Outline

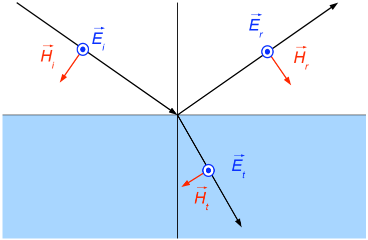
Monochromatic Wave at Interface (from previous lecture)

- using $\vec{H}_0^{i,r,t} = \frac{c}{\omega}\vec{k}^{i,r,t} \times \vec{E}_0^{i,r,t}$
- tangential components of $\vec{E}_0$ and $\vec{H}_0$ are continuous:

$$\left(\vec{E}_0^i + \vec{E}_0^r - \vec{E}_0^t\right) \times \vec{n} = 0$$

$$\left(\vec{k}^i \times \vec{E}_0^i + \vec{k}^r \times \vec{E}_0^r - \vec{k}^t \times \vec{E}_0^t\right) \times \vec{n} = 0$$

Electric Field Perpendicular to Plane of Incidence



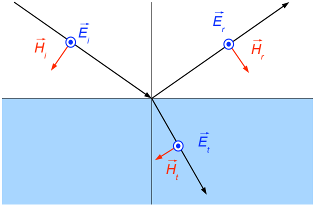
- electric field also perpendicular to interface normal $\vec{n}$
- $\left(\vec{E}_0^i + \vec{E}_0^r - \vec{E}_0^t\right) \times \vec{n} = 0$ becomes (with $E_0^{i,r,t}$ instead of $\vec{E}_0^{i,r,t}$)

$$E_0^i + E_0^r - E_0^t = 0$$

- $\left(\vec{k}^i \times \vec{E}_0^i + \vec{k}^r \times \vec{E}_0^r - \vec{k}^t \times \vec{E}_0^t\right) \times \vec{n} = 0$ becomes

$$\tilde{n}_1 E_0^i \cos\theta_i - \tilde{n}_1 E_0^r \cos\theta_r - \tilde{n}_2 E_0^t \cos\theta_t = 0$$

Electric Field Perpendicular to Plane of Incidence

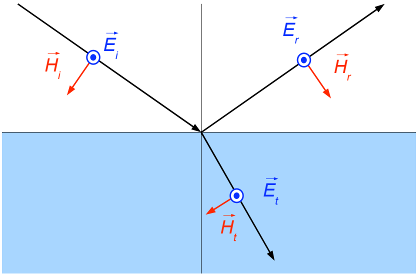


- from previous slide:

$$\tilde{n}_1 E_0^i \cos\theta_i - \tilde{n}_1 E_0^r \cos\theta_r - \tilde{n}_2 E_0^t \cos\theta_t = 0$$

- $\vec{k}^{i,r,t} \times \vec{E}_0^{i,r,t}$ in direction of $\vec{H}_0^{i,r,t}$
- Poynting vector in same direction as wave vector $\Rightarrow$ flip sign of tangential component of magnetic field vector of reflected wave
- reason for minus sign for reflected component in above equation
- $\cos\theta_{i,r,t}$ terms from projecting $\vec{k}^{i,r,t} \times \vec{E}_0^{i,r,t}$ onto interface plane

Electric Field Perpendicular to Plane of Incidence

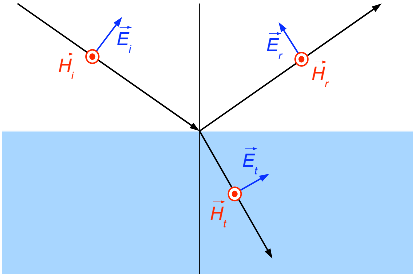


- $\theta_r = \theta_i$
- ratios of reflected and transmitted to incident wave amplitudes

$$r_s = \frac{E_0^r}{E_0^i} = \frac{\tilde{n}_1 \cos\theta_i - \tilde{n}_2 \cos\theta_t}{\tilde{n}_1 \cos\theta_i + \tilde{n}_2 \cos\theta_t}$$

$$t_s = \frac{E_0^t}{E_0^i} = \frac{2\tilde{n}_1 \cos\theta_i}{\tilde{n}_1 \cos\theta_i + \tilde{n}_2 \cos\theta_t}$$

Electric Field in Plane of Incidence

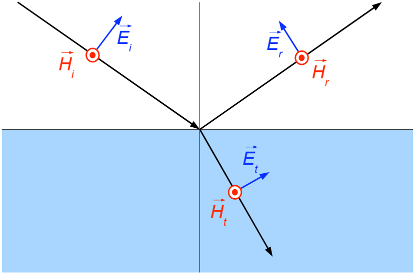


- $\left(\vec{E}_0^i + \vec{E}_0^r - \vec{E}_0^t\right) \times \vec{n} = 0$ becomes

$$E_0^i \cos\theta_i - E_0^r \cos\theta_r - E_0^t \cos\theta_t = 0 \ .$$

- flip tangential component of elecric field vector to align Poynting and wave vectors
- $\cos\theta_{i,r,t}$ terms are due to the cross products $\vec{E}_0^{i,r,t} \times \vec{n}$

Electric Field in Plane of Incidence

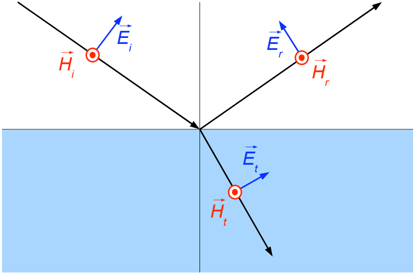


- $\left(\vec{k}^i \times \vec{E}_0^i + \vec{k}^r \times \vec{E}_0^r - \vec{k}^t \times \vec{E}_0^t\right) \times \vec{n} = 0$ becomes

$$\tilde{n}_1 E_0^i + \tilde{n}_1 E_0^r - \tilde{n}_2 E_0^t = 0$$

- $\vec{k}^{i,r,t} \times \vec{E}_0^{i,r,t}$ is in direction of $\vec{H}_0^{i,r,t}$

Electric Field in Plane of Incidence



- ratios of reflected and transmitted to incident wave amplitudes

$$r_p = \frac{E_0^r}{E_0^i} = \frac{\tilde{n}_2 \cos\theta_i - \tilde{n}_1 \cos\theta_t}{\tilde{n}_2 \cos\theta_i + \tilde{n}_1 \cos\theta_t}$$

$$t_p = \frac{E_0^t}{E_0^i} = \frac{2\tilde{n}_1 \cos\theta_i}{\tilde{n}_2 \cos\theta_i + \tilde{n}_1 \cos\theta_t}$$

Summary of Fresnel Equations

- eliminate θ_t using Snell's law $\tilde{n}_2 \cos\theta_t = \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}$
- electric field amplitude transmission $t_{s,p}$, reflection $r_{s,p}$

$$
\begin{aligned}
t_s &= \frac{2\tilde{n}_1 \cos\theta_i}{\tilde{n}_1 \cos\theta_i + \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}} \\
t_p &= \frac{2\tilde{n}_1 \tilde{n}_2 \cos\theta_i}{\tilde{n}_2^2 \cos\theta_i + \tilde{n}_1 \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}} \\
r_s &= \frac{\tilde{n}_1 \cos\theta_i - \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}}{\tilde{n}_1 \cos\theta_i + \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}} \\
r_p &= \frac{\tilde{n}_2^2 \cos\theta_i - \tilde{n}_1 \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}}{\tilde{n}_2^2 \cos\theta_i + \tilde{n}_1 \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}}
\end{aligned}
$$

Consequences of Fresnel Equations

- complex index of refraction $\Rightarrow$ t_s, t_p, r_s, r_p (generally) complex
- real indices $\Rightarrow$ argument of square root in Snell's law $\tilde{n}_2 \cos\theta_t = \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}$ can still be negative $\Rightarrow$ complex t_s, t_p, r_s, r_p
- real indices, arguments of square roots positive (e.g. dielectric without total internal reflection)
  - therefore $t_{s,p} \geq 0$, real $\Rightarrow$ incident and transmitted waves will have same phase
  - therefore $r_{s,p}$ real, but become negative when $n_2 > n_1$ $\Rightarrow$ negative ratios indicate phase change by 180° on reflection by medium with larger index of refraction

Other Form of Fresnel Equations

- using trigonometric identities

$$
\begin{aligned}
t_s &= \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)} \\
t_p &= \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)} \\
r_s &= -\frac{\sin(\theta_i-\theta_t)}{\sin(\theta_i+\theta_t)} \\
r_p &= \frac{\tan(\theta_i-\theta_t)}{\tan(\theta_i+\theta_t)}
\end{aligned}
$$

- refractive indices “hidden” in angle of transmitted wave, θ_t
- can always rework Fresnel equations such that only ratio of refractive indices appears
- $\Rightarrow$ Fresnel equations do not depend on absolute values of indices
- can arbitrarily set index of air to 1; then only use indices of media measured relative to air

Relative Amplitudes for Arbitrary Polarization

- electric field vector of incident wave $\vec{E}_0^i$, length E_0^i, at angle α to plane of incidence
- decompose into 2 components: parallel and perpendicular to interface

$$E_{0,p}^i = E_0^i \cos\alpha\,, \quad E_{0,s}^i = E_0^i \sin\alpha$$

- use Fresnel equations to obtain corresponding (complex) amplitudes of reflected and transmitted waves

$$E_{0,p}^{r,t} = (r_p, t_p)\, E_0^i \cos\alpha\,, \quad E_{0,s}^{r,t} = (r_s, t_s)\, E_0^i \sin\alpha$$

Reflectivity

- Fresnel equations apply to electric field amplitude
- need to determine equations for intensity of waves
- time-averaged Poynting vector $\left\langle\vec{S}\right\rangle = \frac{c}{8\pi}\frac{|\tilde{n}|}{\mu}\left|E_0\right|^2\frac{\vec{k}}{|\vec{k}|}$
- absolute value of complex index of refraction enters
- energy along wave vector and not along interface normal
- each wave propagates in different direction $\Rightarrow$ consider energy of each wave passing through unit surface area on interface
- does not matter for reflected wave $\Rightarrow$ ratio of reflected and incident intensities is independent of these two effects
- relative intensity of reflected wave (*reflectivity*)

$$R = \frac{\left|E_0^r\right|^2}{\left|E_0^i\right|^2}$$

Transmissivity

- transmitted intensity: multiplying amplitude squared ratios with
  - ratios of indices of refraction
  - projected area on interface
- relative intensity of transmitted wave (*transmissivity*)

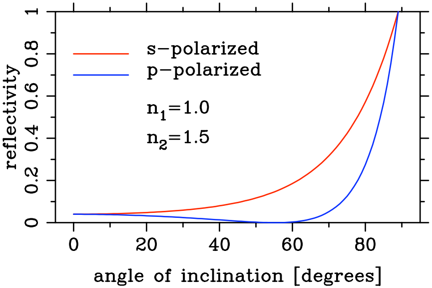
$$T = \frac{|\tilde{n}_2| \cos\theta_t \left|E_0^t\right|^2}{|\tilde{n}_1| \cos\theta_i \left|E_0^i\right|^2}$$

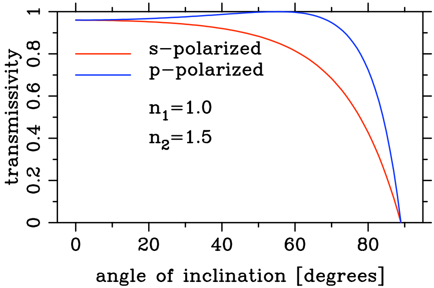
- arbitrarily polarized light with $\vec{E}_0^i$ at angle α to plane of incidence

$$\begin{aligned} R &= |r_p|^2 \cos^2\alpha + |r_s|^2 \sin^2\alpha \\ T &= \frac{|\tilde{n}_2| \cos\theta_t}{|\tilde{n}_1| \cos\theta_i} \left(|t_p|^2 \cos^2\alpha + |t_s|^2 \sin^2\alpha\right) \end{aligned}$$

- $R + T = 1$ for dielectrics, not for conducting, absorbing materials

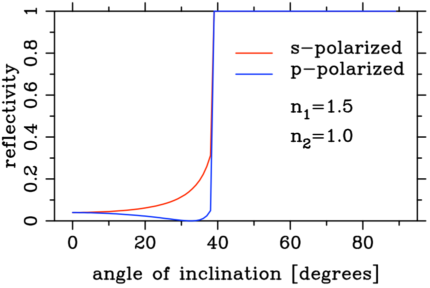
Brewster Angle

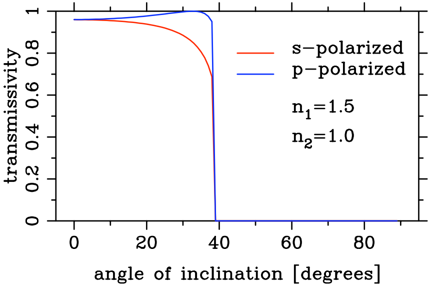




- $r_p = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} = 0$ when $\theta_i + \theta_t = \frac{\pi}{2}$
- corresponds to *Brewster angle* of incidence of $\tan\theta_B = \frac{n_2}{n_1}$
- occurs when reflected wave is perpendicular to transmitted wave
- reflected light is completely s-polarized
- transmitted light is moderately polarized

Total Internal Reflection (TIR)





- Snell's law: $\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i$
- wave from high-index medium into lower index medium (e.g. glass to air): $n_1/n_2 > 1$
- right-hand side > 1 for $\sin\theta_i > \frac{n_2}{n_1}$
- all light is reflected in high-index medium $\Rightarrow$ *total internal reflection*
- transmitted wave has complex phase angle $\Rightarrow$ damped wave along interface

Phase Change on Total Internal Reflection (TIR)

- TIR induces phase change that depends on polarization
- complex ratios: $r_{s,p} = |r_{s,p}| e^{i\delta_{s,p}}$
- phase change $\delta = \delta_s - \delta_p$

$$\tan\frac{\delta}{2} = \frac{\cos\theta_i\sqrt{\sin^2\theta_i - \left(\frac{n_2}{n_1}\right)^2}}{\sin^2\theta_i}$$

- relation valid between critical angle and grazing incidence
- for angles up to critical angle and grazing incidence, phase difference is zero