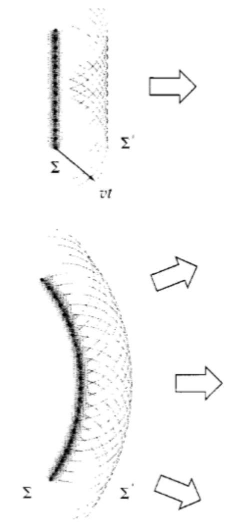
Wave Optics: Diffraction & Interference

- Huygens sources & wave propagation
  - Math: Green's function
- Fraunhofer vs. Fresnel interference

- Applications of diffraction & interference:
  - Diffraction limit of telescopes & microscopes
  - Fresnel diffraction behind halfscreen
  - Fresnel zone plates
  - Interference behind realistic double slit
  - Interference behind realistic N slit

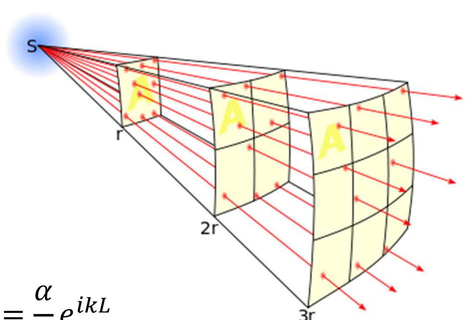
Huygens sources help to visualize field propagation

Rays & Phase fronts



Hecht, Fig. 4.26

Huygens sources: quantitative formulation

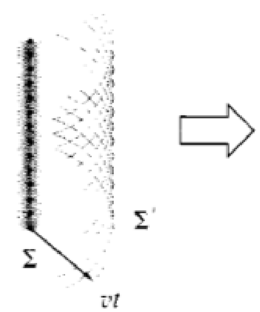


$$E(L) = \frac{\alpha}{L} e^{ikL}$$

Q: What is α ?

$$\alpha = E_s . \frac{dx.dy}{i\lambda}$$

Plane waves propagate undisturbed

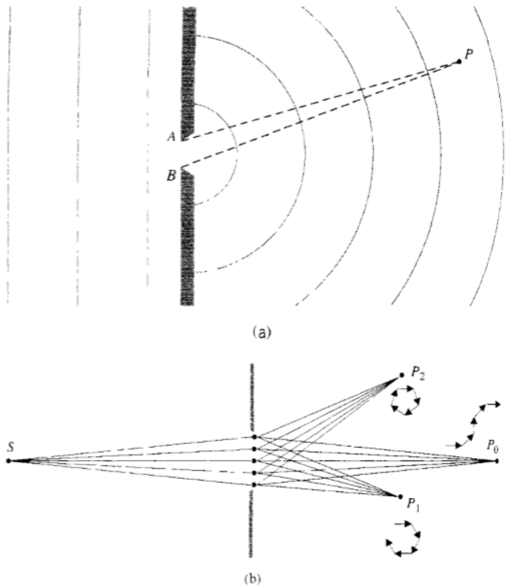


$$E(L_0) = E_s \int_0^\infty \frac{1}{i\lambda L} e^{ikL}\, 2\pi r dr$$

Q: Prove this equals E_S

$$\approx E_s \int_0^\infty \frac{1}{i\lambda L_0} e^{ik\left(L_0 + \frac{r^2}{2L_0}\right)}\, 2\pi r dr = E_s e^{ikL_0} \int_0^\infty \frac{1}{i\lambda L_0} e^{i\pi\frac{r^2}{\lambda L_0}}\, \pi dr^2 = E_s e^{ikL_0}$$

Diffraction: integration over Huygens sources



$$E = \int_0^\infty \frac{E(x,y)}{i\lambda L} e^{ikL}\, dxdy$$

Q: When is this description valid?

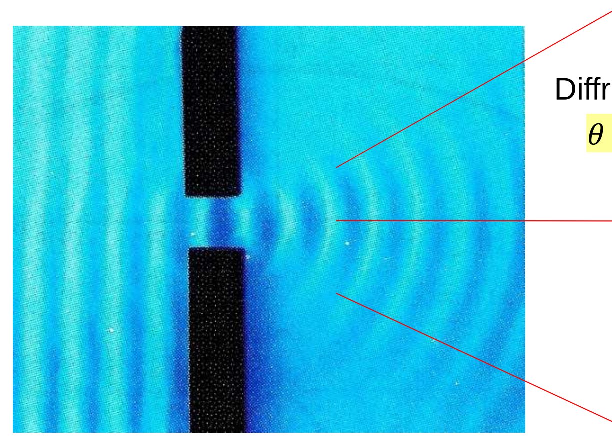
- Spectral coherence (monochromatic)
- Spatial coherence (one coherent source)

Q: How about backward light?

- Factor (1+cosθ)/2

Hecht, Fig. 10.1

Diffraction behind an aperture



NOTE: long-wavelength water wave is not representative for optics

Diffraction pattern changes with distance from slit

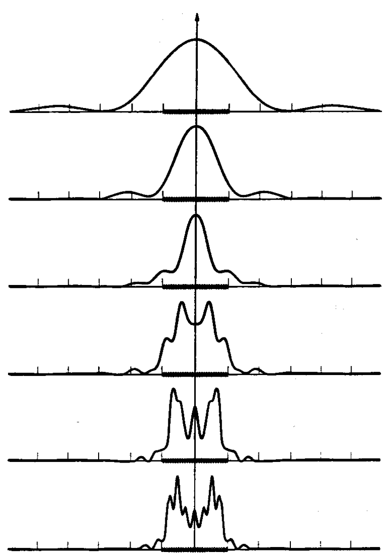
10.2 Fraunhofer diffraction

In **far field**:
- fixed shape
- only scaling ∝ L

In **near field**:
Shape changes with distance from slit

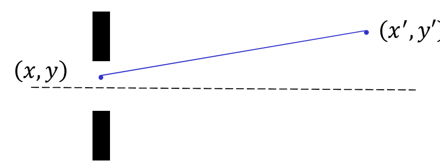
10.3 Fresnel diffraction

Q: What is far enough?



Hecht Fig.10.5

Fraunhofer regime when L >> D²/λ



$$E(x',y') = \int_0^\infty \frac{E(x,y)}{i\lambda L} e^{ikL}\, dxdy$$

$$L \approx L_0 + \frac{(x-x')^2 + \cdots}{2L_0} \approx L_0 + \frac{x^2 - 2xx' + x'^2}{2L_0}$$

$$E(x',\ldots) \approx e^{ikL_0} e^{i\frac{kx'^2}{2L_0}} \int_0^\infty E(x,\ldots) e^{-i2\pi\frac{x.x'}{\lambda L_0}} e^{i\frac{kx^2}{2L_0}} \ldots dxdy/i\lambda L_0$$

Fresnel term only for $L_0 \ll x^2/\lambda$

Fourier transform in Fraunhofer regime: $L_0 \gg x^2/\lambda$

Many examples of Fraunhofer & Fresnel diffraction

- Fresnel diffraction behind slit & half screen
- Fresnel zone plate

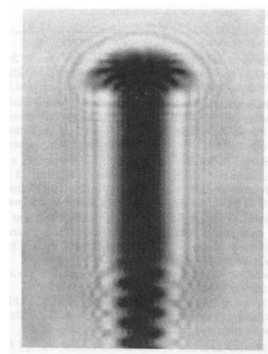
- Fraunhofer diffraction behind slit
- Fraunhofer diffraction behind circular aperture
- Rayleigh criterium for resolution
- Diffraction limit of telescope & microscope

$$\sin\theta = \frac{1.22\lambda}{D} \Rightarrow \theta \approx \frac{1.22\lambda}{D}$$

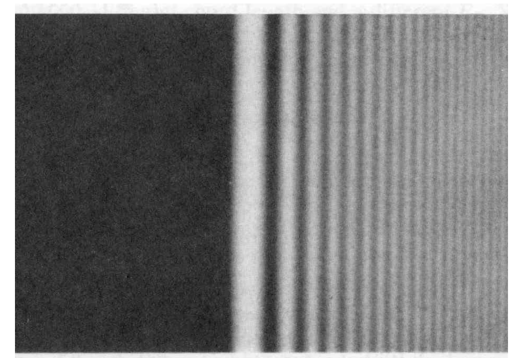
- Fraunhofer interference behind Young's double slit
- Interference behind N identical slits

Interference fringes in Fresnel regime

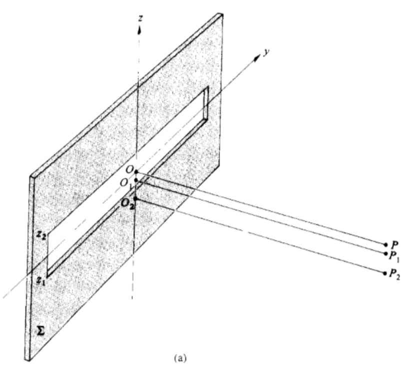
Pattern behind screw



Pattern behind half opaque screen



Fresnel diffraction in 2D and 3D



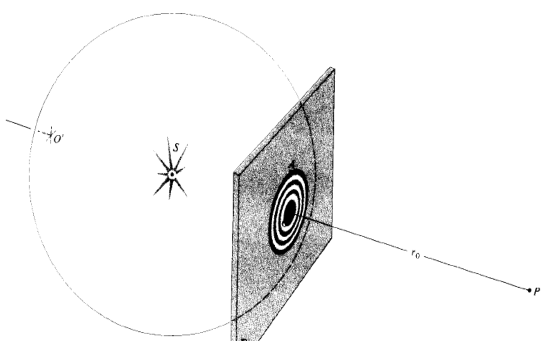


Figure 10.42 A circular aperture.

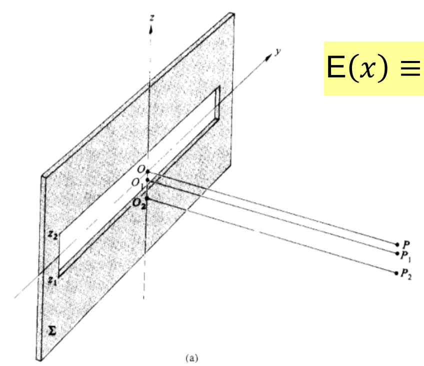
2D: Fresnel integral

$$\mathrm{F}(w) \equiv \int_0^w e^{i\pi x^2/2} dx$$

3D: Trivial integral

$$\int_0^{x^2=r^2} e^{i\pi x^2/2} \pi dx^2$$

Fresnel diffraction in 2D: Fresnel number & Cornu spiral



$$\mathrm{E}(x) \equiv \int_{w_1}^{w_2} e^{i\pi \tilde{x}^2/2} d\tilde{x}$$

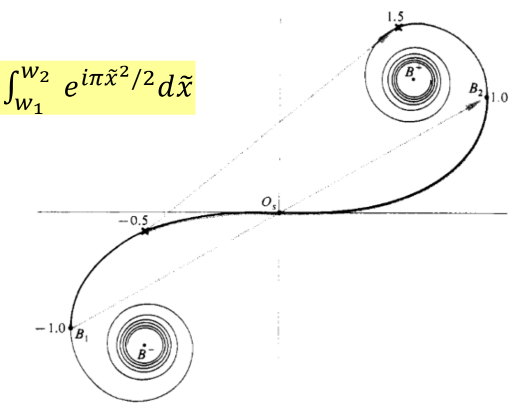
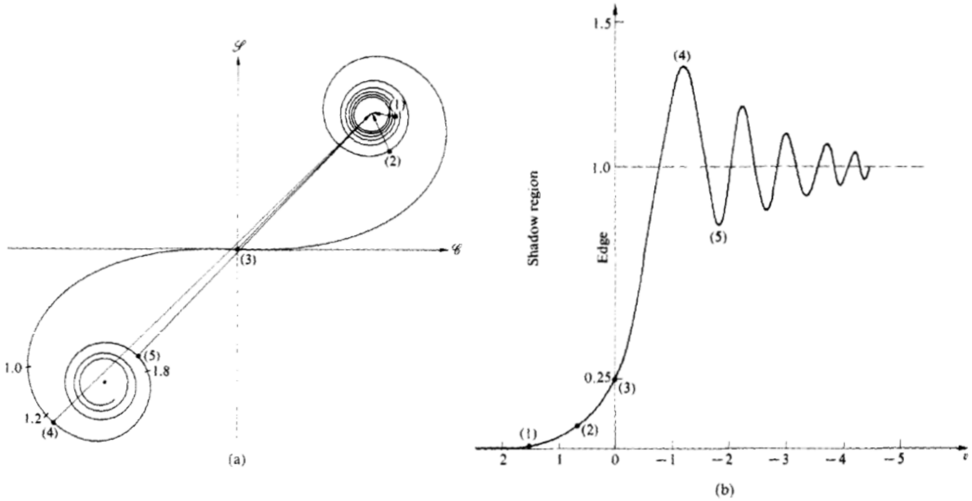


Figure 10.55 Cornu spiral for the slit.

Q: What is w_1 and w_2 ? A: $\frac{\pi\tilde{x}^2}{2} = k\Delta L = \frac{\pi x^2}{\lambda L}$

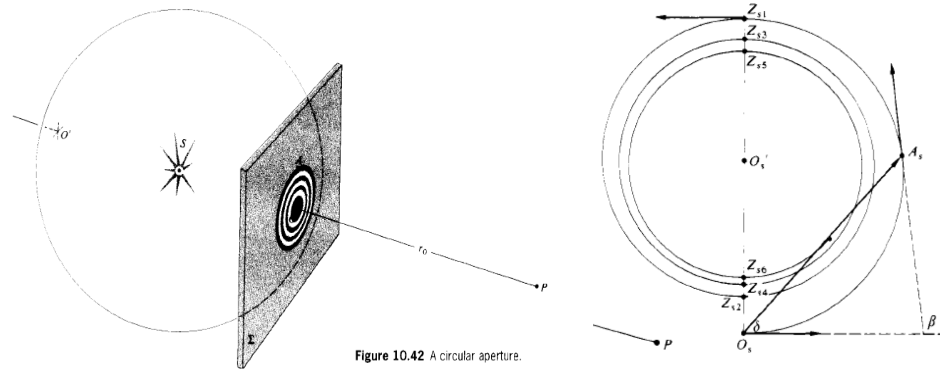
Hecht Fig.10.54 & 10.55

The semi-infinite opaque screen



Hecht Fig.10.59

Fresnel diffraction behind circular aperture

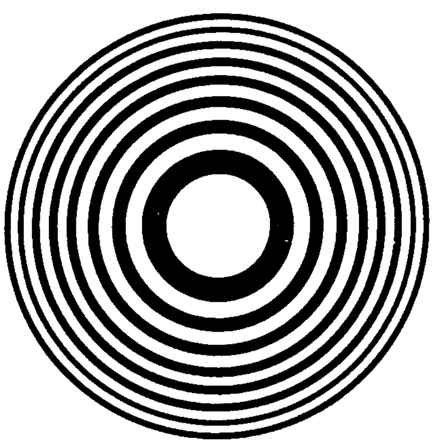


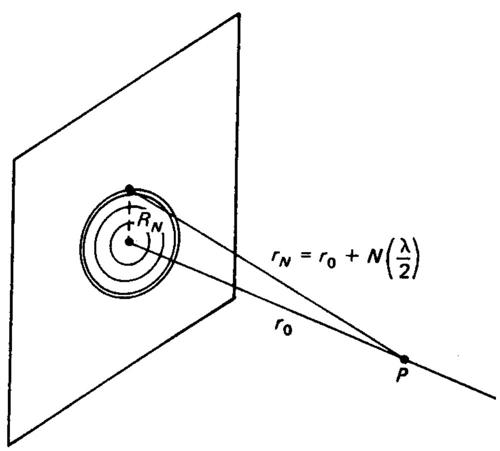
$$E_{centre} = \int_0^{x^2=\frac{2r^2}{\lambda L}} e^{i\pi x^2/2} \pi dx^2$$

Hecht Fig.10.41 & 10.42

Fresnel lens: focusing with black-and-white pattern !!

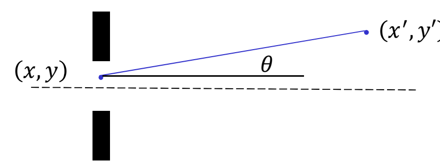
How to increase the intensity in P by blocking Fresnel zones in the beam !





Fresnel number $N_F = b^2/L\lambda$ counts number of “Fresnel zones”

Fraunhofer diffraction behind slit of width w

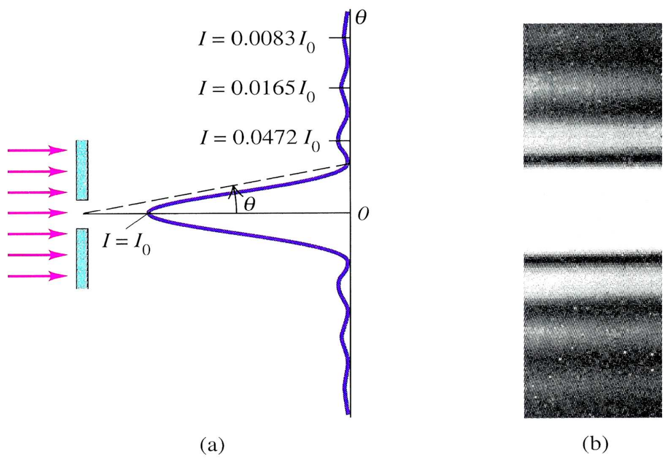


$$E(x') \propto \int_{-b/2}^{b/2} E(x) e^{ik\Delta L}\, dx = \int_{-b/2}^{b/2} E_0\, e^{-ik \sin\theta x}\, dx$$

As graphical construction yields $\Delta L = -\sin\theta\, x$

$$E(\theta) \propto \int_{-b/2}^{b/2} E_0\, e^{-ik \sin\theta x}\, dx = -E_0 b\, \frac{\sin\beta}{\beta} \text{ with } \beta = \frac{1}{2} k \sin\theta\, b$$

Diffraction minima and maxima behind slit

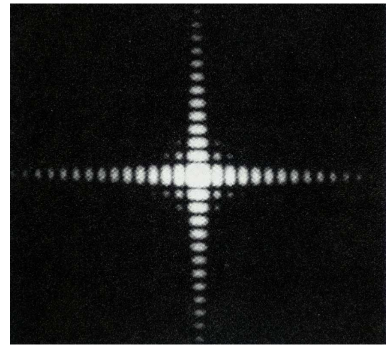


Opening angle of first minimum: $\sin(\theta_{\text{first min.}}) = \dfrac{\lambda}{b}$

Q: What is spacing maxima & minima   A: Only minima equidistant

Fraunhofer diffraction behind rectangular hole

Diffraction behind square hole

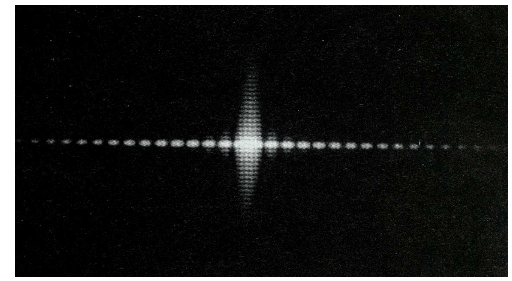


Field profile factorizes !!

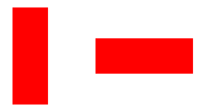
$$E_P = E_0 \frac{\sin\beta_x}{\beta_x}\frac{\sin\beta_y}{\beta_y}$$

$$\beta_x = \tfrac{1}{2}k_0 b_x \sin\theta_x \; ; \; \beta_y = \tfrac{1}{2}k_0 b_y \sin\theta_y$$

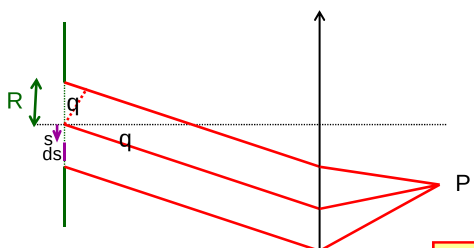
Diffraction behind rectangular hole



What is orientation of hole?



Fraunhofer diffraction behind round hole



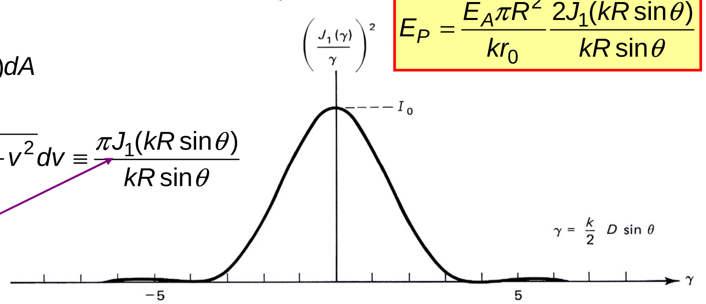
Field per unit Area

$$E_P \propto \frac{E_A}{r_0}\iint_{\text{Area}} \exp(isk\sin\theta)dA$$

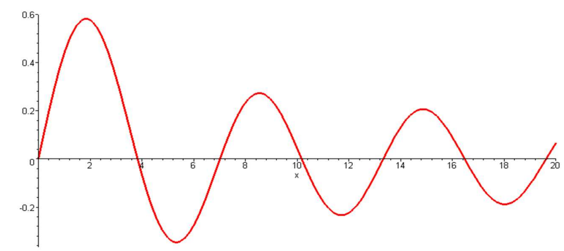
$$\propto \int_{-1}^{1} \exp(ivkR\sin\theta)\sqrt{1-v^2}dv \equiv \frac{\pi J_1(kR\sin\theta)}{kR\sin\theta}$$

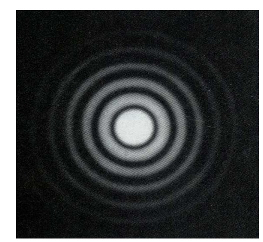
Bessel function

$$E_P = \frac{E_A \pi R^2}{kr_0}\frac{2J_1(kR\sin\theta)}{kR\sin\theta}$$



Properties of Bessel function $J_1(x)$





Central disk = **Airy disk**
Rings = **Airy rings**

$$\lim_{x \downarrow 0} \frac{J_1(x)}{x} = \frac{1}{2}$$

$J_1(x) = 0$ als $x = 3.832, 7.016, 10.173$

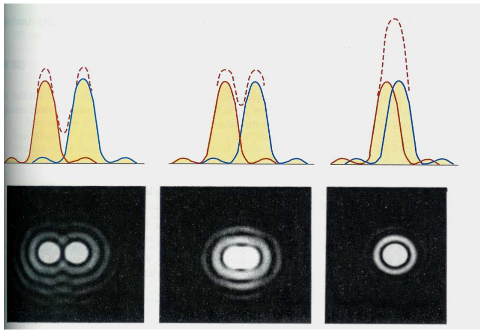
$$I_P = I_0 \left(\frac{2J_1(kR\sin\theta)}{kR\sin\theta}\right)^2$$

$$I = 0 \text{ als } kR\sin\theta = 3.832 \Rightarrow \sin\theta = \frac{1.22\lambda}{2R} = \frac{1.22\lambda}{D}$$

Compare with diffraction from slit of rectangle: $I = 0 \;\text{ als }\; \sin\theta = \dfrac{\lambda}{b}$

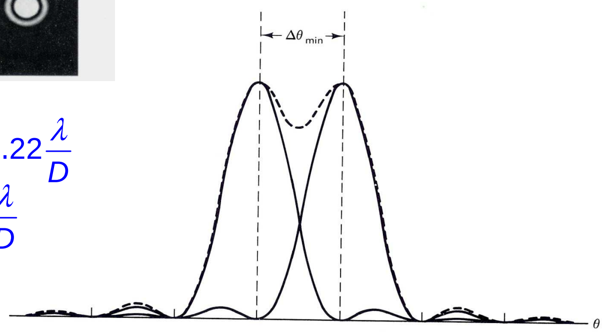
Why is diffraction angle of round hole larger?

Rayleigh criterion for resolution



Q: When can you resolve two maxima?

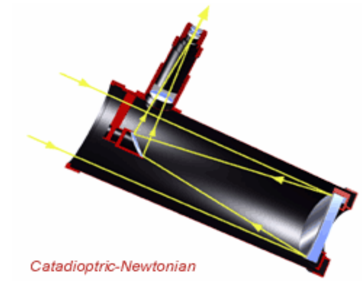
Lord Rayleigh:
- If intensity between peaks drops 19%
- If one diffraction max. coincides with other diffraction minimum

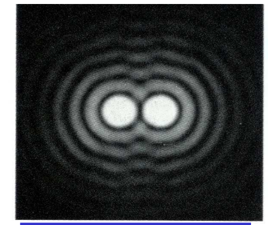


For round holes: $\Delta\theta > 1.22\dfrac{\lambda}{D}$

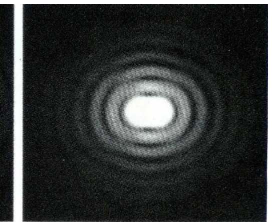
For rectangular holes: $\Delta\theta > \dfrac{\lambda}{D}$

Resolution of telescope





Well distinguishable



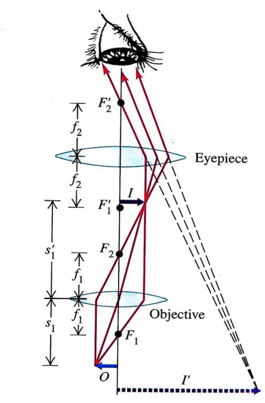
Barely distinguishable

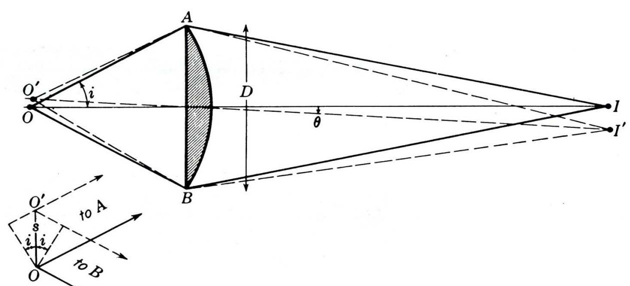
What is angular resolution of telescope with D = 4 cm (for λ=500 nm)?

Answer: $\theta_{res} \approx 15\ \mu$rad (diameter moon ≈ 0.5º ≈ 8 mrad)

Note: Fluctuations in earth atmosphere (“seeing”) typically limit resolution for D > 20 cm

Resolution of microscope





Similar criterium as for telescope:

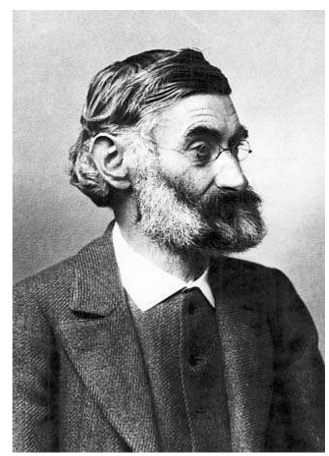
$$\theta_{res} \approx 1.22\frac{\lambda}{D} \Rightarrow x_{res} \approx f\theta_{res} \approx 1.22\frac{\lambda}{D/f} \approx \frac{1.22\lambda}{2N.A.}$$

$$s_{min} = \frac{1.22\lambda}{2n\sin i} = \frac{1.22\lambda}{2N.A.}$$

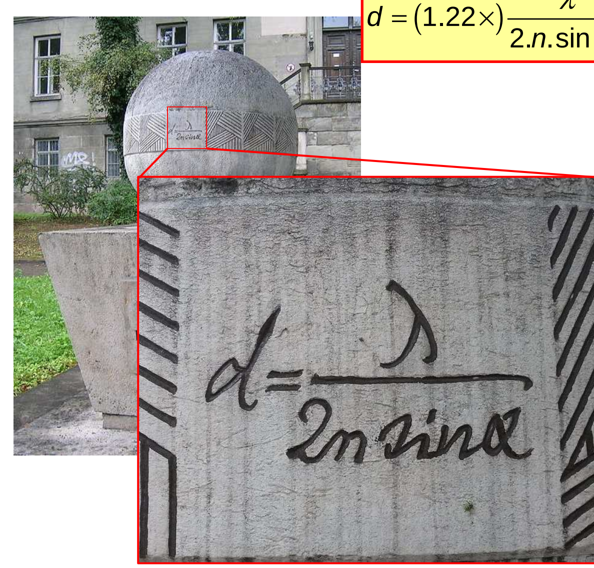
Note: also valid for large angles & immersion objectives

Abbe diffraction limit for microscopes

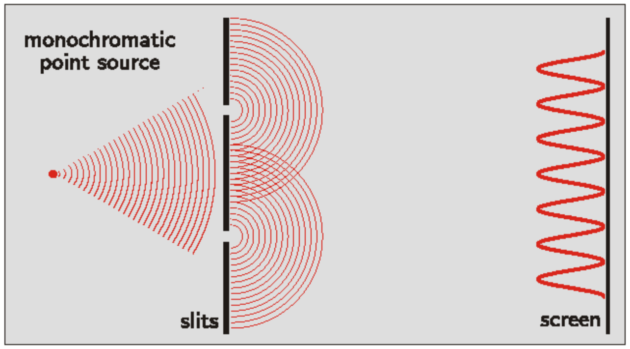
$$d = (1.22\times)\frac{\lambda}{2.n.\sin\theta_{in}}$$



Ernst Abbe (1840-1905)



Young's experiment with a double slit



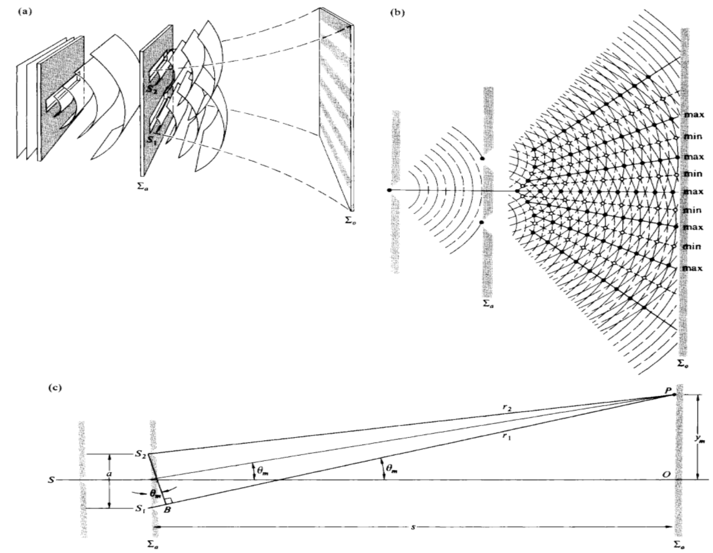
Interference of two diffracted waves

Q: Why did it take until 1803?

- Spectral coherence
- Spatial coherence

Young's double slit experiment

Path length differences are crucial

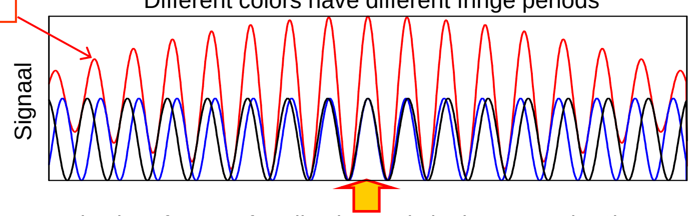


Hecht, Fig. 9.8

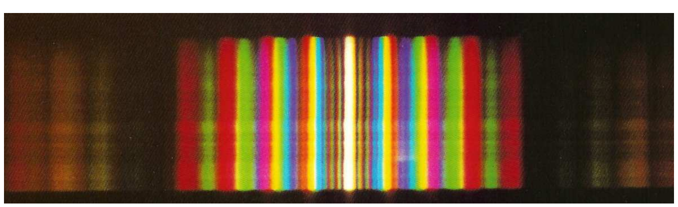
1. Young's experiment with limited spectral coherence

Total intensity

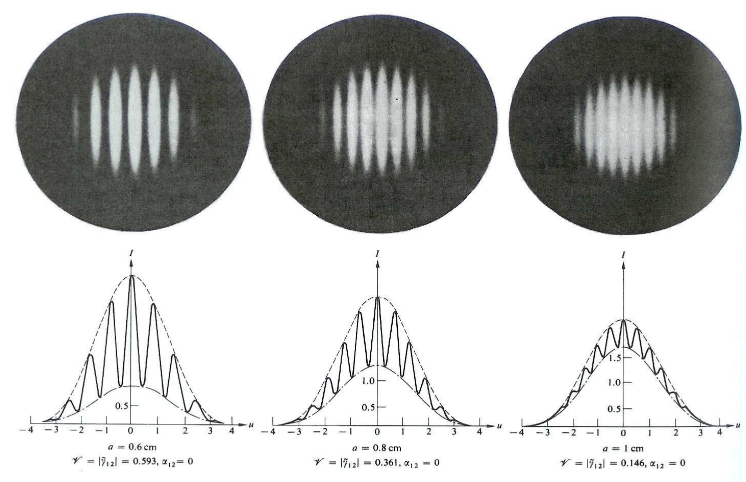
Different colors have different fringe periods



Constructive interference for all colors only in the center (θ=0)



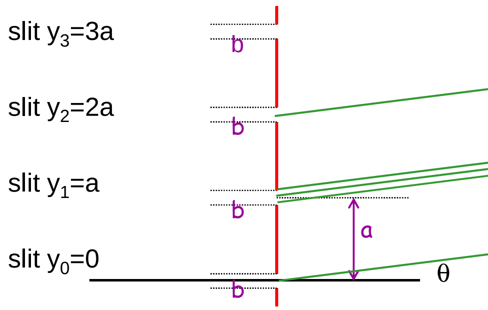
2. Young's experiment with limited spatial coherence



A=0.6 cm, V=0.59 | A=0.8 cm, V=0.36 | A=1.0 cm, V=0.15

⇨ distance between slits increases

Fraunhofer diffraction behind N slits



$$E_P \propto E_0 \int_{\text{Openingen}} \exp(ik_y y)dy$$

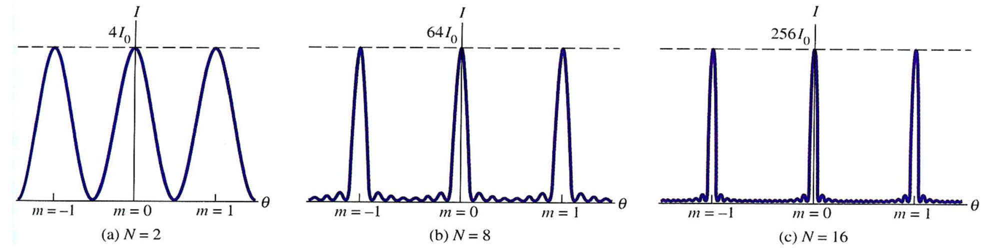
$$= E_0 \int_{-b/2}^{b/2} \exp(ik_y y)dy \sum_{n=0}^{N-1} \exp(ik_y an)$$

$$\propto E_0 b \frac{\sin\beta}{\beta} \frac{\sin(N\alpha)}{\sin(\alpha)}$$

Diffraction ($\frac{\sin\beta}{\beta}$) — Interference ($\frac{\sin(N\alpha)}{\sin(\alpha)}$)

Again: factorization of diffraction from single slit & mutual interference

Fraunhofer diffraction behind N small slits (b → 0)



$$E(\theta) \propto \frac{\sin(N\alpha)}{\sin(\alpha)} \quad ; \quad \alpha = \tfrac{1}{2} k_y a$$

Q: What happens at α= mπ?

A: Diffraction maximum (orde m with a.sinθ=mλ)

Q: How does peak intensity scale with N?

A: Intensiteit ∝ N^2 (coherent sources)

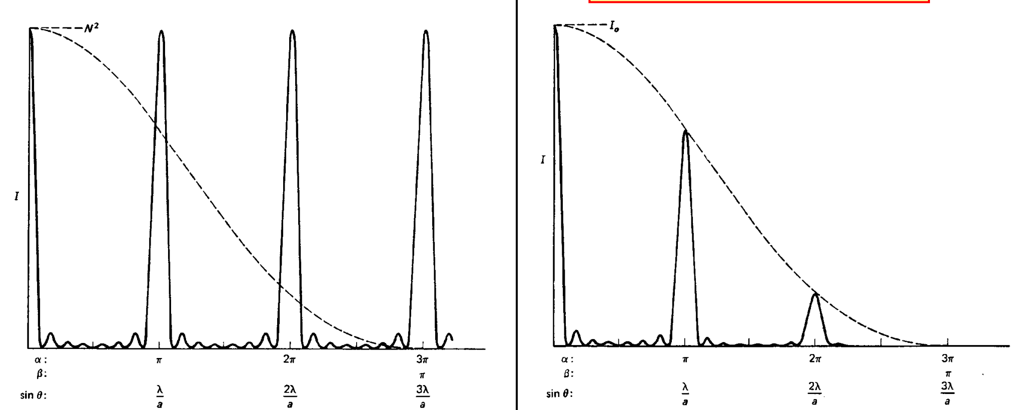
$$\lim_{\alpha \to m\pi} \frac{\sin(N\alpha)}{\sin(\alpha)} = N$$

Better angular resolution with multiple sources/detectors

Westerbork telescope array: λ ≈ 21 cm (ν ≈1.4 GHz)

Angular resolution of array

$$I_{tot}(\theta) = I_{dif.}(\theta) \times I_{N-slit}(\theta)$$



P&P Fig. 16-14