Outline

Homogeneous, Anisotropic Media

Introduction

- material equations for homogeneous, anisotropic media

$$\vec{D} = \epsilon \vec{E}$$
$$\vec{B} = \mu \vec{H}$$

- tensors of rank 2, written as 3 by 3 matrices
  - ϵ: *dielectric tensor*
  - μ: *magnetic permeability tensor*
- for the following, assume $\mu = 1$
- examples:
  - crystals, liquid crystals
  - external electric, magnetic fields acting on isotropic materials (glass, fluids, gas)
  - anisotropic mechanical forces acting on isotropic materials

Properties of Dielectric Tensor

- Maxwell equations imply symmetric dielectric tensor

$$\epsilon = \epsilon^T = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{pmatrix}$$

- symmetric tensor of rank 2 $\Rightarrow$ coordinate system exists where tensor is diagonal
- orthogonal axes of this coordinate system: *principal axes*
- elements of diagonal tensor: *principal dielectric constants*
- 3 *principal indices of refraction* in coordinate system spanned by principal axes

$$\vec{D} = \begin{pmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{pmatrix} \vec{E}$$

- x, y, z because principal axes form Cartesian coordinate system

Uniaxial Materials

- isotropic materials: $n_x = n_y = n_z$
- anisotropic materials: $n_x \neq n_y \neq n_z$
- *uniaxial materials*: $n_x = n_y \neq n_z$
- *ordinary index of refraction*: $n_o = n_x = n_y$
- *extraordinary index of refraction*: $n_e = n_z$
- rotation of coordinate system around z has no effect
- most materials used in polarimetry are (almost) uniaxial

Crystals

Crystal Axes Terminology

- *optic axis* is the axis that has a different index of refraction
- also called *c* or *crystallographic axis*
- *fast axis*: axis with smallest index of refraction
- ray of light going through uniaxial crystal is (generally) split into two rays
- *ordinary ray* (*o-ray*) passes the crystal without any deviation
- *extraordinary ray* (*e-ray*) is deviated at air-crystal interface
- two emerging rays have orthogonal polarization states
- common to use indices of refraction for ordinary ray (n_o) and extraordinary ray (n_e) instead of indices of refraction in crystal coordinate system
- $n_e < n_o$: *negative* uniaxial crystal
- $n_e > n_o$: *positive* uniaxial crystal

Plane Waves in Anisotropic Media

Displacement and Electric Field Vectors

- plane-wave ansatz for $\vec{D}$, $\vec{E}$, $\vec{H}$

$$\vec{E} = \vec{E}_0 e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$
$$\vec{D} = \vec{D}_0 e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$
$$\vec{H} = \vec{H}_0 e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

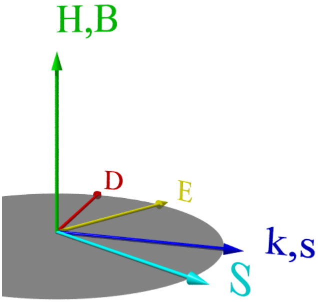
- no net charges in medium ($\nabla \cdot \vec{D} = 0$)

$$\vec{D} \cdot \vec{k} = 0$$

  $\vec{D}$ perpendicular to $\vec{k}$
- $\vec{D}$ and $\vec{E}$ not parallel $\Rightarrow$ $\vec{E}$ not perpendicular to $\vec{k}$
- wave normal $\vec{s} = \vec{k}/|\vec{k}|$, energy flow in different directions, at different speeds

Magnetic Field

- constant, scalar μ, vanishing current density $\Rightarrow \vec{H} \parallel \vec{B}$
- $\nabla \cdot \vec{H} = 0 \Rightarrow \vec{H} \perp \vec{k}$
- $\nabla \times \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \Rightarrow \vec{H} \perp \vec{D}$
- $\nabla \times \vec{E} = -\frac{\mu}{c}\frac{\partial \vec{H}}{\partial t} \Rightarrow \vec{H} \perp \vec{E}$
- $\vec{D}$, $\vec{E}$, and $\vec{k}$ all in one plane
- $\vec{H}$, $\vec{B}$ perpendicular to that plane
- Poynting vector $\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{H}$ perpendicular to $\vec{E}$ and $\vec{H} \Rightarrow \vec{S}$ (in general) not parallel to $\vec{k}$
- energy (in general) not transported in direction of wave vector $\vec{k}$



Relation between $\vec{D}$ and $\vec{E}$

- combine Maxwell, material equations in principal coordinate system

$$D_i = \epsilon_i E_i = n^2 \left(E_i - s_i \left(\vec{E} \cdot \vec{s} \right) \right) \quad i = 1 \cdots 3$$

- $\vec{s} = \vec{k}/|\vec{k}|$: unit vector in direction of wave vector $\vec{k}$
- n: refractive index associated with direction $\vec{s}$, i.e. $n = n(\vec{s})$
- 3 equations for 3 unknowns E_i
- eliminate $\vec{E}$ assuming $\vec{E} \neq \vec{0} \Rightarrow$ *Fresnel equation*

$$\frac{s_x^2}{n^2 - \epsilon_x} + \frac{s_y^2}{n^2 - \epsilon_y} + \frac{s_z^2}{n^2 - \epsilon_z} = \frac{1}{n^2}$$

- with $n_i^2 = \epsilon_i$

$$s_x^2 n_x^2 \left(n^2 - n_y^2\right)\left(n^2 - n_z^2\right) + s_y^2 n_y^2 \left(n^2 - n_x^2\right)\left(n^2 - n_z^2\right) + s_z^2 n_z^2 \left(n^2 - n_x^2\right)\left(n^2 - n_y^2\right) = 0$$

Electric Field in Anisotropic Material

- electric field can also be written as

$$E_k = \frac{n^2 s_k \left(\vec{E} \cdot \vec{s}\right)}{n^2 - \epsilon_k}$$

- equivalent to (*a* a constant)

$$\vec{E} = a \begin{pmatrix} \frac{s_x}{n^2 - n_x^2} \\ \frac{s_y}{n^2 - n_y^2} \\ \frac{s_z}{n^2 - n_z^2} \end{pmatrix}$$

- quadratic equation in $n \Rightarrow$ generally two solutions for given direction $\vec{s}$
- system of 3 equations can be solved for E_k
- denominator vanishes if $\vec{k}$ parallel to a principal axis $\Rightarrow$ treat separately

Non-Absorbing, Non-Active, Anisotropic Materials

- $\vec{k}$ not parallel to a principal axis $\Rightarrow$ ratio of 2 electric field components k and l

$$\frac{E_k}{E_l} = \frac{s_k\left(n^2 - \epsilon_l\right)}{s_l\left(n^2 - \epsilon_k\right)}$$

- ratio is independent of electric field components
- n^2 and ϵ_i real $\Rightarrow$ ratios are real $\Rightarrow$ electric field is linearly polarized
- in non-absorbing, non-active, anisotropic material, 2 waves propagate that have different linear polarization states and different directions of energy flows
- direction of vibration of $\vec{D}$ corresponding to 2 solutions are orthogonal to each other (without proof)

$$\vec{D}_1 \cdot \vec{D}_2 = 0$$

Introduction

- uniaxial media ⇒ dielectric constants:

$$\begin{aligned} \epsilon_x = \epsilon_y &= n_o^2 \\ \epsilon_z &= n_e^2 \end{aligned}$$

- second form of Fresnel equation reduces to

$$\left(n^2 - n_o^2\right)\left[n_o^2\left(s_x^2 + s_y^2\right)\left(n^2 - n_e^2\right) + s_z^2 n_e^2\left(n^2 - n_o^2\right)\right] = 0$$

- two solutions n_1, n_2 given by

$$\begin{aligned} n_1^2 &= n_o^2 \\ \frac{1}{n_2^2} &= \frac{s_x^2 + s_y^2}{n_e^2} + \frac{s_z^2}{n_o^2} \end{aligned}$$

Propagation in General Direction

- (unit) wave vector direction in spherical coordinates

$$\vec{s} = \begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

  - θ: angle between wave vector and optic axis
  - ϕ: azimuth angle in plane perpendicular to optic axis
-

$$\frac{1}{n_2^2} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2}$$

$$n_2(\theta) = \frac{n_o n_e}{\sqrt{n_o^2\sin^2\theta + n_e^2\cos^2\theta}}$$

- take positive root, negative value corresponds to waves propagating in opposite direction

Ordinary and Extraordinary Rays

- from before

$$\frac{1}{n_2^2} = \frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2}$$

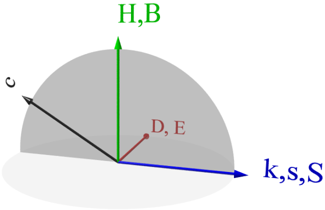
$$n_2(\theta) = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2\theta + n_e^2 \cos^2\theta}}$$

- n_2 varies between n_o for $\theta = 0$ and n_e for $\theta = 90^\circ$
- first solution propagates according to ordinary index of refraction, independent of direction $\Rightarrow$ *ordinary* beam or ray
- second solution corresponds to *extraordinary* beam or ray
- index of refraction of extraordinary beam is (in general) *not* the extraordinary index of refraction

Ordinary Beam

- ordinary beam speed independent of wave vector direction
- for $D_i = \epsilon_i \vec{E}_i = n^2 \left(\vec{E}_i - s_i \left(\vec{E} \cdot \vec{s} \right) \right), i = 1 \cdots 3$ to hold for any direction $\vec{s}$, $\vec{E}_o \cdot \vec{s} = 0$ and $E_{o,z} = 0$
- electric field vector of ordinary beam (with real constant $a_o \neq 0$)

$$\vec{E}_o = a_o \begin{pmatrix} \sin\phi \\ -\cos\phi \\ 0 \end{pmatrix}$$



- ordinary beam is linearly polarized
- $\vec{E}_o$ perpendicular to plane formed by wave vector $\vec{k}$ and c-axis
- displacement vector $\vec{D}_o = n_o \vec{E}_o \parallel \vec{E}_o$
- Poynting vector $\vec{S}_o \parallel \vec{k}$

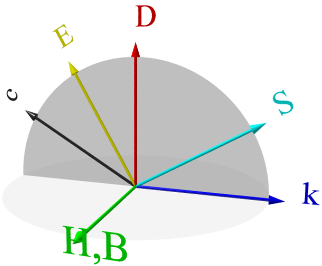
Extraordinary Ray

- since $\vec{D}_e \cdot \vec{k} = 0$ and $\vec{D}_e \cdot \vec{D}_o = 0 \Rightarrow$ unique solution (up to real constant a_e)

$$\vec{D}_e = a_e \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}$$

- since $E_e \cdot D_o = 0$, $D_e = \epsilon \vec{E}_e$

$$\vec{E}_e = a \begin{pmatrix} n_e^2 \cos\theta\cos\phi \\ n_e^2 \cos\theta\sin\phi \\ -n_o^2 \sin\theta \end{pmatrix}$$



- uniaxial medium $\Rightarrow \vec{E}_o \cdot \vec{E}_e = 0$
- however, $\vec{E}_e \cdot \vec{k} \neq 0$

Dispersion Angle

- angle between $\vec{k}$ and Poynting vector $\vec{S}$ = angle between $\vec{E}$ and $\vec{D}$ = *dispersion angle*

$$\tan\alpha = \frac{\left|\vec{E}_e \times \vec{D}_e\right|}{\vec{E}_e \cdot \vec{D}_e} = \frac{(n_e^2 - n_o^2)\tan\theta}{n_e^2 + n_o^2 \tan^2\theta} = \frac{\sin 2\theta}{2}\,\frac{(n_e^2 - n_o^2)}{n_o^2 \sin^2\theta + n_e^2 \cos^2\theta}$$

- equivalent expression

$$\alpha = \theta - \arctan\left(\frac{n_o^2}{n_e^2}\tan\theta\right)$$

- for given $\vec{k}$ in principal axis system, α fully determines direction of energy propagation in uniaxial medium
- for θ approaching $\pi/2$, $\alpha = 0$
- for $\theta = 0$, $\alpha = 0$

Propagation Direction of Extraordinary Beam

- angle θ' between Poynting vector $\vec{S}$ and optic axis

$$\tan\theta' = \frac{n_o^2}{n_e^2}\tan\theta$$

- ordinary and extraordinary wave do (in general) not travel at the same speed
- phase difference in radians between the two waves given by

$$\frac{\omega}{c}\left(n_2(\theta)d_e - n_o d_o\right)$$

- $d_{o,e}$: geometrical distances traveled by ordinary and extraordinary rays

Propagation Along *c* Axis

- plane wave propagating along *c*-axis $\Rightarrow \theta = 0$
- ordinary and extraordinary beams propagate at same speed $\frac{c}{n_o}$
- electric field vectors are perpendicular to *c*-axis and only depend on azimuth ϕ
- ordinary and extraordinary rays are indistinguishable
- uniaxial medium behaves like an isotropic medium
- example: "c-cut" sapphire windows

Propagation Perpendicular to *c* Axis

- plane wave propagating perpendicular to *c*-axis $\Rightarrow \theta = \pi/2$

$$\vec{E}_o = \begin{pmatrix} \sin\phi \\ -\cos\phi \\ 0 \end{pmatrix}$$

- $\vec{E}_o$ perpendicular to plane formed by $\vec{k}$ and *c*-axis
- electric field vector of extraordinary wave

$$\vec{E}_e = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- $\vec{E}_e$ parallel to *c*-axis
- direction of energy propagation of extraordinary wave parallel to $\vec{k}$ since $\vec{E}_e \parallel \vec{D}_e$

Phase Delay between Ordinary and Extraordinary Rays

- ordinary and extraordinary wave propagate in same direction
- ordinary ray propagates with speed $\frac{c}{n_o}$
- extraordinary beam propagates at different speed $\frac{c}{n_e}$
- $\vec{E}_o$, $\vec{E}_e$ perpendicular to each other $\Rightarrow$ plane wave with arbitrary polarization can be (coherently) decomposed into components parallel to $\vec{E}_o$ and $\vec{E}_e$
- 2 components will travel at different speeds
- (coherently) superposing 2 components after distance $d \Rightarrow$ phase difference between 2 components $\frac{\omega}{c}(n_e - n_o)d$ radians
- phase difference $\Rightarrow$ change in polarization state
- basis for constructing linear retarders

Summary: Wave Propagation in Uniaxial Media

- ordinary ray propagates like in an isotropic medium with index n_o
- extraordinary ray sees direction-dependent index of refraction

$$n_2(\theta) = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2\theta + n_e^2 \cos^2\theta}}$$

n_2 direction-dependent index of refraction of the extraordinary ray
n_o ordinary index of refraction
n_e extraordinary index of refraction
θ angle between extraordinary wave vector and optic axis

- extraordinary ray is not parallel to its wave vector
- angle between the two is *dispersion angle*

$$\tan\alpha = \frac{(n_e^2 - n_o^2)\tan\theta}{n_e^2 + n_o^2 \tan^2\theta}$$

Reflection and Transmission at Uniaxial Interfaces

General case

- from isotropic medium (n_I) into uniaxial medium (n_o, n_e)
- θ_I: angle between surface normal and $\vec{k}_I$ for incoming beam
- $\theta_{1,2}$: angles between surface normal and wave vectors of (refracted) ordinary wave $\vec{k}_1$ and extraordinary wave $\vec{k}_2$
- phase matching at interface requires

$$\vec{k}_I \cdot \vec{x} = \vec{k}_1 \cdot \vec{x} = \vec{k}_2 \cdot \vec{x}$$

- $\vec{x}$: position vector of a point on interface surface

$$n_I \sin\theta_I = n_1 \sin\theta_1 = n_2 \sin\theta_2$$

- $n_1 = n_o$: index of refraction of ordinary wave
- n_2: index of refraction of extraordinary wave

Ordinary and Extraordinary Rays

- ordinary wave $\Rightarrow$ Snell's law

$$\sin\theta_1 = \frac{n_I}{n_1}\sin\theta_I$$

- law for extraordinary ray not trivial

$$n_I \sin\theta_I = n_2\left(\theta(\theta_2)\right)\sin\theta_2$$

- (in general) θ_2 and therefore $\vec{k}_2$ will *not* determine direction of extraordinary beam since Poynting vector (in general) not parallel to wave vector
- solve for $\theta_2 \Rightarrow$ determine direction of Poynting vector
- special cases reduce complexity of equations

Extraordinary Ray Refraction for General Case

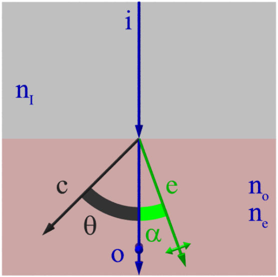
$$\cot\theta_2 = \frac{c_x c_y\left(n_o^2 - n_e^2\right) \pm n_o\sqrt{\frac{n_o^2 n_e^2 + n_e^2 c_x^2\left(n_e^2 - n_o^2\right)}{\sin^2\theta_I} - n_o^2 - \left(n_e^2 - n_o^2\right)\left(c_x^2 + c_y^2\right)}}{n_o^2 + c_x^2\left(n_e^2 - n_o^2\right)}$$

propagation vector of extraordinary ray

$$\begin{aligned}
S_x &= \cos\alpha\cos\theta_2 + \frac{\sin\alpha\sin\theta_2\left(c_x\sin\theta_2 - c_y\cos\theta_2\right)}{\sqrt{c_z^2 + \left(c_x\sin\theta_2 - c_y\cos\theta_2\right)^2}} \\
S_y &= \cos\alpha\sin\theta_2 - \frac{\sin\alpha\cos\theta_2\left(c_x\sin\theta_2 - c_y\cos\theta_2\right)}{\sqrt{c_z^2 + \left(c_x\sin\theta_2 - c_y\cos\theta_2\right)^2}} \\
S_z &= c_z * \frac{\sin\alpha}{\sqrt{c_z^2 + \left(c_x\sin\theta_2 - c_y\cos\theta_2\right)^2}}
\end{aligned}$$

- $\vec{c}$ optic axis vector $\vec{c} = (c_x, c_y, c_z)^T$
- $\vec{S}$ propagation direction of extraordinary ray $\vec{S} = (S_x, S_y, S_z)^T$
- θ_I angle between $\vec{k}_I$ and interface normal
- θ_2 angle between $\vec{k}_e$ and interface normal
- α dispersion angle

Normal Incidence



- normal incidence $\Rightarrow \theta_I = 0$, $\theta_1 = \theta_2 = 0$
- choose plane formed by surface normal and crystal axis
- both wave vectors and ordinary ray not refracted
- extraordinary ray refracted by dispersion angle α

$$\alpha = \theta - \arctan\left(\frac{n_o^2}{n_e^2}\tan\theta\right)$$

Optic Axis in Plane of Incidence and Plane of Interface

- $\theta + \theta_2 = \pi/2 \Rightarrow \cot\theta_2 = \frac{n_e}{n_o}\cot\theta_1$
- θ_1: angle between surface normal and *ordinary* ray or wave vector ($\sin\theta_I = n_o \sin\theta_1$)
- extraordinary wave sees equivalent refractive index

$$n_y = \sqrt{n_e^2 + \sin^2\theta_I \left(1 - \frac{n_e^2}{n_o^2}\right)}$$

- direction of Poynting vector

$$\begin{aligned} S_x &= \cos(\theta_2 + \alpha) \\ S_y &= \sin(\theta_2 + \alpha) \\ S_z &= 0 \end{aligned}$$

- determine dispersion angle α and *add* to θ_2 to obtain direction of extraordinary *ray*

Optic Axis Perpendicular to Plane of Incidence

- c-axis perpendicular to plane of incidence $\Rightarrow \theta = \frac{\pi}{2}$, $n_2\left(\frac{\pi}{2}\right) = n_e$

$$n_l \sin\theta_l = n_e \sin\theta_2$$

- extraordinary wave vector obeys Snell's law with index n_e
- $\theta = \frac{\pi}{2} \Rightarrow$ dispersion angle $\alpha = 0$
- Poynting vector $\parallel$ wave vector, extraordinary beam itself obeys Snell's law with n_e
- double refraction only for non-normal incidence

Interface from Uniaxial Medium to Isotropic Medium

- ordinary ray follows Snell's law
- transmitted extraordinary wave vector and ray coincide
- exit of extraordinary *wave* on interface defined by extraordinary *ray*
- extraordinary wave vector follows Snell's law with index $n_2(\theta)$

$$n_I \sin\theta_E = n_2 \sin\theta_U$$

  - n_I index of isotropic medium
  - θ_E angle of wave/ray vector with surface normal in isotropic medium
  - n_2, θ_U corresponding values for extraordinary *wave vector* in uniaxial medium
- n_2 is function of θ normally already known from beam propagation in uniaxial medium
- θ_U is function of geometry of interface,
- plane-parallel slab of uniaxial medium, $\theta_E = \theta_I$, (in general) extraordinary beam displaced on exit

Total Internal Reflection (TIR)

- TIR also in anisotropic media
- $n_o \neq n_e \Rightarrow$ one beam may be totally reflected while other is transmitted
- principal of most crystal polarizers
- example: calcite prism, normal incidence, optic axis parallel to first interface, exit face inclined by 40°
- $\Rightarrow$ extraordinary ray not refracted, two rays propagate according to indices n_o,n_e
- at second interface rays (and wave vectors) at 40° to surface
- 632.8 nm: $n_o = 1.6558$, $n_e = 1.4852$
- requirement for total reflection $\frac{n_U}{n_I} \sin\theta_U > 1$
- with $n_I = 1 \Rightarrow$ extraordinary ray transmitted, ordinary ray undergoes TIR