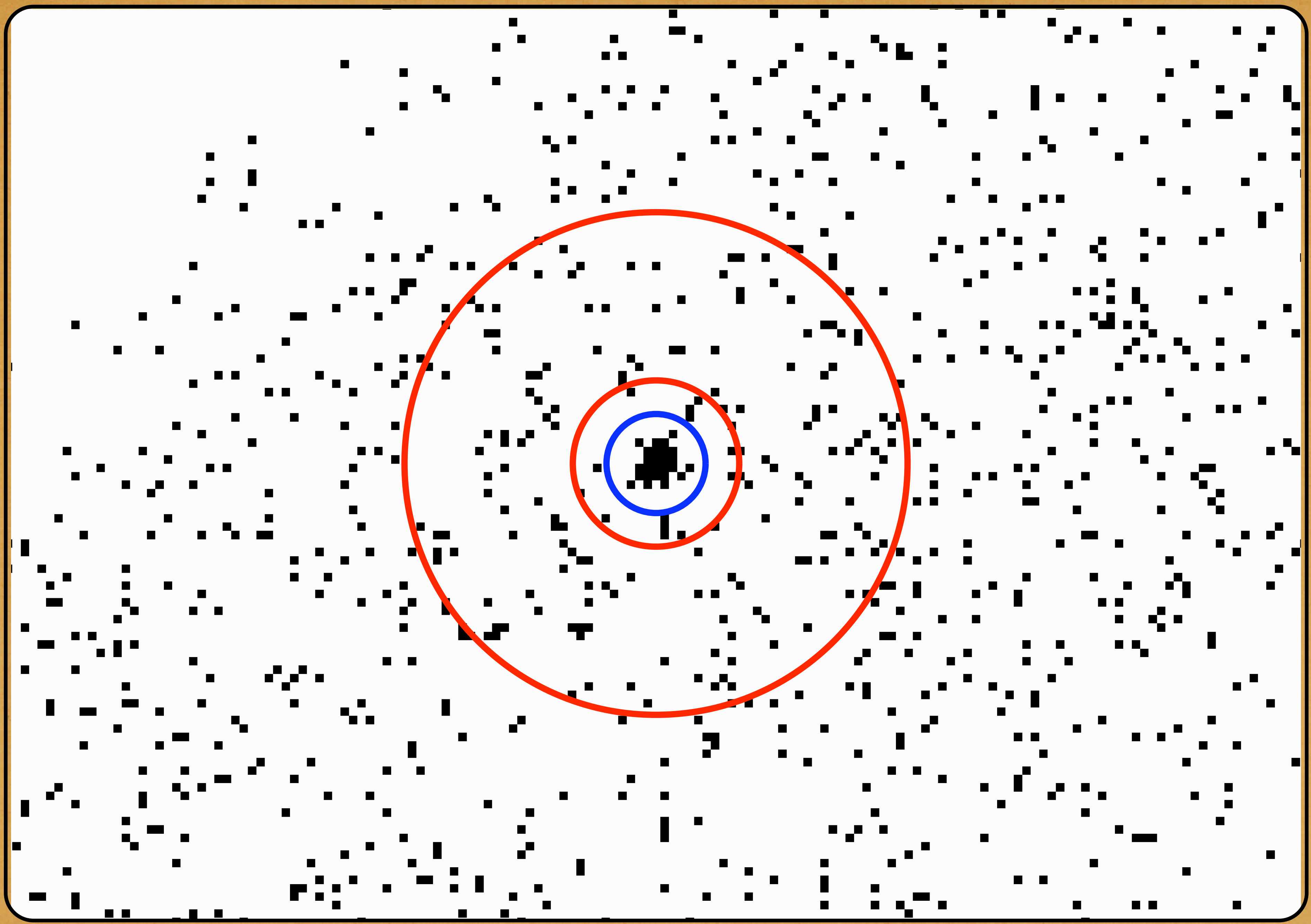
$$P(\chi^2_{obs}) = \mathrm{gammq}(\frac{N-M}{2}, \frac{\chi^2_{obs}}{2})$$

Value and Poisson errors

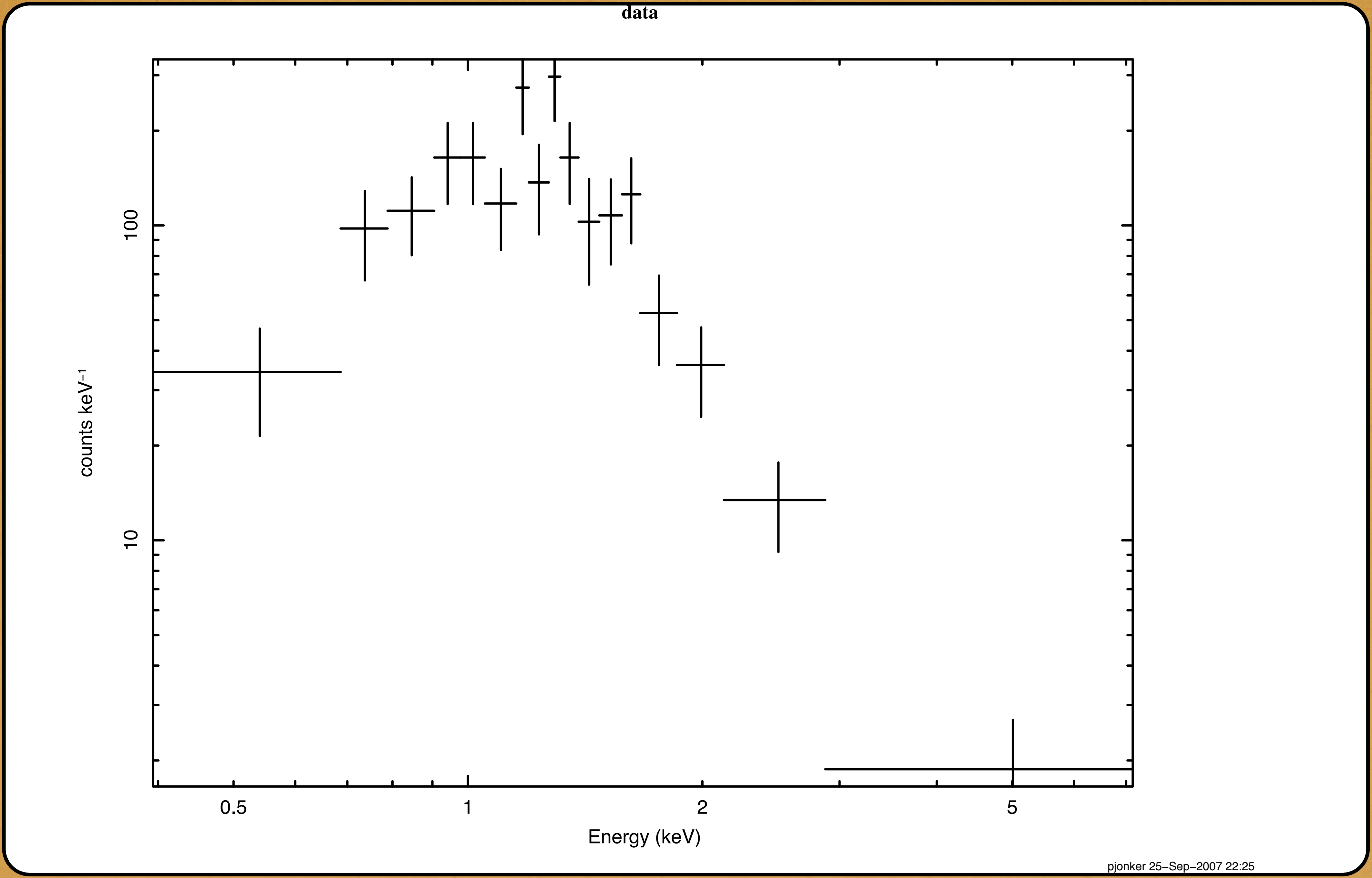


Example from Bevington & Robertson 1992

Chandra CCD (ACIS) observation of an X-ray binary



Same data as before



Gaussian approximation for errors but at low counts Gauss and Poisson errors differ

χ^2 FITTING PROVIDES:

BEST-FITTING PARAMETERS

AN ERROR ESTIMATE OF THE UNCERTAINTY OF THE FITTED PARAMETERS

A PROBABILITY THAT THE DATA IS DRAWN FROM A PARENT POPULATION DESCRIBED BY THE MODEL PARAMETERS

Fitting a straight line to the data

(see also exercise)

$$y_m(x_i, a, b) = a + bx_i$$

minimise χ_i^2 to find best-fitting parameters

$$\frac{\partial \sum_{i=1}^{N} (\frac{y_i - a - bx_i}{\sigma_i})^2}{\partial a} = 0 \quad \Rightarrow$$

$$\sum \frac{y_i}{\sigma_i^2} - a \sum \frac{1}{\sigma_i^2} - b \sum \frac{x_i}{\sigma_i^2} = 0$$

&

$$\sum \frac{x_i y_i}{\sigma_i^2} - a \sum \frac{x_i}{\sigma_i^2} - b \sum \frac{x_i^2}{\sigma_i^2} = 0$$

$$a = \frac{\sum_i \frac{x_i^2}{\sigma_i^2} \sum_i \frac{y_i}{\sigma_i^2} - \sum_i \frac{x_i}{\sigma_i^2} \sum_i \frac{x_i y_i}{\sigma_i^2}}{\sum_i \frac{1}{\sigma_i^2} \sum_i \frac{x_i^2}{\sigma_i^2} - (\sum_i \frac{x_i}{\sigma_i^2})^2}$$

DETERMINE ERRORS ON THE BEST-FITTING PARAMETERS

REMEMBER $$\sigma_f^2 = \sigma_u^2 (\frac{\partial f}{\partial u})^2 + \sigma_v^2 (\frac{\partial f}{\partial v})^2 + ...$$

$$\sigma_a^2 = \sum_{i=1}^{N} [\sigma_i^2 \frac{\partial a}{\partial y_i}]^2$$

$\partial u \,\&\, \partial v$ ETC ARE THE DIFFERENT MEASUREMENT VALUES y_i

$$\sigma_a^2 = \frac{\sum_i \frac{x_i^2}{\sigma_i^2}}{\sum_i \frac{1}{\sigma_i^2} \sum_i \frac{x_i^2}{\sigma_i^2} - (\sum_i \frac{x_i}{\sigma_i^2})^2}$$

SIMILARLY

$$\sigma_b^2 = \frac{\sum_i \frac{1}{\sigma_i^2}}{\sum_i \frac{1}{\sigma_i^2} \sum_i \frac{x_i^2}{\sigma_i^2} - (\sum_i \frac{x_i}{\sigma_i^2})^2}$$

Finally calculate the probability of obtaining the χ^2 by chance

$$P(\chi^2_{obs}) = \mathrm{gammq}(\frac{N-M}{2}, \frac{\chi^2_{obs}}{2})$$

for the straight line fit M=2

$\nu = N - M$ degrees of freedom

reduced χ^2_ν

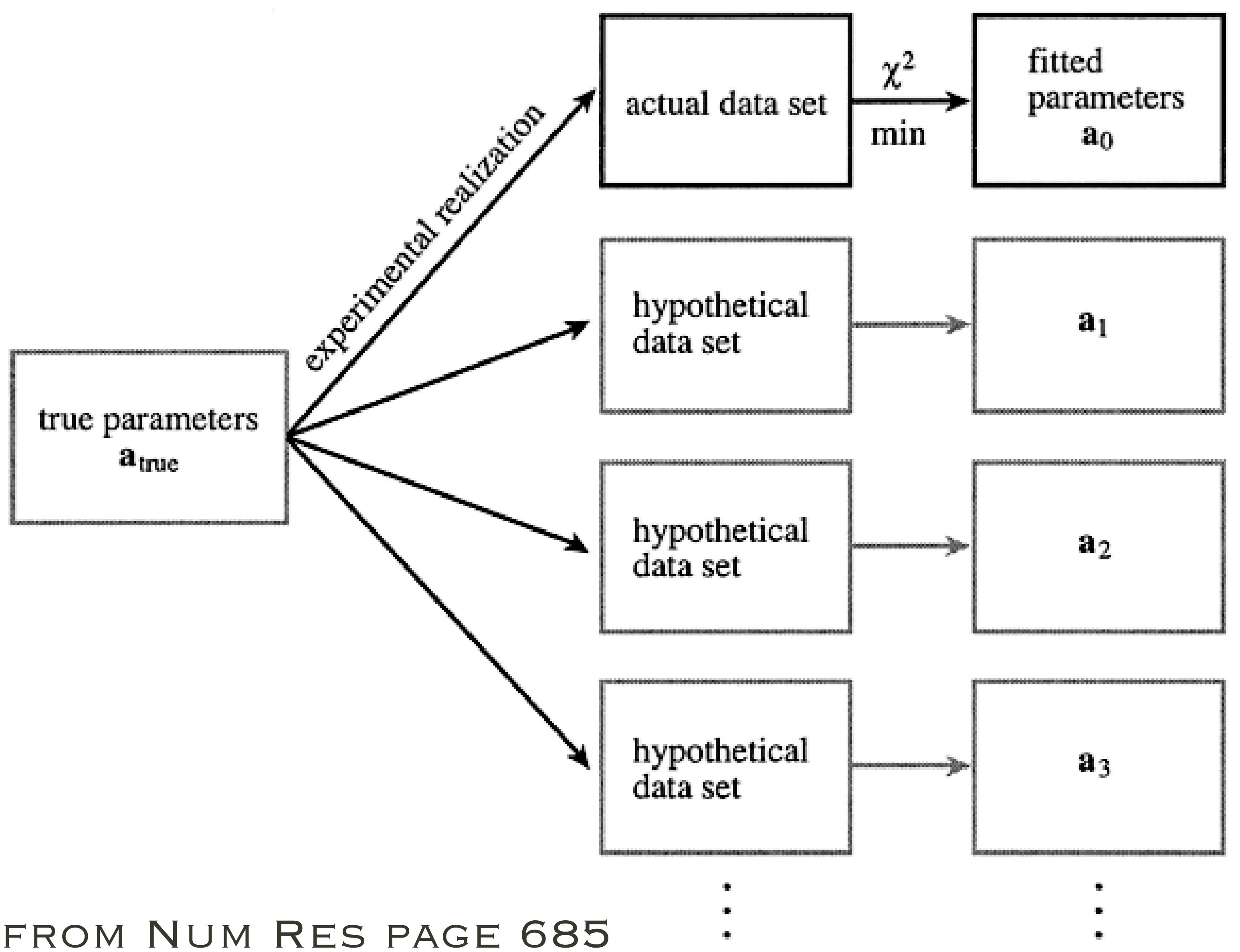
$$\chi^2_\nu \equiv \frac{\chi^2}{\nu}$$

for data fitting: $\chi^2_\nu \sim 1$ $\chi^2 \approx \nu$

$$\sigma_{\chi^2} = \sqrt{2\nu}$$

Estimating confidence limits

Monte Carlo simulations



from Num Res page 685

$\vec{a}_0$

} one draw from the distribution of a's

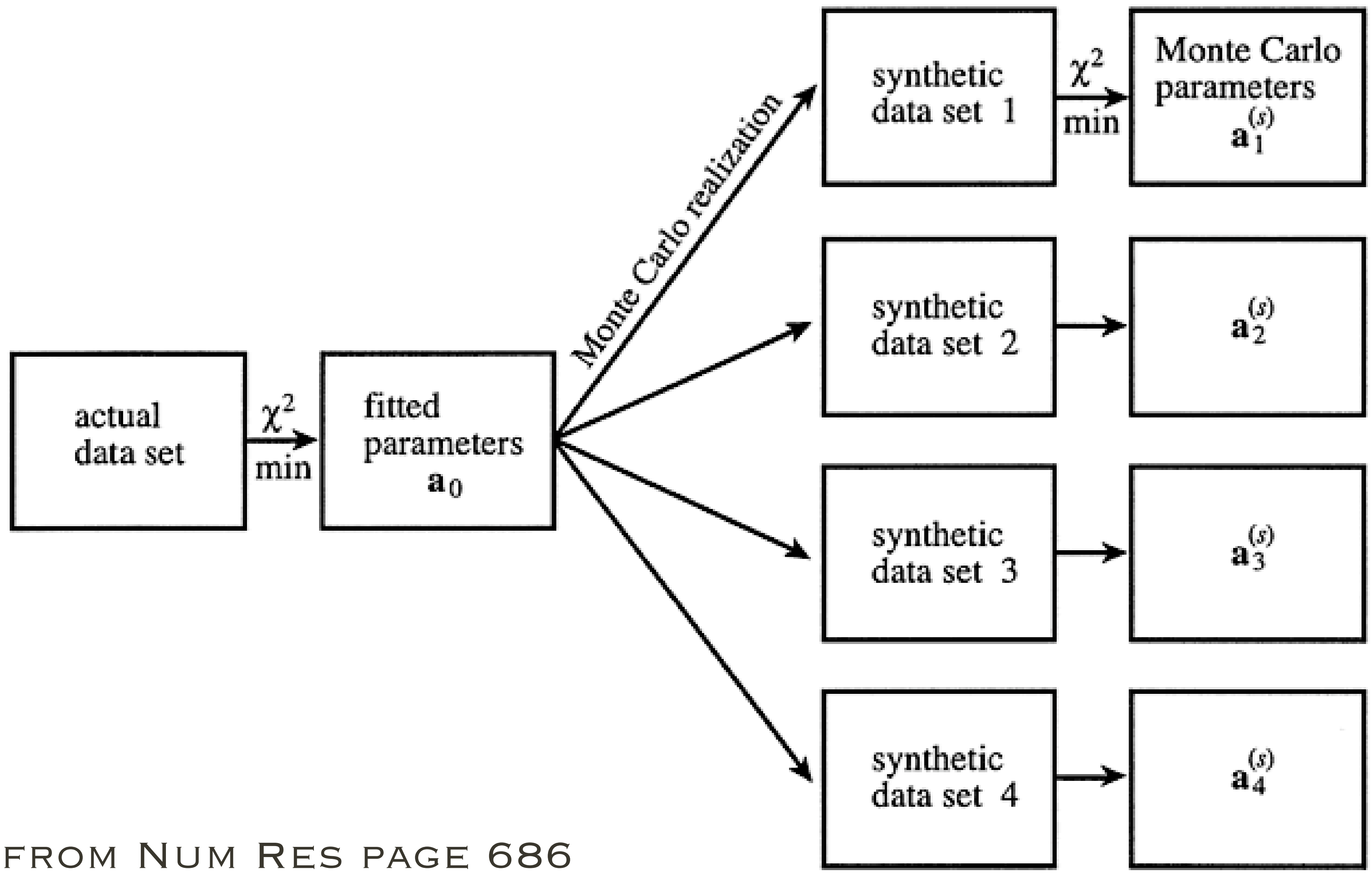
assume that the distribution of

$$a_i - a_0$$

is close to the probability distribution

$$a_i - a_{true}$$

Monte Carlo simulations



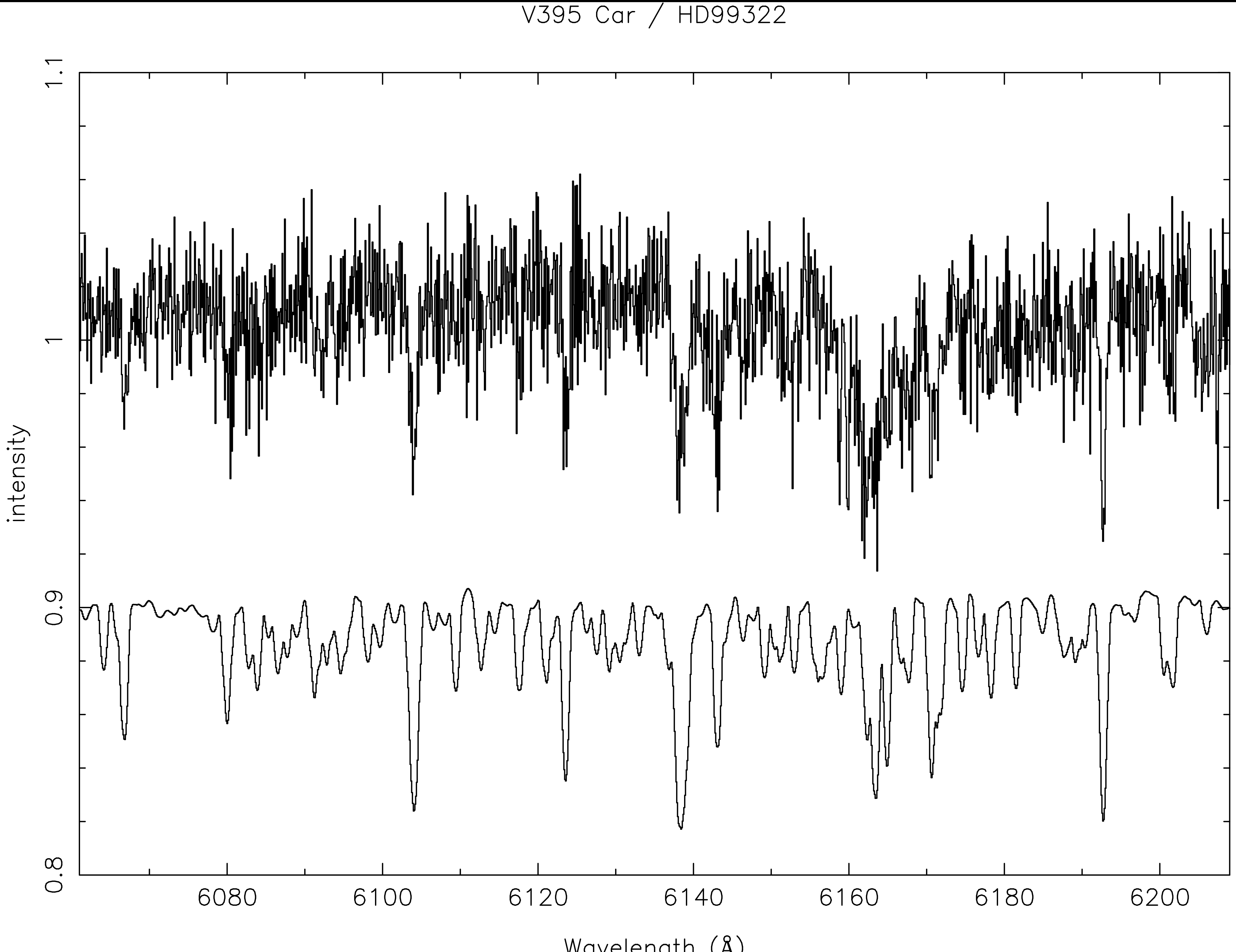
from Num Res page 686

calculate distribution of $a_i - a_0$

by simulating many sets of data and using χ^2 fitting to determine a_i

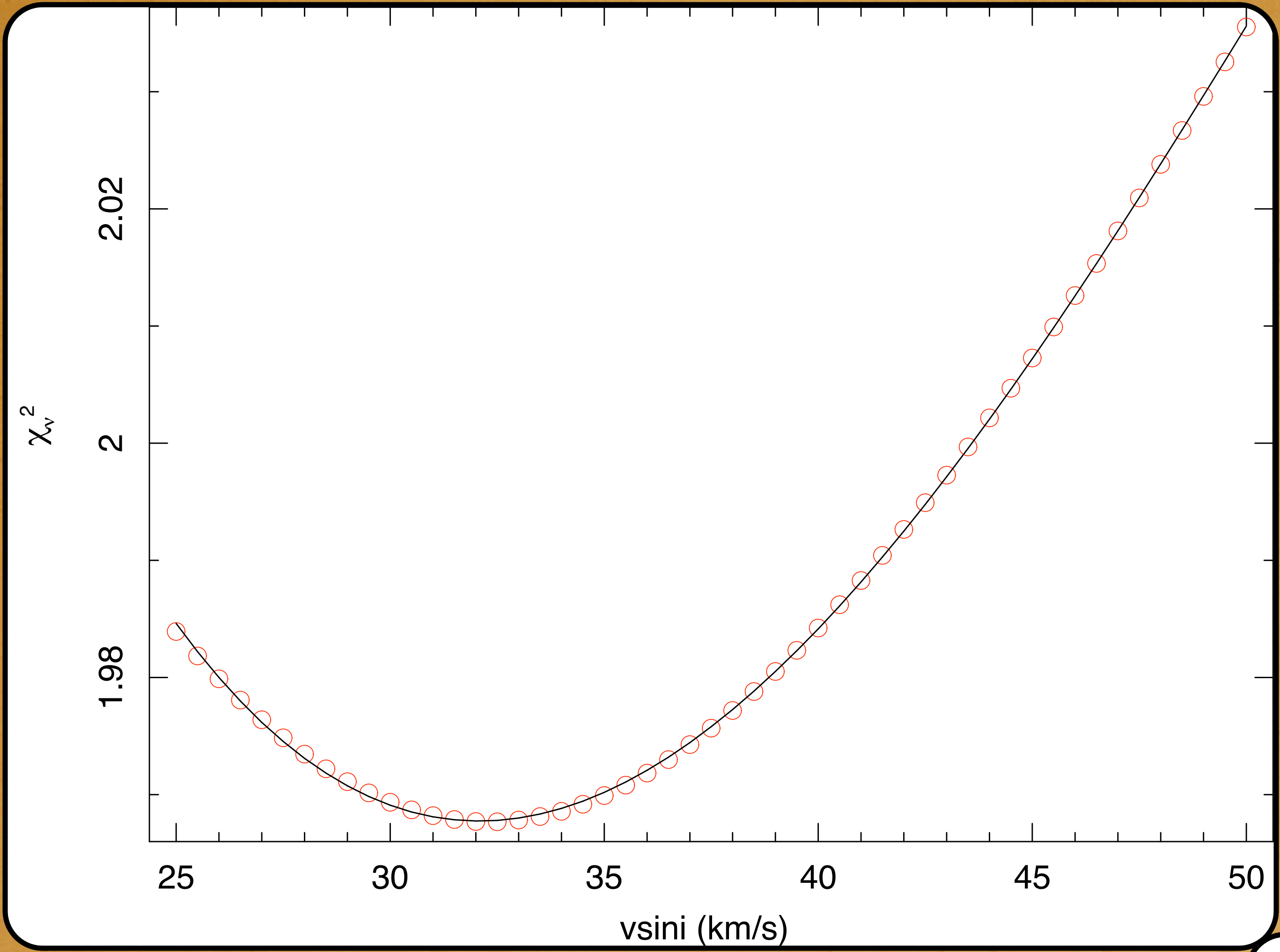
BOOTSTRAP METHOD AND APPLICATION

X-RAY BINARY V395 CAR



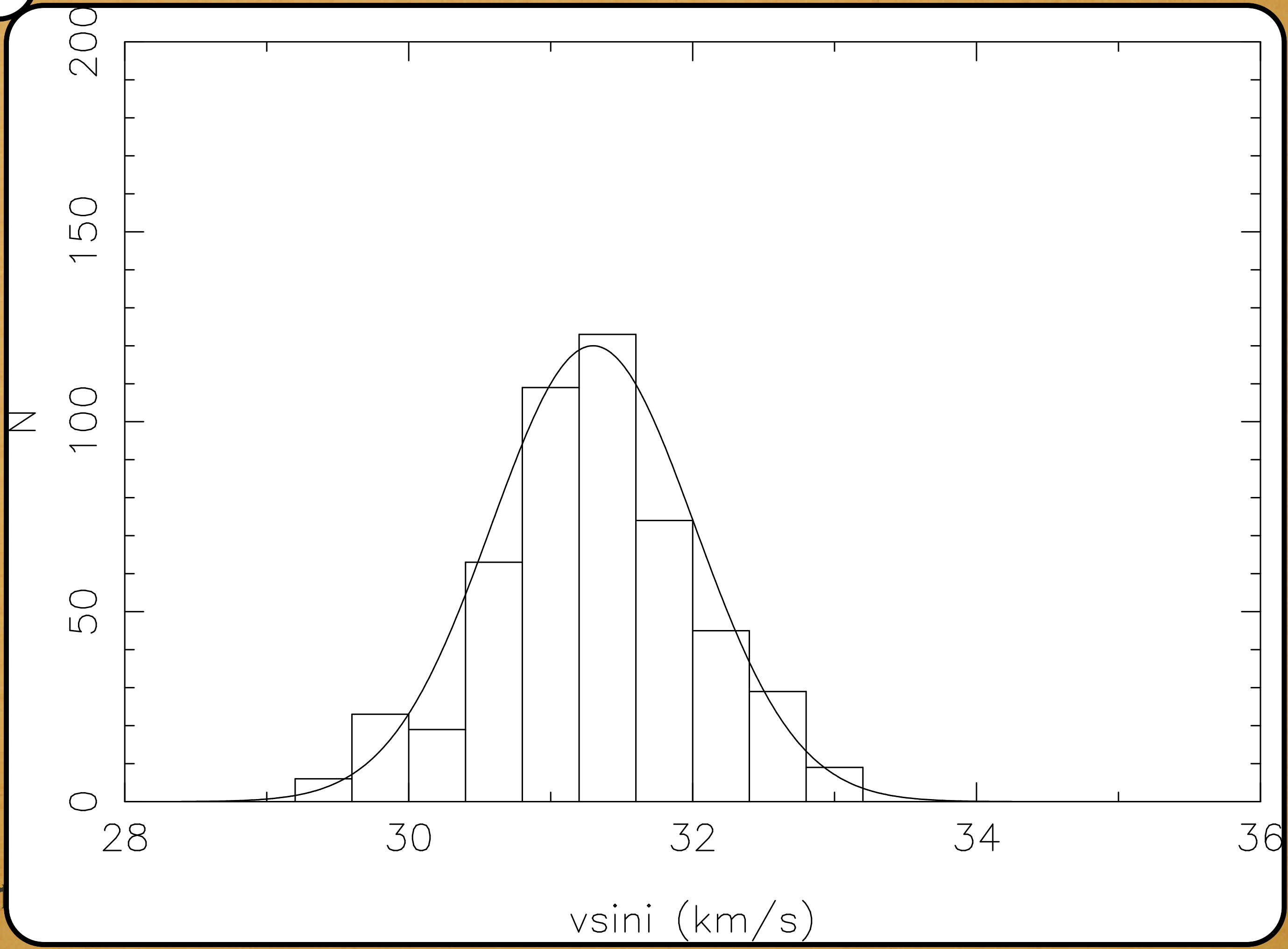
from Steeghs & Jonker 2007

Broadening and optimal subtraction

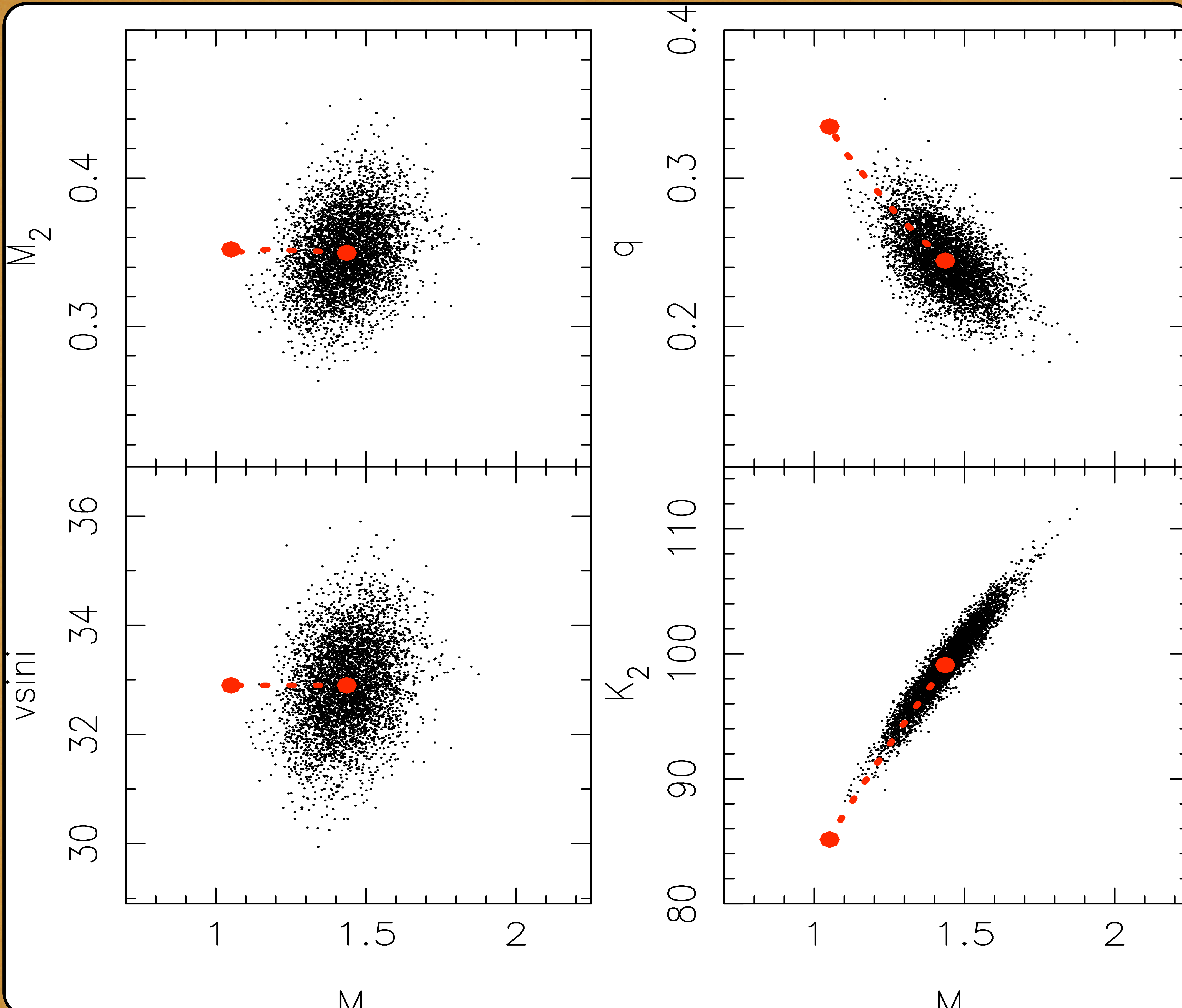


NOTE REDUCED χ_ν^2

Bootstrap determined rotational velocity

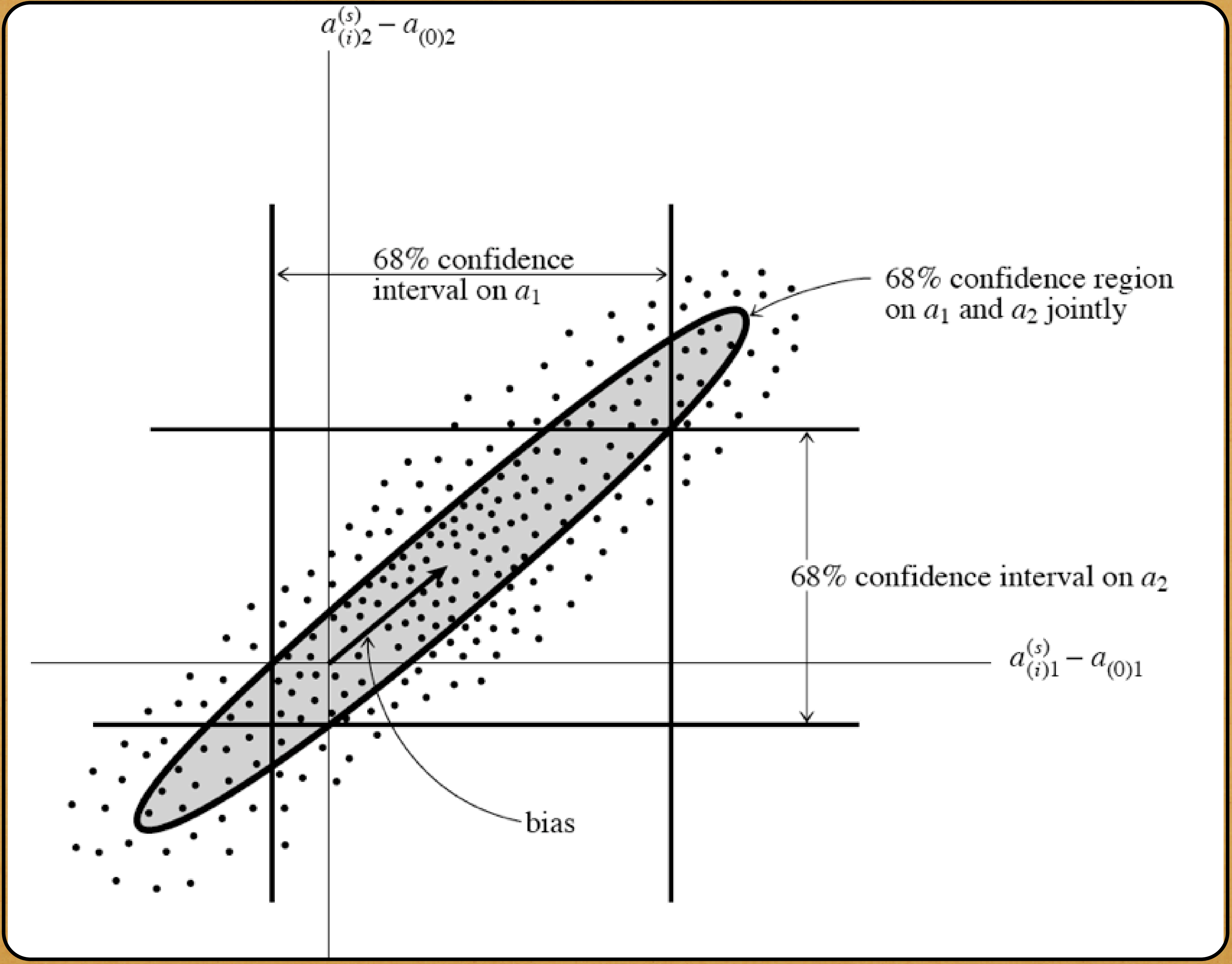


BOOTSTRAP METHOD: AN APPLICATION



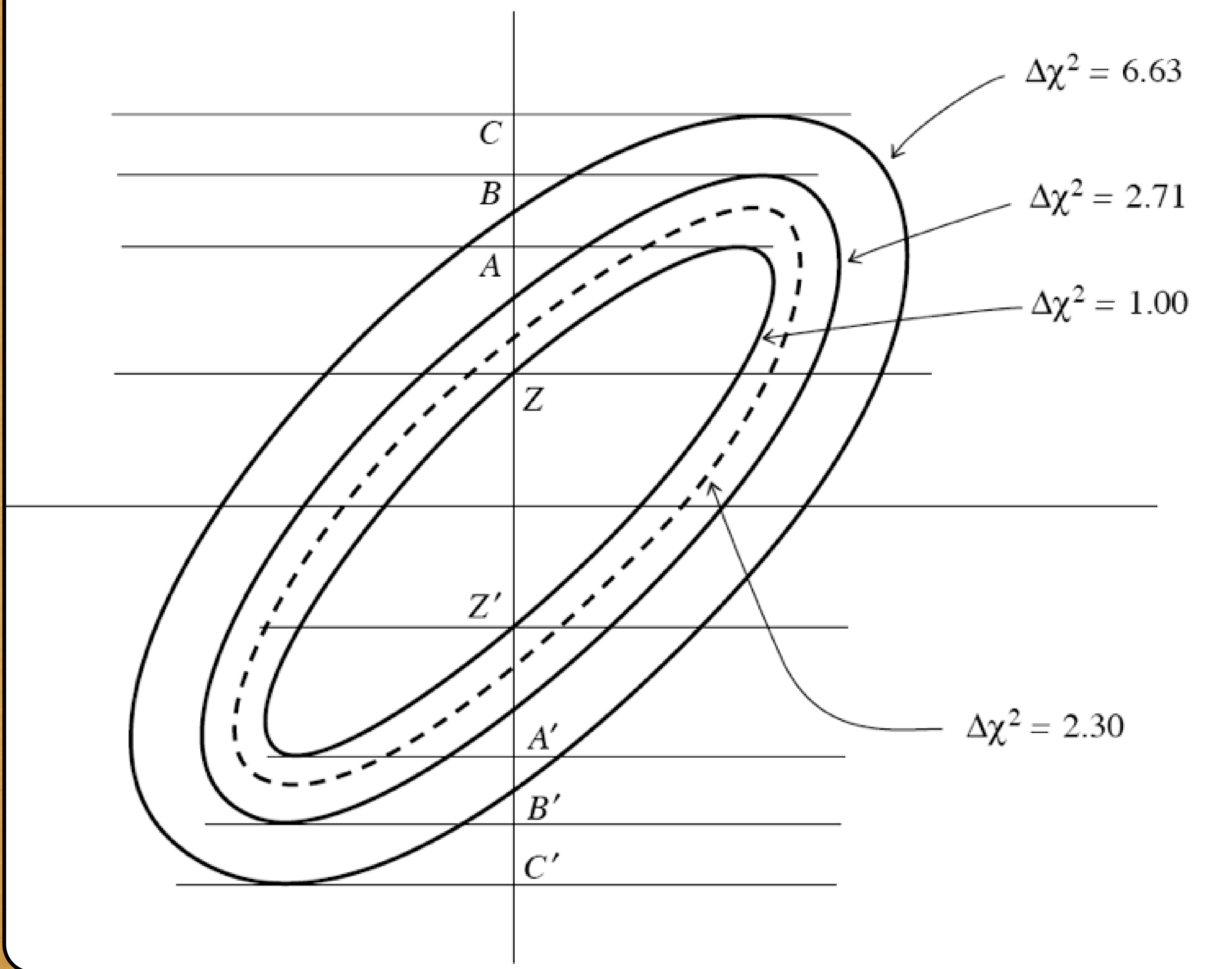
Confidence limits

single vs. multiple parameter confidence region



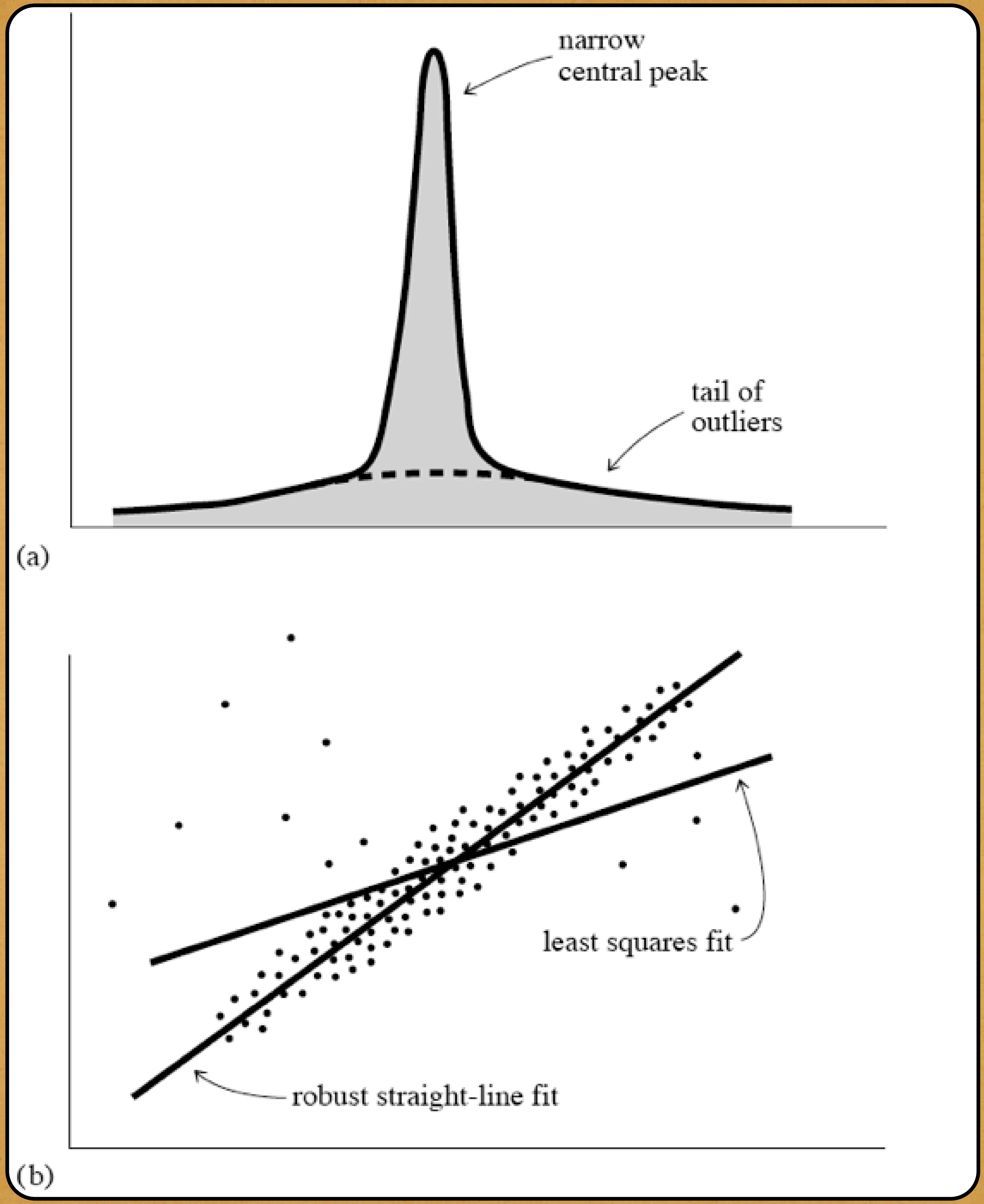
from Num Res page 688

Projections



from Num Res page 689

Be aware of non-gaussian distributions



from Num Res page 695