Outline

1. Electromagnetic Waves
2. Material Properties
3. Electromagnetic Waves Across Interfaces
4. Fresnel Equations

Electromagnetic Waves

Electromagnetic Waves in Matter

- *Maxwell's equations* $\Rightarrow$ electromagnetic waves
- optics: interaction of electromagnetic waves with matter as described by *material equations*
- polarization of electromagnetic waves are integral part of optics

Maxwell's Equations in Matter

$$\nabla \cdot \vec{D} = 4\pi\rho$$

$$\nabla \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c}\vec{j}$$

$$\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{B} = 0$$

Symbols

$\vec{D}$	*electric displacement*
ρ	*electric charge density*
$\vec{H}$	*magnetic field*
c	*speed of light in vacuum*
$\vec{j}$	*electric current density*
$\vec{E}$	*electric field*
$\vec{B}$	*magnetic induction*
t	time

Linear Material Equations

$$\vec{D} = \epsilon \vec{E}$$
$$\vec{B} = \mu \vec{H}$$
$$\vec{j} = \sigma \vec{E}$$

Symbols

- ϵ *dielectric constant*
- μ *magnetic permeability*
- σ *electrical conductivity*

Isotropic and Anisotropic Media

- isotropic media: ϵ and μ are scalars
- anisotropic media: ϵ and μ are tensors of rank 2
- isotropy of medium broken by
  - anisotropy of material itself (e.g. crystals)
  - external fields (e.g. Kerr effect)

Wave Equation in Matter

- static, homogeneous medium with no net charges: $\rho = 0$
- for most materials: $\mu = 1$
- combine Maxwell, material equations $\Rightarrow$ differential equations for damped (vector) wave

$$\nabla^2 \vec{E} - \frac{\mu\epsilon}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{4\pi\mu\sigma}{c^2}\frac{\partial \vec{E}}{\partial t} = 0$$

$$\nabla^2 \vec{H} - \frac{\mu\epsilon}{c^2}\frac{\partial^2 \vec{H}}{\partial t^2} - \frac{4\pi\mu\sigma}{c^2}\frac{\partial \vec{H}}{\partial t} = 0$$

- damping controlled by conductivity σ
- $\vec{E}$ and $\vec{H}$ are equivalent $\Rightarrow$ sufficient to consider $\vec{E}$
- interaction with matter almost always through $\vec{E}$
- but: at interfaces, boundary conditions for $\vec{H}$ are crucial

Plane-Wave Solutions

- Plane Vector Wave ansatz $\vec{E} = \vec{E}_0 e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$
  - $\vec{k}$ spatially and temporally constant *wave vector*
  - $\vec{k}$ normal to surfaces of constant phase
  - $|\vec{k}|$ *wave number*
  - $\vec{x}$ spatial location
  - ω *angular frequency* ($2\pi\times$ frequency ν)
  - t time
  - $\vec{E}_0$ (generally complex) vector independent of time and space
- could also use $\vec{E} = \vec{E}_0 e^{-i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$
- damping if $\vec{k}$ is complex
- $\vec{E}_0$ describes the polarization and the absolute phase
- real electric field vector given by real part of $\vec{E}$
- sum of solutions is also a solution

Complex Index of Refraction

- temporal derivatives $\Rightarrow$ Helmholtz equation

$$\nabla^2 \vec{E} + \frac{\omega^2 \mu}{c^2} \left(\epsilon + i \frac{4\pi\sigma}{\omega} \right) \vec{E} = 0$$

- spatial derivatives $\Rightarrow$ *dispersion relation* between $\vec{k}$ and ω

$$\vec{k} \cdot \vec{k} = \frac{\omega^2 \mu}{c^2} \left(\epsilon + i \frac{4\pi\sigma}{\omega} \right)$$

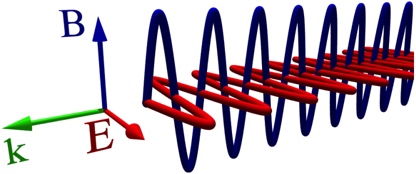
- *complex index of refraction*

$$\tilde{n}^2 = \mu \left(\epsilon + i \frac{4\pi\sigma}{\omega} \right), \quad \vec{k} \cdot \vec{k} = \frac{\omega^2}{c^2} \tilde{n}^2$$

- split into real (n: *index of refraction*) and imaginary parts (k: *extinction coefficient*)

$$\tilde{n} = n + ik$$

Transverse Waves



- plane-wave solution must also fulfill Maxwell's equations

$$\vec{E}_0 \cdot \vec{k} = 0, \quad \vec{H}_0 \cdot \vec{k} = 0, \quad \vec{H}_0 = \frac{\tilde{n}}{\mu} \frac{\vec{k}}{|\vec{k}|} \times \vec{E}_0$$

- isotropic media: electric, magnetic field vectors normal to wave vector $\Rightarrow$ transverse waves
- $\vec{E}_0$, $\vec{H}_0$, and $\vec{k}$ orthogonal to each other, right-handed vector-triple
- conductive medium $\Rightarrow$ complex $\tilde{n}$, $\vec{E}_0$ and $\vec{H}_0$ out of phase
- $\vec{E}_0$ and $\vec{H}_0$ have constant relationship $\Rightarrow$ consider only $\vec{E}$

Energy Propagation in Isotropic Media

- time-averaged *Poynting vector*

$$\left\langle \vec{S} \right\rangle = \frac{c}{8\pi} \mathrm{Re} \left(\vec{E}_0 \times \vec{H}_0^* \right)$$

Re real part of complex expression
$*$ complex conjugate
$\langle . \rangle$ time average

- energy flow parallel to wave vector in isotropic media

$$\left\langle \vec{S} \right\rangle = \frac{c}{8\pi} \frac{|\tilde{n}|}{\mu} |E_0|^2 \frac{\vec{k}}{|\vec{k}|}$$

- energy flow is proprtional to index of refraction!
- in anisotropic materials (e.g. crystals), energy propagation and wave vector are not parallel!

Quasi-Monochromatic Light

- monochromatic light: purely theoretical concept
- monochromatic light wave always fully polarized
- real life: light includes range of wavelengths $\Rightarrow$ *quasi-monochromatic light*
- quasi-monochromatic: superposition of mutually incoherent monochromatic light beams whose wavelengths vary in narrow range $\delta\lambda$ around central wavelength λ_0

$$\frac{\delta\lambda}{\lambda} \ll 1$$

- measurement of quasi-monochromatic light: integral over measurement time t_m
- amplitude, phase (slow) functions of time for given spatial location
- *slow*: variations occur on time scales much longer than the mean period of the wave

Polychromatic Light or White Light

- wavelength range comparable wavelength ($\frac{\delta\lambda}{\lambda} \sim 1$)
- incoherent sum of quasi-monochromatic beams that have large variations in wavelength
- cannot write electric field vector in a plane-wave form
- must take into account frequency-dependent material characteristics
- intensity of polychromatic light is given by sum of intensities of constituting quasi-monochromatic beams

Material Properties

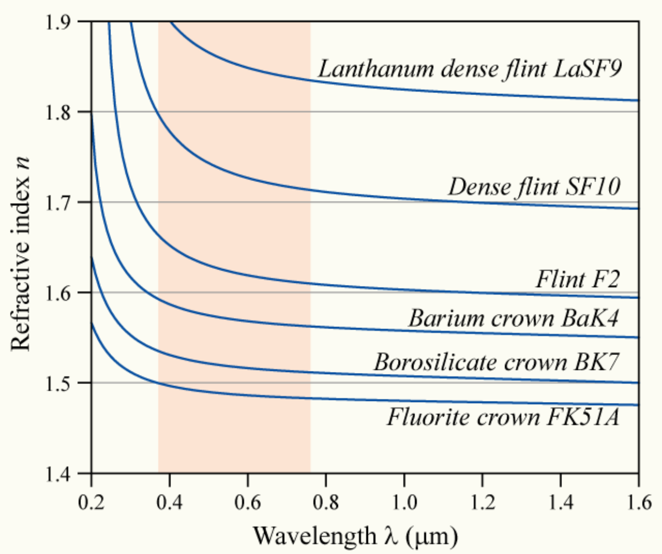
Index of Refraction

- complex index of refraction

$$\tilde{n}^2 = \mu \left(\epsilon + i \frac{4\pi\sigma}{\omega} \right), \quad \vec{k} \cdot \vec{k} = \frac{\omega^2}{c^2} \tilde{n}^2$$

- no electrical conductivity $\Rightarrow$ real index of refraction
- transparent, dielectric materials: real index of refraction
- conducting materials (metal): complex index of refraction
- also transparent, conducting materials, e.g. ITO
- index of refraction depends on wavelength (dispersion)
- index of refraction depends on temperature
- index of refraction roughly proportional to density

Glass Dispersion



http://en.wikipedia.org/wiki/File:Dispersion-curve.png

Wavelength Dependence of Index of Refraction

- tabulated by glass manufacturer
- various approximations to express wavelength dependence with a few parameters
- typically index increases with decreasing wavelength
- Abbé number:

$$\nu_d = \frac{n_d - 1}{n_F - n_C}$$

- n_d: index of refraction at Fraunhofer d line (587.6 nm)
- n_F: index of refraction at Fraunhofer F line (486.1 nm)
- n_C: index of refraction at Fraunhofer C line (656.3 nm)
- low dispersion materials have high values of ν_d
- Abbe diagram: ν_d vs n_d

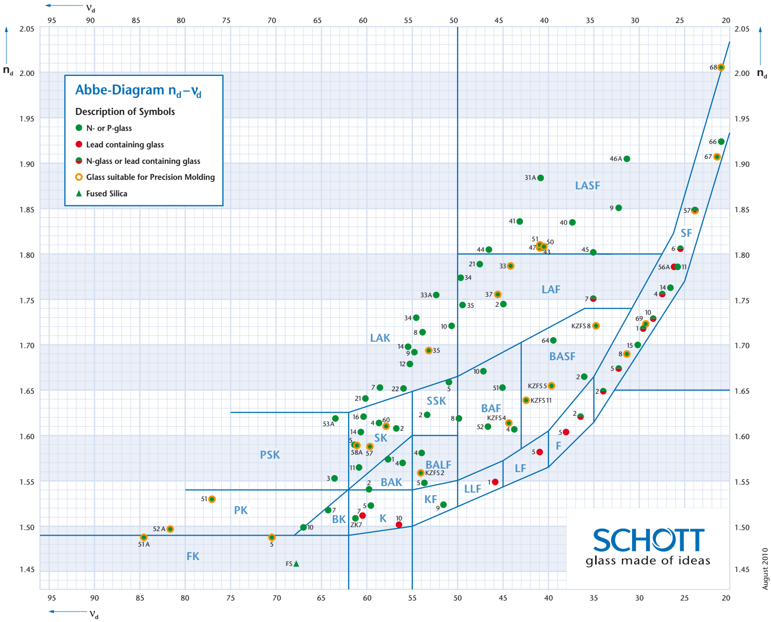
Empirical Models of Index of Refraction

- tabulated, measured index as function of wavelength
- approximations to express smooth wavelength dependence
- most common: Sellmeier equation and coefficients:

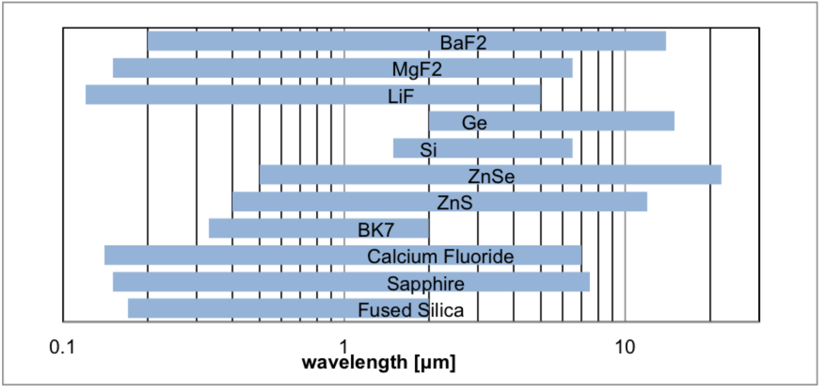
$$n^2(\lambda) = 1 + \frac{B_1\lambda^2}{\lambda^2 - C_1} + \frac{B_2\lambda^2}{\lambda^2 - C_2} + \frac{B_3\lambda^2}{\lambda^2 - C_3}$$

- reflects resonances that drive index variation with wavelength
- only applicable over a limited wavelength range
- BK7: B_1 =1.03961212, C_1 =6.00069867e-3, B_2 =2.31792344e-1, C_2 =2.00179144e-2, B_3 =1.01046945, C_3 =1.03560653e2

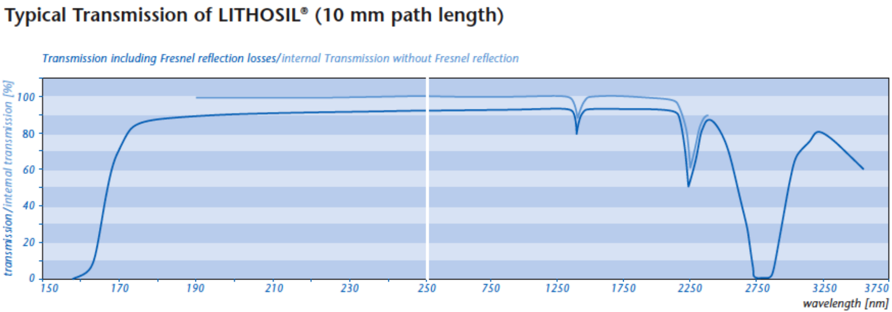
Glasses


Abbe-Diagram $n_d - \nu_d$
Description of Symbols
N- or P-glass
Lead containing glass
N-glass or lead containing glass
Glass suitable for Precision Molding
Fused Silica
LASF
LAF
SF
LAK
BASF
SSK
BAF
SK
PSK
BALF
BAK
LF
F
LLF
KF
PK
BK
K
FK
SCHOTT
glass made of ideas
August 2010

Transparent Materials

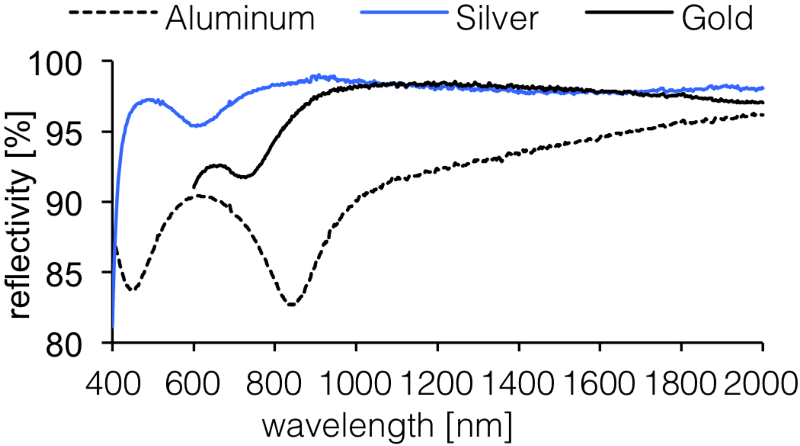


Internal Transmission



- internal transmission per cm
- typically strong absorption in the blue and UV
- almost all glass absorbs above 2 μm

Metal Reflectivity



Introduction

- classical optics due to interfaces between 2 different media
- from Maxwell's equations in integral form at interface from medium 1 to medium 2

$$
\begin{aligned}
\left(\vec{D}_2 - \vec{D}_1\right) \cdot \vec{n} &= 4\pi\Sigma \\
\left(\vec{B}_2 - \vec{B}_1\right) \cdot \vec{n} &= 0 \\
\left(\vec{E}_2 - \vec{E}_1\right) \times \vec{n} &= 0 \\
\left(\vec{H}_2 - \vec{H}_1\right) \times \vec{n} &= -\frac{4\pi}{c}\vec{K}
\end{aligned}
$$

$\vec{n}$ normal on interface, points from medium 1 to medium 2
Σ surface charge density on interface
$\vec{K}$ surface current density on interface

Fields at Interfaces

- $\Sigma = 0$ in general, $\vec{K} = 0$ for dielectrics
- *complex* index of refraction includes effects of currents $\Rightarrow \vec{K} = 0$
- requirements at interface between media 1 and 2

$$
\begin{aligned}
\left(\vec{D}_2 - \vec{D}_1\right) \cdot \vec{n} &= 0 \\
\left(\vec{B}_2 - \vec{B}_1\right) \cdot \vec{n} &= 0 \\
\left(\vec{E}_2 - \vec{E}_1\right) \times \vec{n} &= 0 \\
\left(\vec{H}_2 - \vec{H}_1\right) \times \vec{n} &= 0
\end{aligned}
$$

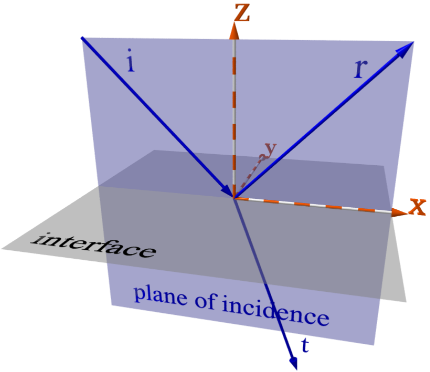
- normal components of $\vec{D}$ and $\vec{B}$ are continuous across interface
- tangential components of $\vec{E}$ and $\vec{H}$ are continuous across interface

Plane of Incidence

- plane wave onto interface
- incident (^{i}), reflected (^{r}), and transmitted (^{t}) waves

$$\vec{E}^{i,r,t} = \vec{E}_0^{i,r,t} e^{i\left(\vec{k}^{i,r,t}\cdot\vec{x}-\omega t\right)}$$
$$\vec{H}^{i,r,t} = \frac{c}{\mu\omega}\vec{k}^{i,r,t} \times \vec{E}^{i,r,t}$$

- interface normal $\vec{n} \parallel z$-axis



- spatial, temporal behavior at interface the same for all 3 waves

$$(\vec{k}^i \cdot \vec{x})_{z=0} = (\vec{k}^r \cdot \vec{x})_{z=0} = (\vec{k}^t \cdot \vec{x})_{z=0}$$

- valid for all $\vec{x}$ in interface $\Rightarrow$ all 3 wave vectors in one plane, *plane of incidence*

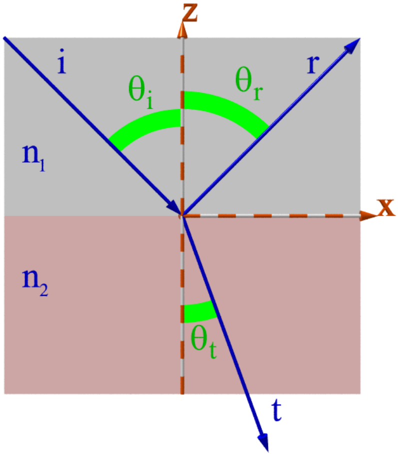
Snell's Law

- spatial, temporal behavior the same for all three waves

$$(\vec{k}^i \cdot \vec{x})_{z=0} = (\vec{k}^r \cdot \vec{x})_{z=0} = (\vec{k}^t \cdot \vec{x})_{z=0}$$

- $\left|\vec{k}\right| = \frac{\omega}{c}\tilde{n}$
- ω, c the same for all 3 waves
- *Snell's law*

$$\tilde{n}_1 \sin\theta_i = \tilde{n}_1 \sin\theta_r = \tilde{n}_2 \sin\theta_t$$



Monochromatic Wave at Interface

- $$\vec{H}_0^{i,r,t} = \frac{c}{\omega\mu}\vec{k}^{i,r,t} \times \vec{E}_0^{i,r,t}, \quad \vec{B}_0^{i,r,t} = \frac{c}{\omega}\vec{k}^{i,r,t} \times \vec{E}_0^{i,r,t}$$
- boundary conditions for monochromatic plane wave:

$$\left(\tilde{n}_1^2\vec{E}_0^i + \tilde{n}_1^2\vec{E}_0^r - \tilde{n}_2^2\vec{E}_0^t\right) \cdot \vec{n} = 0$$

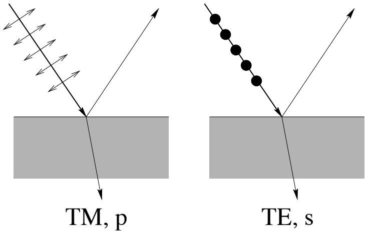
$$\left(\vec{k}^i \times \vec{E}_0^i + \vec{k}^r \times \vec{E}_0^r - \vec{k}^t \times \vec{E}_0^t\right) \cdot \vec{n} = 0$$

$$\left(\vec{E}_0^i + \vec{E}_0^r - \vec{E}_0^t\right) \times \vec{n} = 0$$

$$\left(\frac{1}{\mu_1}\vec{k}^i \times \vec{E}_0^i + \frac{1}{\mu_1}\vec{k}^r \times \vec{E}_0^r - \frac{1}{\mu_2}\vec{k}^t \times \vec{E}_0^t\right) \times \vec{n} = 0$$

- 4 equations are not independent
- only need to consider last two equations (tangential components of $\vec{E}_0$ and $\vec{H}_0$ are continuous)

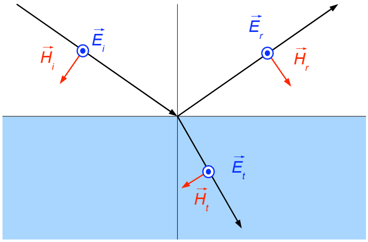
Two Special (Polarization) Cases



1. electric field **p**arallel to plane of incidence $\Rightarrow$ magnetic field is transverse to plane of incidence (TM)
2. electric field particular (German: **s**enkrecht) or transverse to plane of incidence (TE)

- general solution as (coherent) superposition of two cases
- choose direction of magnetic field vector such that Poynting vector parallel, same direction as corresponding wave vector

Electric Field Perpendicular to Plane of Incidence



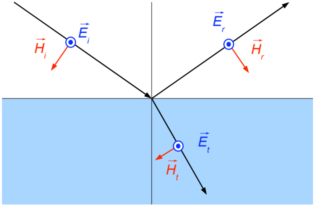
- electric field also perpendicular to interface normal $\vec{n}$
- $\left(\vec{E}_0^i + \vec{E}_0^r - \vec{E}_0^t\right) \times \vec{n} = 0$ becomes (with $E_0^{i,r,t}$ instead of $\vec{E}_0^{i,r,t}$)

$$E_0^i + E_0^r - E_0^t = 0$$

- $\left(\vec{k}^i \times \vec{E}_0^i + \vec{k}^r \times \vec{E}_0^r - \vec{k}^t \times \vec{E}_0^t\right) \times \vec{n} = 0$ becomes

$$\tilde{n}_1 E_0^i \cos\theta_i - \tilde{n}_1 E_0^r \cos\theta_r - \tilde{n}_2 E_0^t \cos\theta_t = 0$$

Electric Field Perpendicular to Plane of Incidence

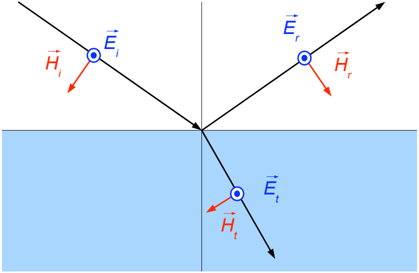


- from previous slide:

$$\tilde{n}_1 E_0^i \cos\theta_i - \tilde{n}_1 E_0^r \cos\theta_r - \tilde{n}_2 E_0^t \cos\theta_t = 0$$

- $\vec{k}^{i,r,t} \times \vec{E}_0^{i,r,t}$ in direction of $\vec{H}_0^{i,r,t}$
- Poynting vector in same direction as wave vector $\Rightarrow$ flip sign of tangential component of magnetic field vector of reflected wave
- reason for minus sign for reflected component in above equation
- $\cos\theta_{i,r,t}$ terms from projecting $\vec{k}^{i,r,t} \times \vec{E}_0^{i,r,t}$ onto interface plane

Electric Field Perpendicular to Plane of Incidence

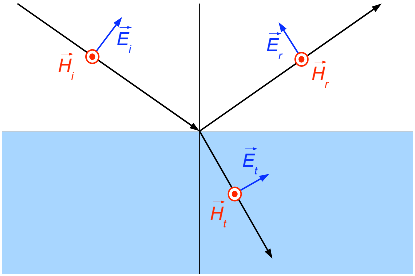


- $\theta_r = \theta_i$
- ratios of reflected and transmitted to incident wave amplitudes

$$r_s = \frac{E_0^r}{E_0^i} = \frac{\tilde{n}_1 \cos\theta_i - \tilde{n}_2 \cos\theta_t}{\tilde{n}_1 \cos\theta_i + \tilde{n}_2 \cos\theta_t}$$

$$t_s = \frac{E_0^t}{E_0^i} = \frac{2\tilde{n}_1 \cos\theta_i}{\tilde{n}_1 \cos\theta_i + \tilde{n}_2 \cos\theta_t}$$

Electric Field in Plane of Incidence

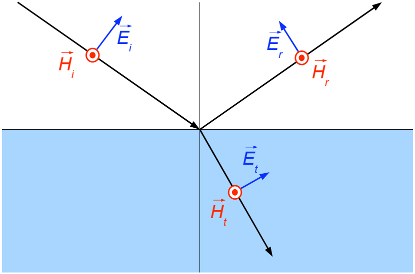


- $\left(\vec{E}_0^i + \vec{E}_0^r - \vec{E}_0^t\right) \times \vec{n} = 0$ becomes

$$E_0^i \cos\theta_i - E_0^r \cos\theta_r - E_0^t \cos\theta_t = 0 \ .$$

- flip tangential component of electric field vector to align Poynting and wave vectors
- $\cos\theta_{i,r,t}$ terms are due to the cross products $\vec{E}_0^{i,r,t} \times \vec{n}$

Electric Field in Plane of Incidence

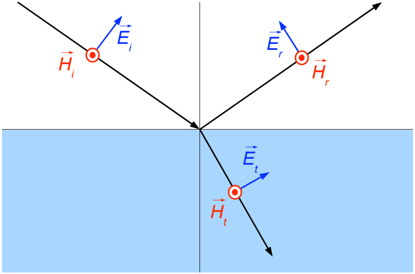


- $\left(\vec{k}^i \times \vec{E}_0^i + \vec{k}^r \times \vec{E}_0^r - \vec{k}^t \times \vec{E}_0^t\right) \times \vec{n} = 0$ becomes

$$\tilde{n}_1 E_0^i + \tilde{n}_1 E_0^r - \tilde{n}_2 E_0^t = 0$$

- $\vec{k}^{i,r,t} \times \vec{E}_0^{i,r,t}$ is in direction of $\vec{H}_0^{i,r,t}$

Electric Field in Plane of Incidence



- ratios of reflected and transmitted to incident wave amplitudes

$$r_p = \frac{E_0^r}{E_0^i} = \frac{\tilde{n}_2 \cos\theta_i - \tilde{n}_1 \cos\theta_t}{\tilde{n}_2 \cos\theta_i + \tilde{n}_1 \cos\theta_t}$$

$$t_p = \frac{E_0^t}{E_0^i} = \frac{2\tilde{n}_1 \cos\theta_i}{\tilde{n}_2 \cos\theta_i + \tilde{n}_1 \cos\theta_t}$$

Summary of Fresnel Equations

- eliminate θ_t using Snell's law $\tilde{n}_2 \cos\theta_t = \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}$
- for most materials $\mu_1/\mu_2 \approx 1$
- electric field amplitude transmission $t_{s,p}$, reflection $r_{s,p}$

$$
\begin{aligned}
t_s &= \frac{2\tilde{n}_1 \cos\theta_i}{\tilde{n}_1 \cos\theta_i + \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}} \\
t_p &= \frac{2\tilde{n}_1 \tilde{n}_2 \cos\theta_i}{\tilde{n}_2^2 \cos\theta_i + \tilde{n}_1 \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}} \\
r_s &= \frac{\tilde{n}_1 \cos\theta_i - \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}}{\tilde{n}_1 \cos\theta_i + \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}} \\
r_p &= \frac{\tilde{n}_2^2 \cos\theta_i - \tilde{n}_1 \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}}{\tilde{n}_2^2 \cos\theta_i + \tilde{n}_1 \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}}
\end{aligned}
$$

Consequences of Fresnel Equations

- complex index of refraction $\Rightarrow t_s, t_p, r_s, r_p$ (generally) complex
- real indices $\Rightarrow$ argument of square root in Snell's law $\tilde{n}_2 \cos\theta_t = \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}$ can still be negative $\Rightarrow$ complex t_s, t_p, r_s, r_p
- real indices, arguments of square roots positive (e.g. dielectric without total internal reflection)
  - therefore $t_{s,p} \geq 0$, real $\Rightarrow$ incident and transmitted waves will have same phase
  - therefore $r_{s,p}$ real, but become negative when $n_2 > n_1 \Rightarrow$ negative ratios indicate phase change by 180° on reflection by medium with larger index of refraction

Other Form of Fresnel Equations

- using trigonometric identities

$$
\begin{aligned}
t_s &= \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)} \\
t_p &= \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)} \\
r_s &= -\frac{\sin(\theta_i-\theta_t)}{\sin(\theta_i+\theta_t)} \\
r_p &= \frac{\tan(\theta_i-\theta_t)}{\tan(\theta_i+\theta_t)}
\end{aligned}
$$

- refractive indices “hidden” in angle of transmitted wave, θ_t
- can always rework Fresnel equations such that only ratio of refractive indices appears
- $\Rightarrow$ Fresnel equations do not depend on absolute values of indices
- can arbitrarily set index of air to 1; then only use indices of media measured relative to air

Relative Amplitudes for Arbitrary Polarization

- electric field vector of incident wave $\vec{E}_0^i$, length E_0^i, at angle α to plane of incidence
- decompose into 2 components: parallel and perpendicular to interface

$$E_{0,p}^i = E_0^i \cos\alpha\,, \quad E_{0,s}^i = E_0^i \sin\alpha$$

- use Fresnel equations to obtain corresponding (complex) amplitudes of reflected and transmitted waves

$$E_{0,p}^{r,t} = (r_p, t_p)\, E_0^i \cos\alpha\,, \quad E_{0,s}^{r,t} = (r_s, t_s)\, E_0^i \sin\alpha$$

Reflectivity

- Fresnel equations apply to electric field amplitude
- need to determine equations for intensity of waves
- time-averaged Poynting vector $\left\langle \vec{S} \right\rangle = \frac{c}{8\pi} \frac{|\tilde{n}|}{\mu} \left| E_0 \right|^2 \frac{\vec{k}}{|\vec{k}|}$
- absolute value of complex index of refraction enters
- energy along wave vector and not along interface normal
- each wave propagates in different direction $\Rightarrow$ consider energy of each wave passing through unit surface area on interface
- does not matter for reflected wave $\Rightarrow$ ratio of reflected and incident intensities is independent of these two effects
- relative intensity of reflected wave (*reflectivity*)

$$R = \frac{\left| E_0^r \right|^2}{\left| E_0^i \right|^2}$$

Transmissivity

- transmitted intensity: multiplying amplitude squared ratios with
  - ratios of indices of refraction (different speed of light)
  - projected area on interface
- relative intensity of transmitted wave (*transmissivity*)

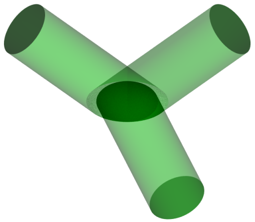
$$T = \frac{|\tilde{n}_2| \cos\theta_t \left|E_0^t\right|^2}{|\tilde{n}_1| \cos\theta_i \left|E_0^i\right|^2}$$

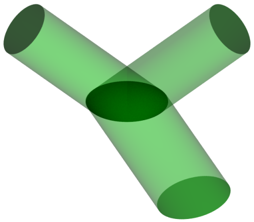
- arbitrarily polarized light with $\vec{E}_0^i$ at angle α to plane of incidence

$$\begin{aligned} R &= |r_p|^2 \cos^2\alpha + |r_s|^2 \sin^2\alpha \\ T &= \frac{|\tilde{n}_2| \cos\theta_t}{|\tilde{n}_1| \cos\theta_i} \left(|t_p|^2 \cos^2\alpha + |t_s|^2 \sin^2\alpha\right) \end{aligned}$$

- $R + T = 1$ for dielectrics, not for conducting, absorbing materials

Projection Correction





Reflection and Transmission