Today:

remainder of Chapter 1.6-1.7
alternative thermodynamic view

Relation of 'special' functions to Gaussian and Poissonian distributions
OAF2 chapter 5.1 &
Num Res chapter 6.1 & 6.2

Error propagation
OAF2 chapter 5.2

Recap lecture 3

High freq. filtering due to detector response +

Windowing in time domain +

Sampling(=running average in time domain) →

for μ=0 the variance in the measurement

$$C_{X_T}(0) \equiv \sigma^2_{X_T} = \frac{1}{T}\int_{-T}^{+T}(1-\frac{|\tau|}{T})C_{X(t)}(\tau)d\tau$$

B-E distribution (cf. Planck/Fermi-Dirac)

Fluctuations in the number of photons per s per Hz

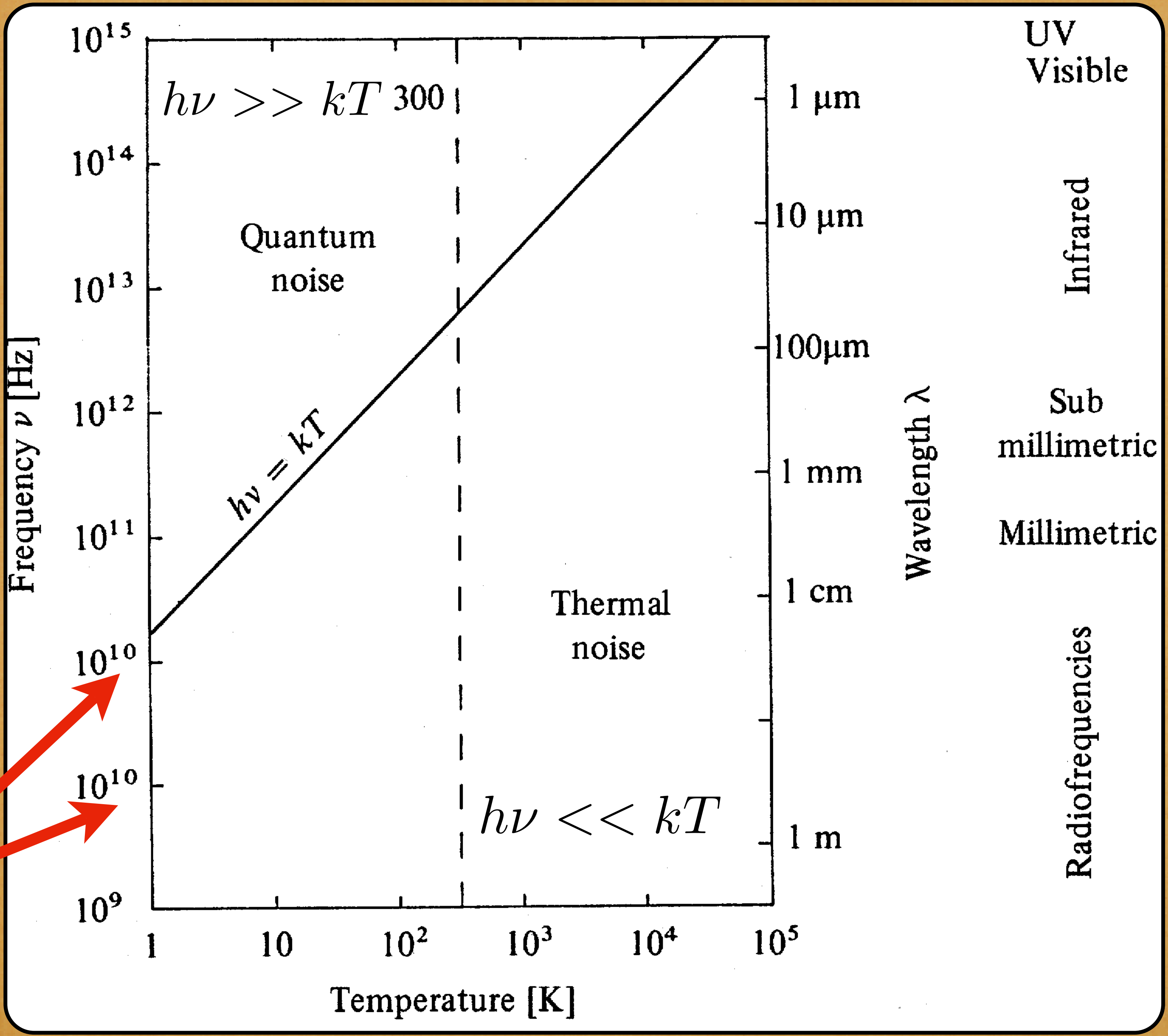
$$\Delta n^2(\nu) = n_\nu(1+\frac{1}{exp(\frac{h\nu}{kT})-1})$$

Power: $P(\nu) = h\nu\ n(\nu)$

Two limits: $h\nu >> kT$ Quantum limit

$h\nu << kT$ Thermal limit $\overline{\Delta P^2}(\nu) = \bar{P}^2(\nu)$

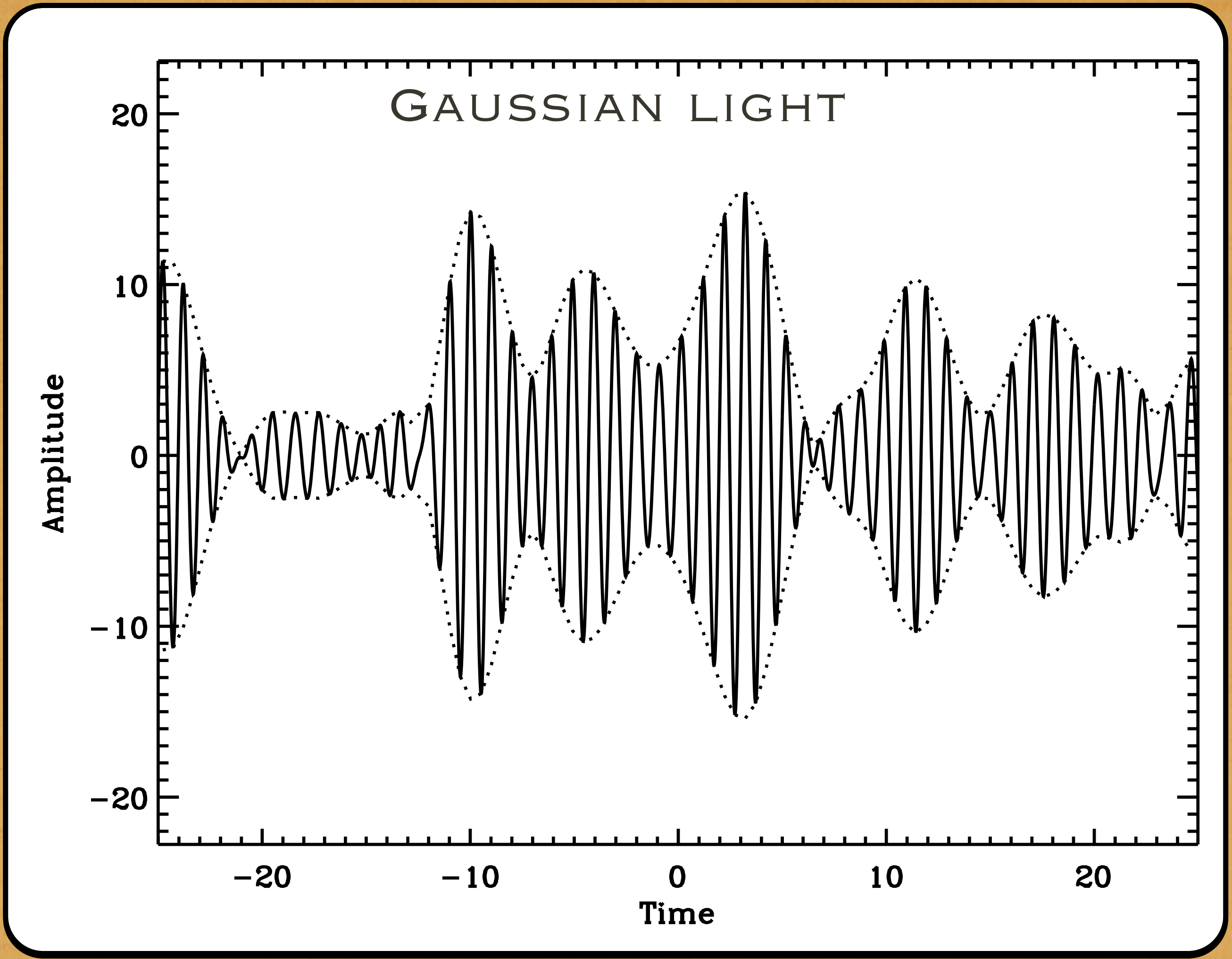
Difference between thermal and quantum limit explains the difference between the principles behind/limitations of radio and optical/X-ray observations



NOTE
ERROR

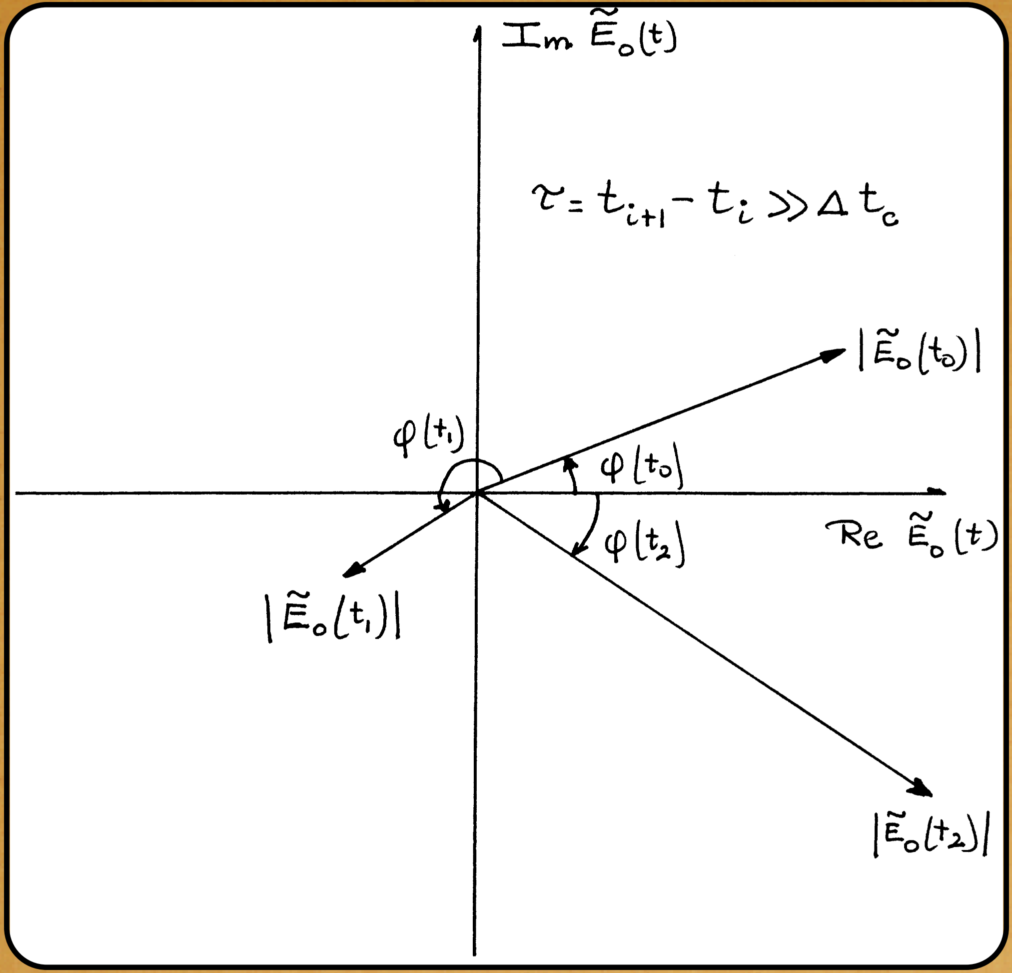
Stochastic description of thermal limit

of quasi-monochromatic radiation
from a thermal source



Describe electric field by $\tilde{E}(t) = \tilde{E_0}(t)\ e^{2\pi i \bar{\nu} t}$
where $\tilde{E_0}(t)$ is the phasor

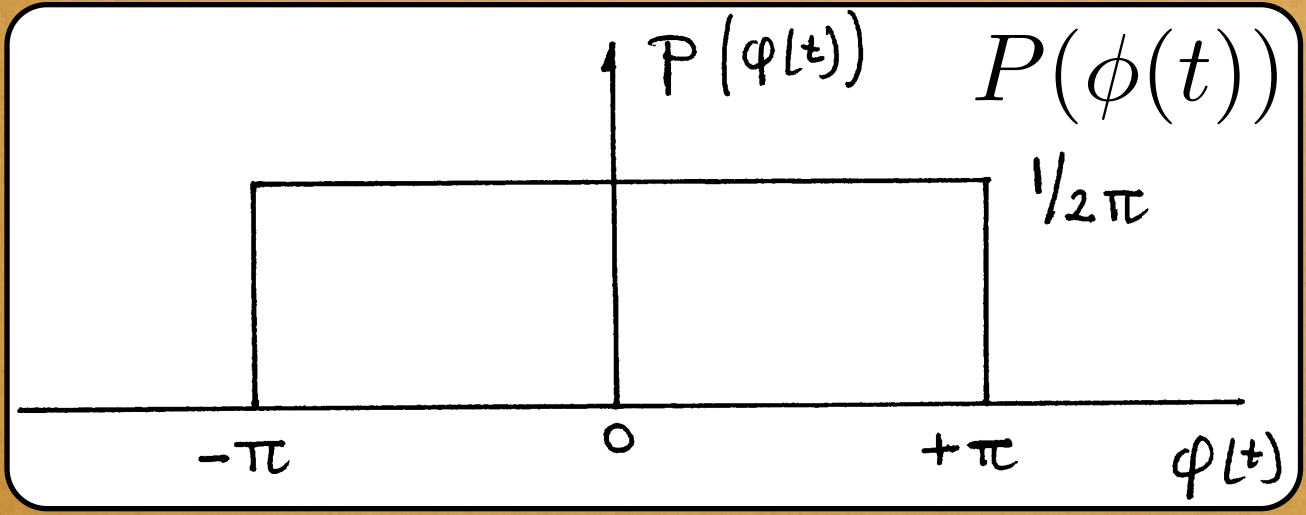
The phasor is described by amplitude $|\tilde{E_0}(t)|$
and phase $\phi(t)$



RANDOM VARIATIONS ON TIME SCALES >> THE COHERENCE TIME ASSOCIATED WITH (ATOMIC) TRANSITIONS

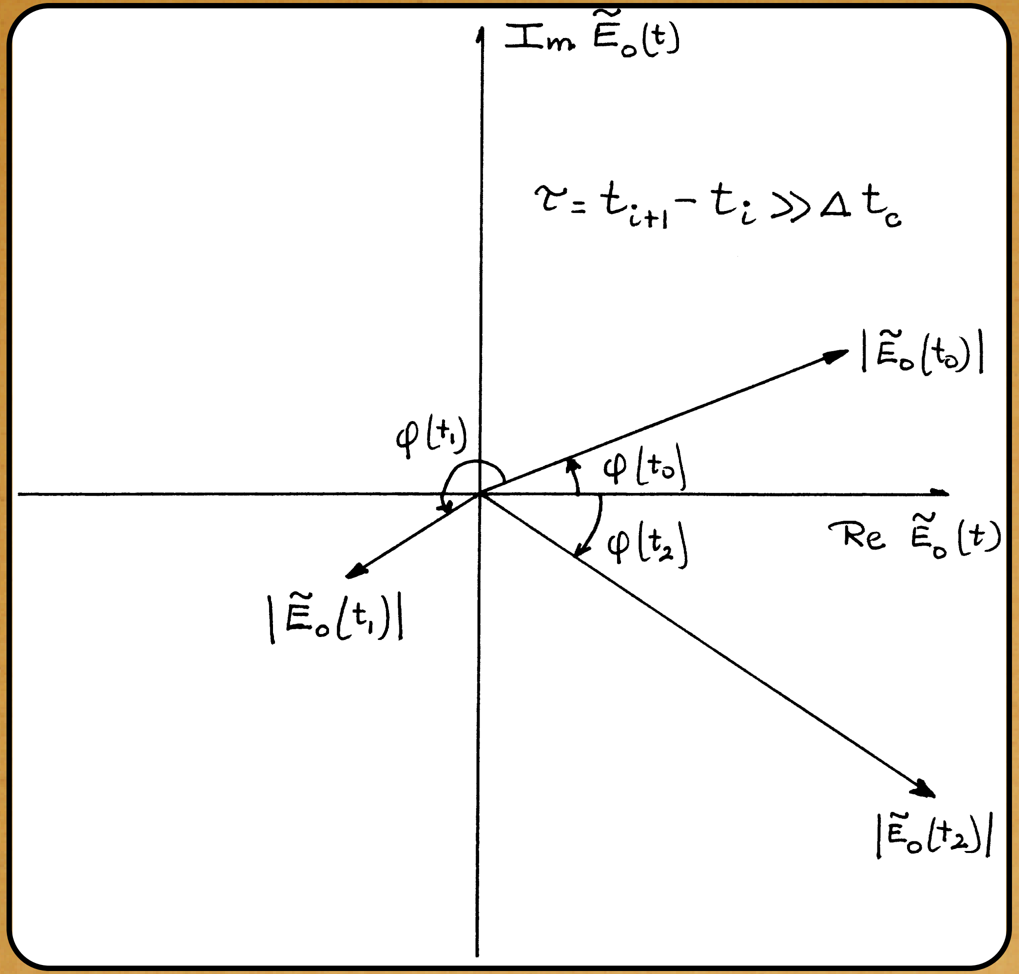
$$P(|E_0|(t), \phi(t))d|E_0|d\phi = \frac{E_0}{2\pi\sigma^2} e^{-\frac{E_0^2}{2\sigma^2}} d|E_0|d\phi$$

PROBABILITY DENSITY FOR:



AGAIN ONE FINDS THAT THE FLUCTUATIONS IN THE POWER FLUX DENSITY ARE:

$$\overline{\Delta I^2} = \bar{I}^2$$



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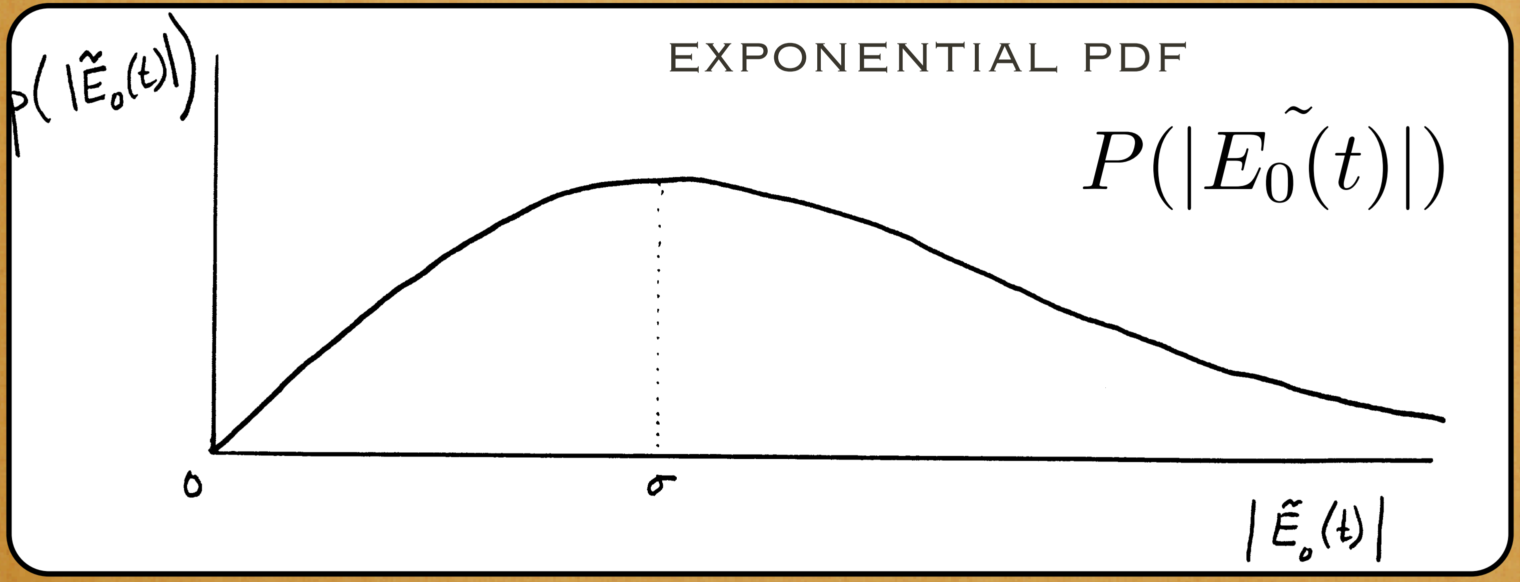

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Alternative, thermodynamic view

connection via $S \equiv k \ln(W)$

→ DEFINE: $a(t) = (\epsilon_0 c \lambda^2)^{1/2} E(t)$

$$R(\tau) = \frac{1}{T}\int_0^T a(t)a^*(t+\tau)d\tau \quad [power][= \text{Watts}]$$

$$P(\nu) = R(\tau) \times \ time \quad [\text{Joule} \ = \ \text{Watts Hz}^{-1}]$$

$$P(\nu) = \frac{1}{2}\bar{u(\nu)}\frac{\lambda^2}{4\pi}c = h\nu[exp\frac{h\nu}{kT} - 1]^{-1}$$

$$where \ \bar{u(\nu)} = \frac{8\pi h\nu^3}{c^3}[exp(\frac{h\nu}{kT}) - 1]^{-1}$$ =ENERGY DENSITY PHOTON FIELD

→ THERMODYNAMICS (W HERE MEAN ENERGY):

$$\langle \Delta W^2 \rangle = kT^2 \frac{d\langle W \rangle}{dT}$$

$$\langle \Delta P(\nu)^2 \rangle = kT^2 \frac{d\langle P(\nu) \rangle}{dT} = P(\nu)h\nu\{1 + [exp(\frac{h\nu}{kT}) - 1]^{-1}\}$$

Two limits again: quantum noise & thermal noise

QUANTUM NOISE LIMIT

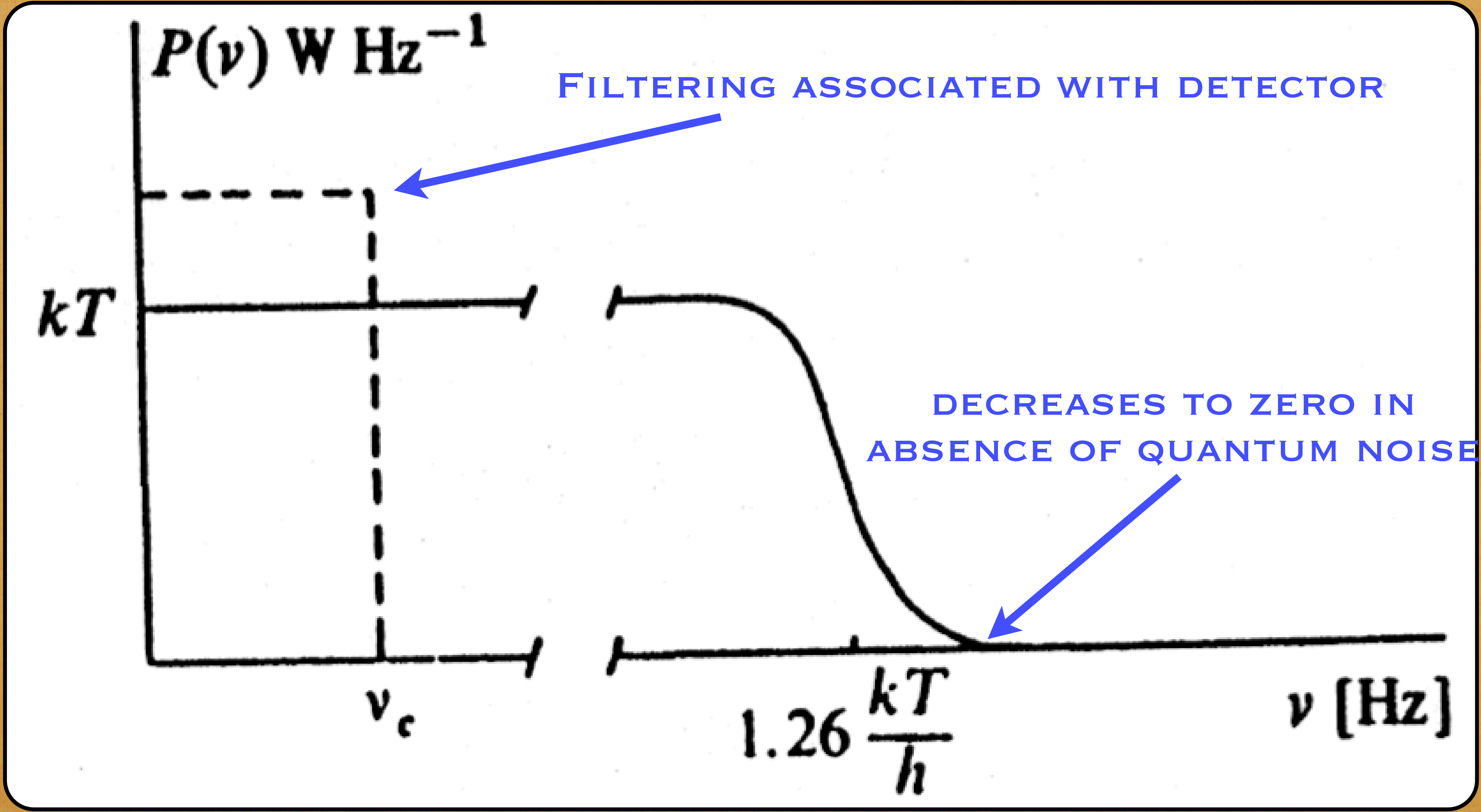
$$h\nu >> kT \quad \rightarrow \langle [\Delta P(\nu)]^2 \rangle \approx P(\nu) h\nu$$

THERMAL NOISE LIMIT

$$h\nu << kT \quad \rightarrow \langle [\Delta P(\nu)]^2 \rangle \approx P(\nu) h\nu [exp(\frac{h\nu}{kT}) - 1]^{-1}$$

$$\approx (kT)^2 \; since \; e^{\epsilon} - 1 = 1 + \epsilon - 1$$

$$and \; P(\nu) \approx kT$$



Three different ways to derive the size of the fluctuation in the thermal limit

bose-Einstein

Stochastic description E-M wave

Thermodynamic

$$\overline{\Delta P^2}(\nu) = (kT)^2$$

Some (computational) math

stirling's approximation

$$\ln x! = x \ln x - x$$

in code use Gamma Function

$$\Gamma(z+1) = z!$$

$$\Gamma(z+1) = z\Gamma(z)$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

Gamma function has a computationally simple accurate approximation

Numerical Recipes Chap 6.1-6.2

Incomplete Gamma Functions:

$$P(a,x) \equiv \frac{\gamma(a,x)}{\Gamma(a)} \equiv \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t} dt$$

$$Q(a,x) \equiv 1 - P(a,x) = \frac{\gamma(a,x)}{\Gamma(a)} \equiv \frac{1}{\Gamma(a)} \int_x^\infty t^{a-1} e^{-t} dt$$

Error Functions:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad = P(\frac{1}{2}, x^2)$$

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \quad = Q(\frac{1}{2}, x^2)$$

DISTRIBUTION FUNCTION $F(x) = P\{x \leq y\}$

PROBABILITY DENSITY FUNCTION $\frac{dF(x)}{dx} = f(x)$

$\hookrightarrow$ GAUSS, POISSON, χ^2 ETC

GAUSSIAN OR NORMAL DISTRIBUTION AND PROBABILITY DENSITY FUNCTION

$$F(x, \eta, \sigma) = 0.5 + erf\frac{x - \eta}{\sigma^2}$$

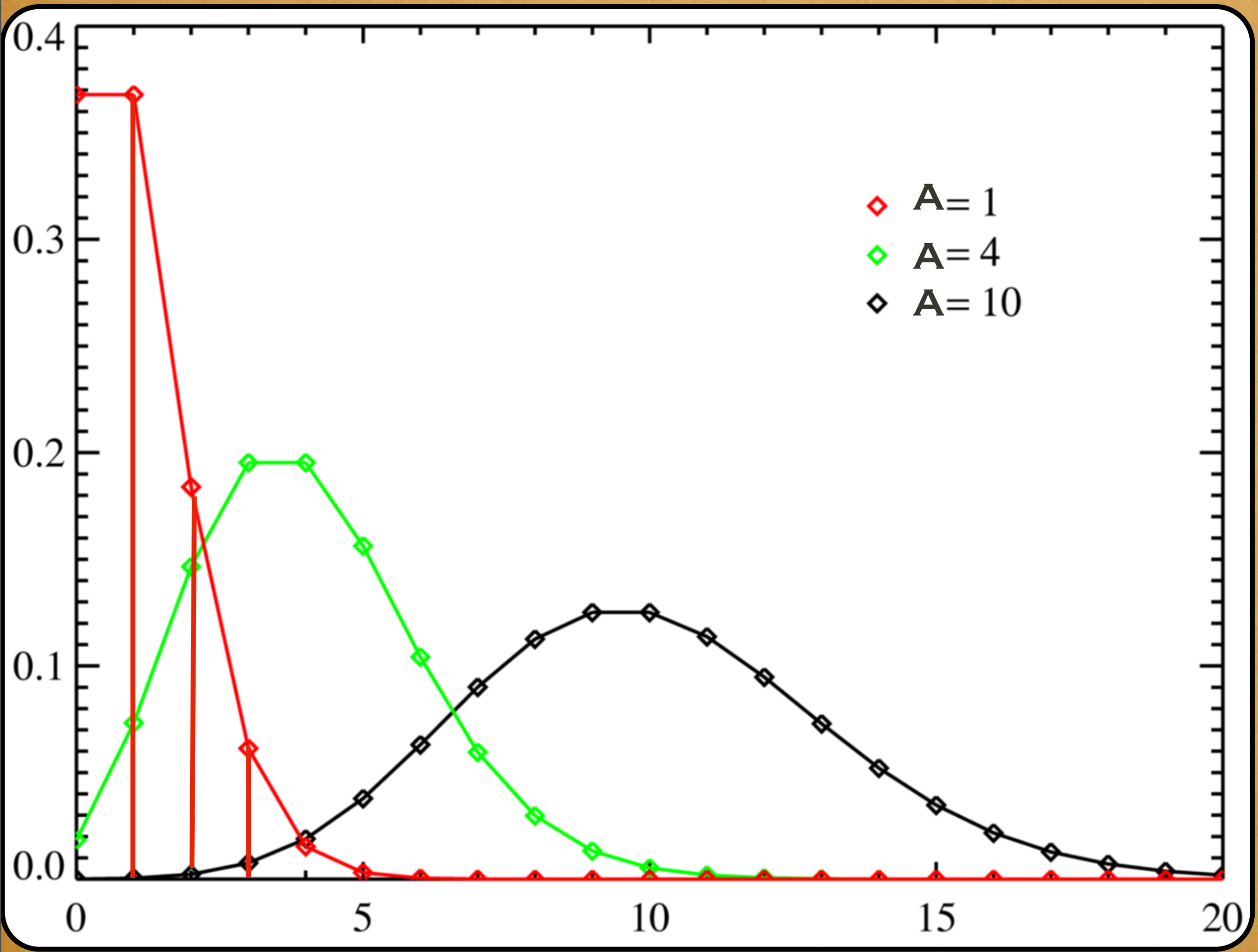
$$f(x) = \frac{1}{\sigma\sqrt{(2\pi)}} exp(-\frac{1}{2}\frac{(x-\eta)^2}{\sigma^2})$$

Poisson cumulative distribution and probability density function

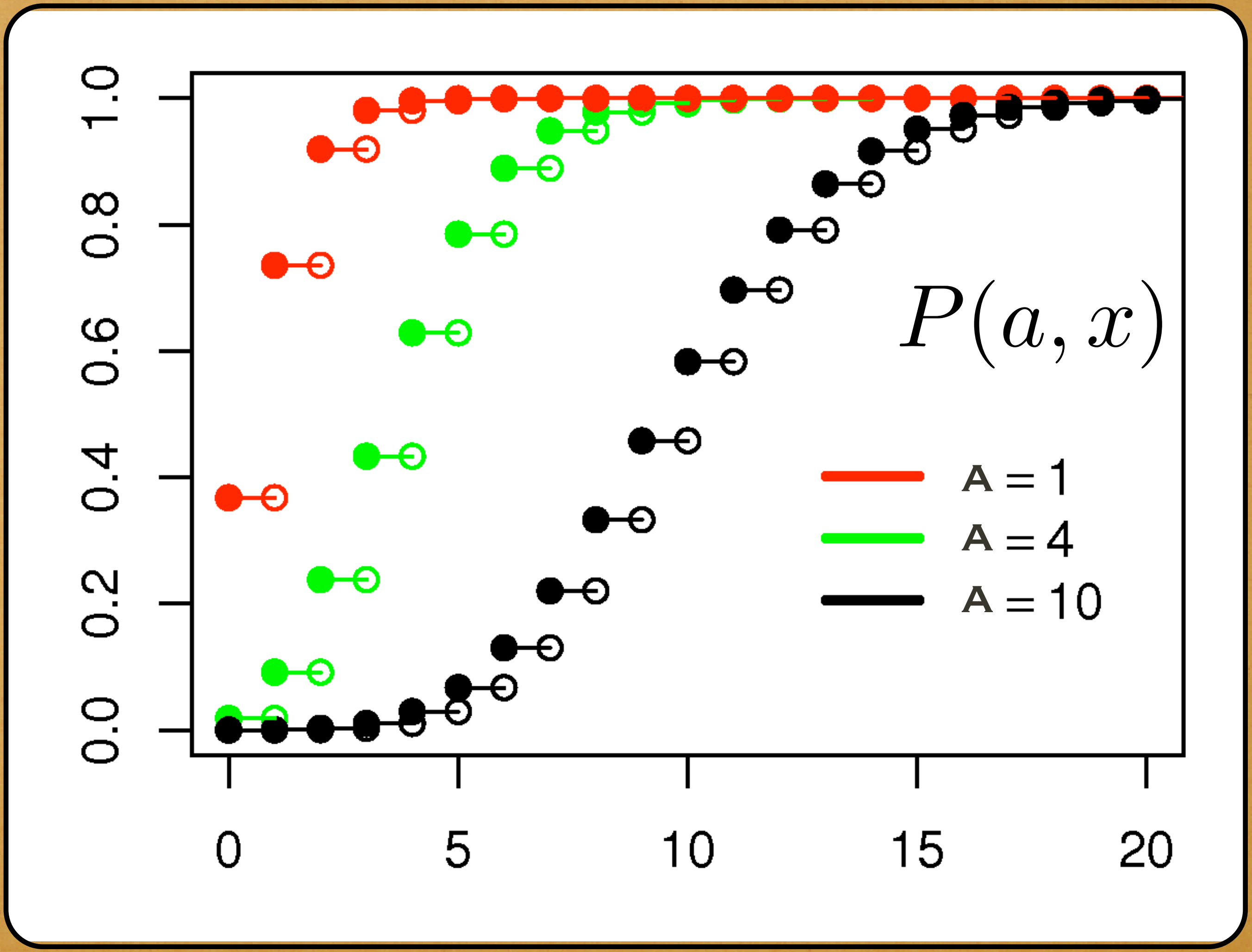
$$F(x) = 1 - P(a,x) \equiv \frac{1}{\Gamma(a)} \int_x^\infty e^{-t} t^{a-1} dt$$

$$f(x) = \frac{a^k}{k!} e^{-a} \rightarrow \ (discrete)\ e^{-a} \sum_{k=0}^{\infty} \frac{a^k}{k!} \delta(x-k)$$

Probability density



Cumulative distribution



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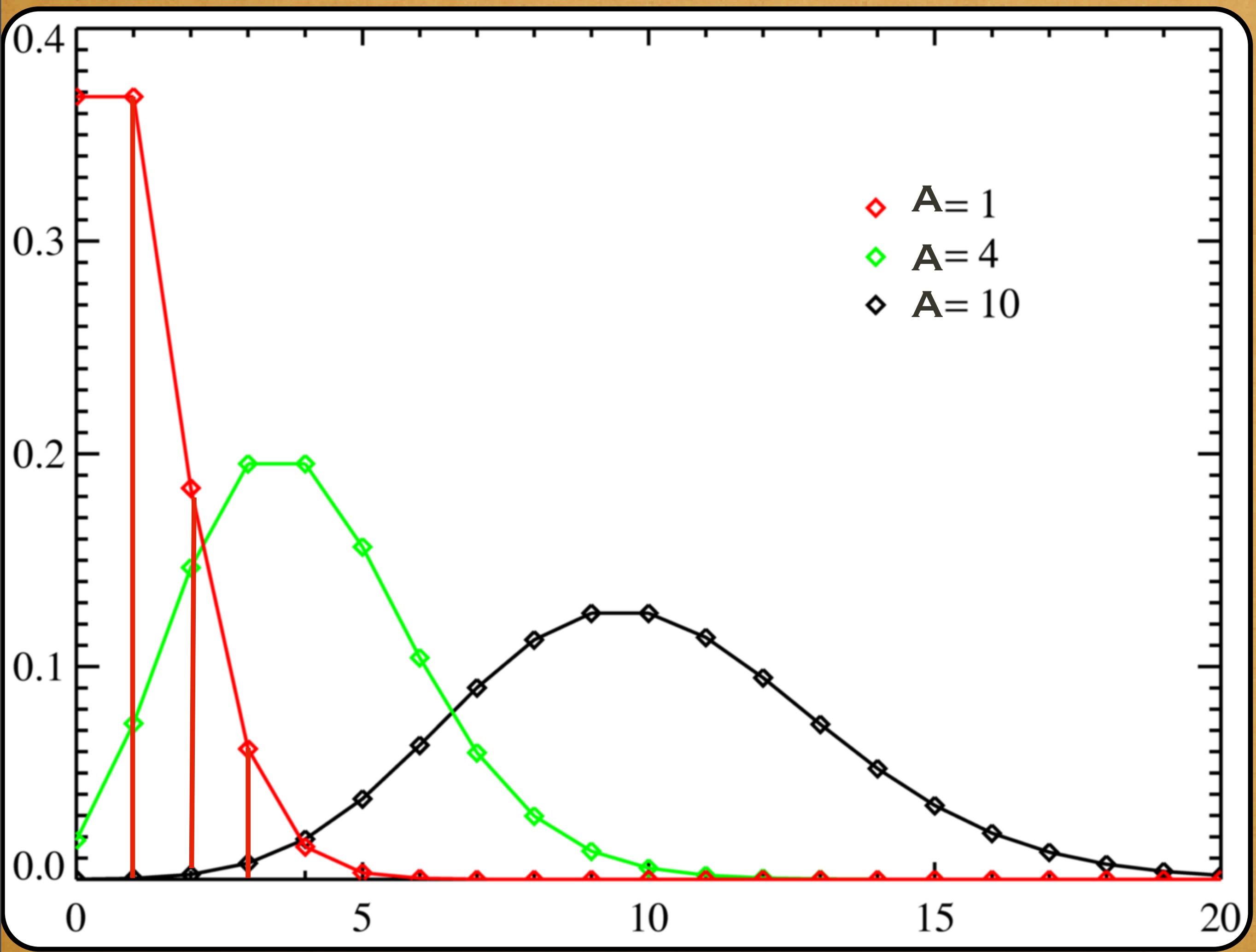
Incomplete Gamma Function

Probability density



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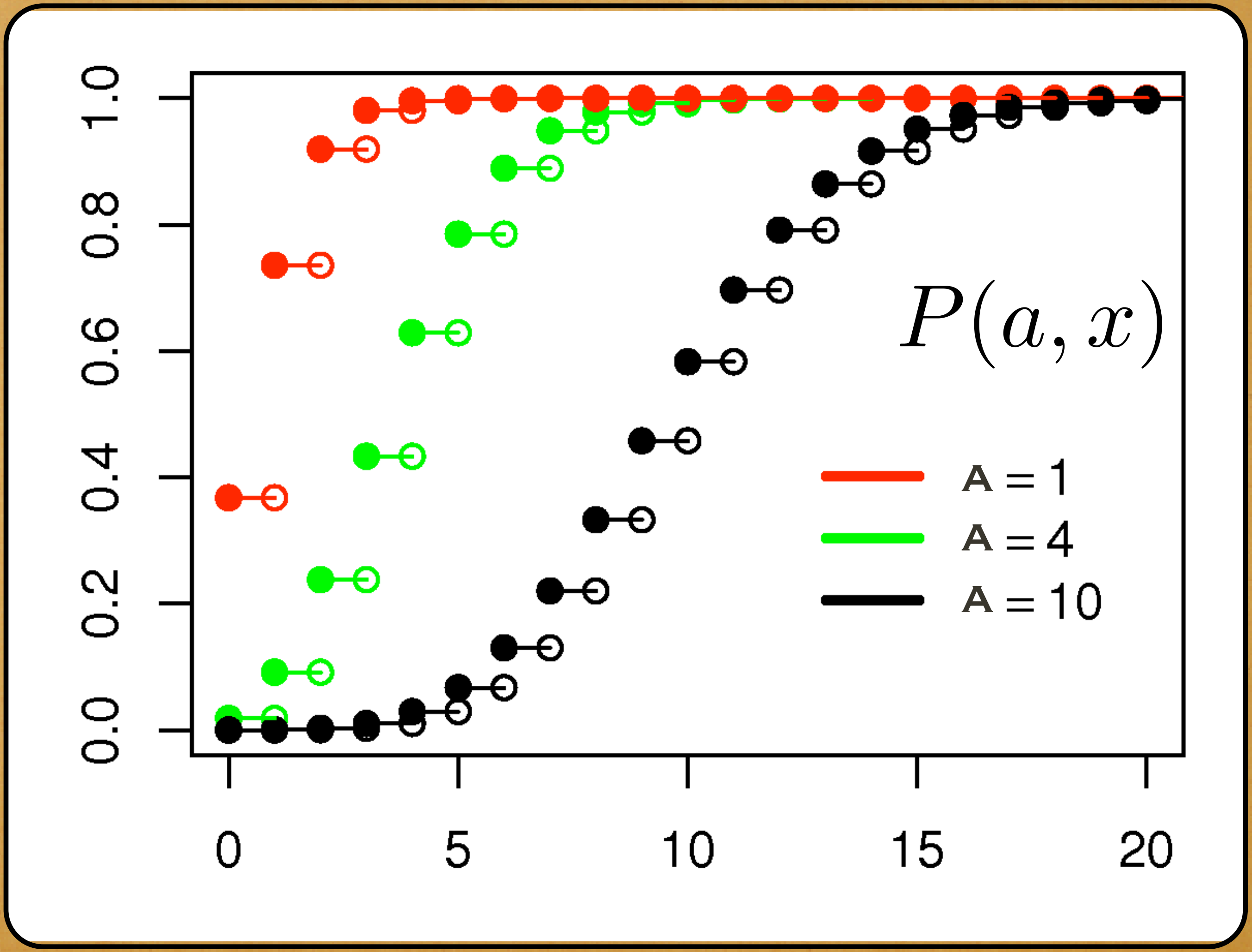

In computer codes dealing with Poisson and Gaussian distributions incomplete Gamma Functions are used

Error propagation

Besides noise intrinsic to the s.p. noise is added due to the detector, background etc.

→ how to determine the resultant variance

Main assumption is that the average of the function f is well represented by the value for f at the averages for the variables

$$\bar{f} = f(\bar{u}, \bar{v}, ..)$$

Taylor expansion to first order around the average for each variable

$$f_i - \bar{f} \approx (u_i - \bar{u})\frac{\partial f}{\partial u} + (v_i - \bar{v})\frac{\partial f}{\partial v} + ...$$

REMEMBER THAT THE VARIANCE

$$\sigma^2 = \frac{1}{N-1}\sum_{i=1}^{N}(f_i - \bar{f})^2$$

FILL-IN THE TAYLOR EXPANSION HERE

ASSUME THAT THE VARIABLES ARE INDEPENDENT SUCH THAT THEIR CROSS PRODUCT CANCEL

$$\sigma_f^2 = \sigma_u^2 (\frac{\partial f}{\partial u})^2 + \sigma_v^2 (\frac{\partial f}{\partial v})^2 + ...$$