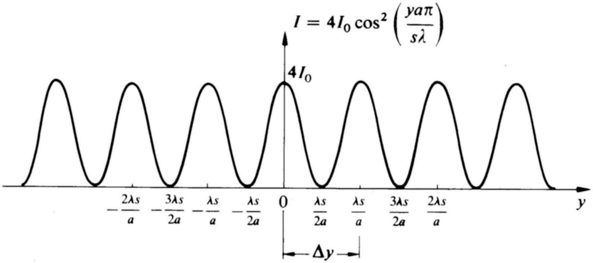
Lecture 11: Interferometers

Outline

1. Interference and Interferometers
2. VLT Interferometer
3. Aperture Synthesis
4. Earth-Rotation Aperture Synthesis
5. WSRT
6. Temporal and Spatial Coherence
7. Etendue of Coherence
8. Intensity Interferometry

Summary: Interference

2-Point Interferometer Fringes from Monochromatic Point Source



- λ wavelength
- s distance to observing plane
- y coordinate perpendicular to fringes
- a interferometer spacing

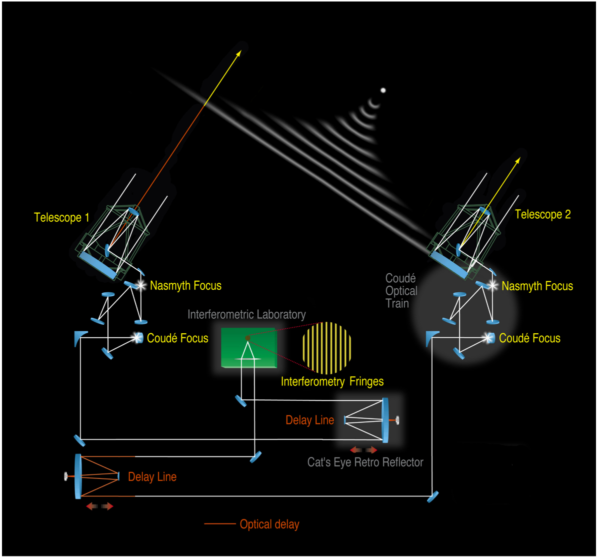
Introduction

- angular resolution of telescope given by λ/D
- small telescopes with diameter *d* and spacing *D* have the same angular resolution
- telescope positions must be known to accuracy $<< \lambda$
- amplitudes must be combined, not intensities
- series of small telescopes with large spacings are cheaper than one large telescope
- standard for radio telescopes, difficult for optical telescopes
- much eassier for infrared than for visible wavelengths
- have to deal with complex point-spread functions that change in time

VLTI, the VLT Interferometer

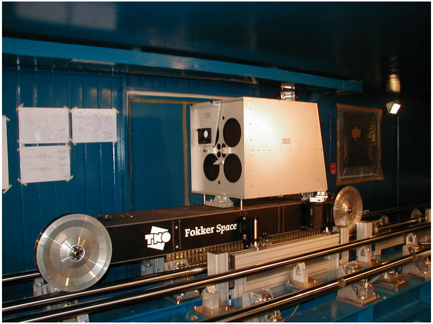
www.mpifr-bonn.mpg.de/public/pr/VLTI-gw2.jpg

VLTI Operating Principle



www.eso.org/public/news/eso0111

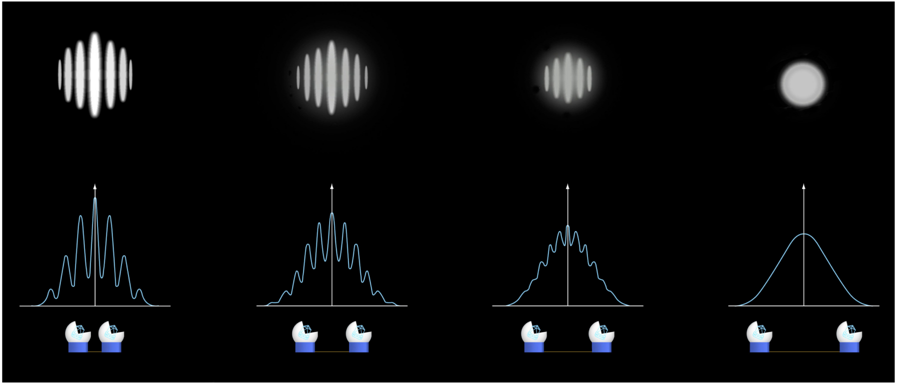
Delay Lines



www.eso.org/public/news/eso0032/

- compensate for telescopes not being on a common mount independent of pointing on sky
- length of the light path changes continuously
- use mirrors on a cart to adjust length of light path continuously
- carts succeed in positioning mirrors with an accuracy of better than 10 nanometers on a track of 60 meters

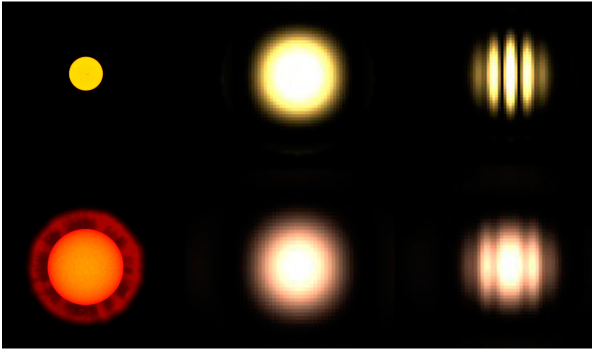
VLTI Single Star Fringes



www.eso.org/public/news/eso0111

- interferometric observations of single star
- increasing distance between telescopes
- change of fringe pattern with separation

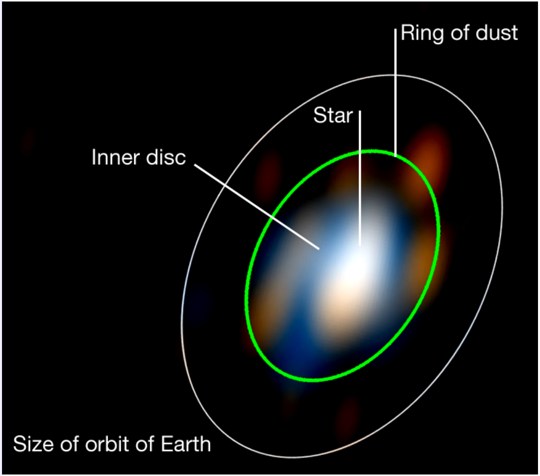
VLTI Stellar Diameter Measurements



www.eso.org/public/news/eso0111

- two stars with different angular sizes (left)
- image from single telescope (middle)
- image from interferometer (right)
- fringe pattern related to star's angular diameter

VLTI Result



- field of view: 25 mas by 25 mas
- H and K band data combined
- Herbig Ae star HD 163296
- combination of VLTI, IOTA, KeckI and CHARA interferometer data

Renard et al. (2010)

Summary: Transfer Functions

- image of a point source: *Point Spread Function (PSF)*

$$\mathrm{PSF} = |FT(A)|^2$$

A aperture (or pupil) function

FT Fourier Transform

- image of arbitrary object is a convolution of object with PSF

$$i = o * s$$

- *Optical Transfer Function (OTF)* is Fourier transform of PSF and vice versa

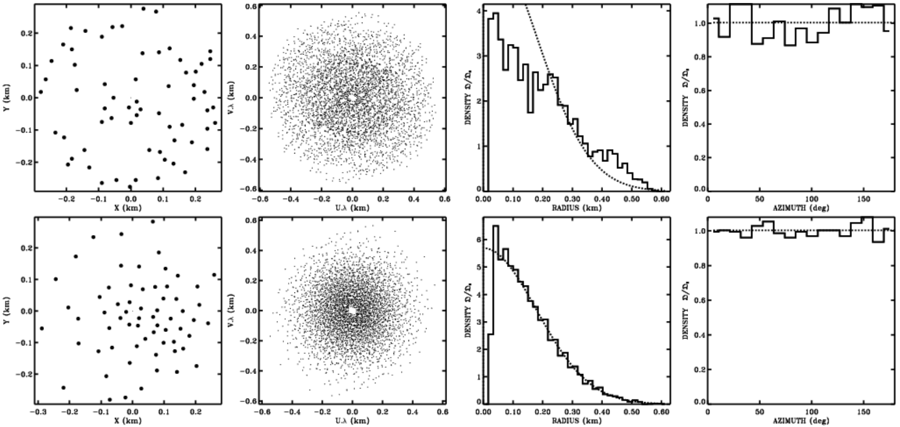
$$I = O \cdot S$$

- OTF is auto-correlation of aperture function
- also applies to PSF and OTF of interferometer

Arrays and their OTFs

- telescope pairs with equal separations and oriented in same direction have the same OTF $\Rightarrow$ redundant baselines
- create non-redundant array configurations
- maximize coverage of uv plane
- can also use sky (Earth) rotation for better coverage

Optimum Array Configuration



Boone (2001)

Interferometer with Finite Apertures

- optical transfer function (OTF) of 2-point interferometer

$$OTF = 2\left(\frac{\lambda}{R}\right)^2 \left[\delta(\vec{\zeta}) + \frac{1}{2}\delta\left(\vec{\zeta} - \vec{s}/\lambda\right) + \frac{1}{2}\delta\left(\vec{\zeta} + \vec{s}/\lambda\right)\right]$$

- pair of pinholes transmits three spatial frequencies
  - DC-component $\delta(\vec{0})$
  - two high frequencies related to length of baseline vector $\vec{s}$ at $\pm\vec{s}/\lambda$
- 3 spatial frequencies represent three-point sampling of the **uv-plane** in 2-d spatial frequency space
- complete sampling of uv-plane provides sufficient information to completely reconstruct original brightness distribution

Point-Spread Function (PSF)

- PSF is Fourier Transform of OTF

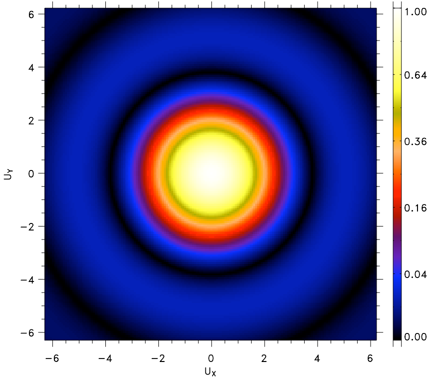
$$\begin{aligned} \delta(\vec{\zeta}) &\Leftrightarrow 1 \\ \delta\left(\vec{\zeta} - \vec{s}/\lambda\right) &\Leftrightarrow e^{i2\pi\vec{\theta}\cdot\vec{s}/\lambda} \\ \delta\left(\vec{\zeta} + \vec{s}/\lambda\right) &\Leftrightarrow e^{-i2\pi\vec{\theta}\cdot\vec{s}/\lambda} \end{aligned}$$

- Point-Spread Function of 2-element interferometer

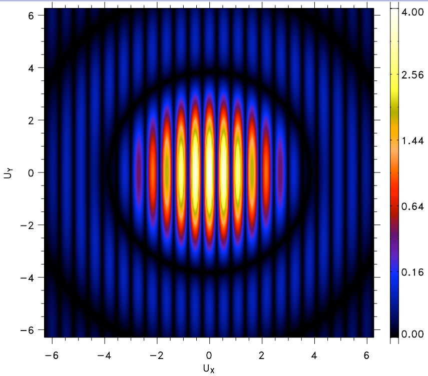
$$\left(\frac{\lambda}{R}\right)^2 \left[2(1 + \cos 2\pi\vec{\theta}\cdot\vec{s}/\lambda)\right] = 4\left(\frac{\lambda}{R}\right)^2 \cos^2 \pi\vec{\theta}\cdot\vec{s}/\lambda$$

- $\vec{\theta}$: 2-d angular coordinate vector
- attenuation factor $(\lambda/R)^2$ from spherical expansion

2-d Brightness Distribution



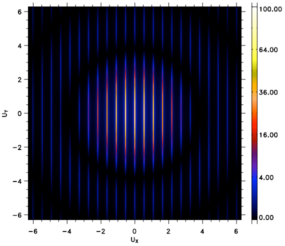
PSF of single circular aperture



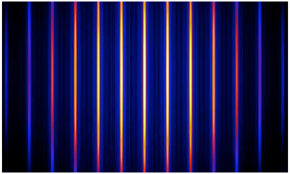
PSF of two-element interferometer, aperture diameter $d = 25$ m, length of baseline vector $|\vec{s}| = 144$ m

- double beam interference fringes showing modulation effect of diffraction by aperture of a single pinhole

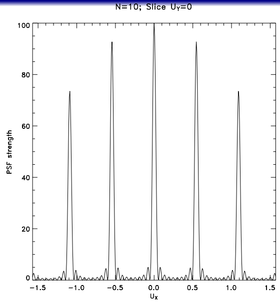
Multi-Aperture Array PSF



PSF of 10-element interferometer with circular apertures



Magnification of central part



Cross-section of central part of PSF for 10-element interferometer

Point-Spread Function of Equally-Spaced Array

- scalar function due to circular symmetry

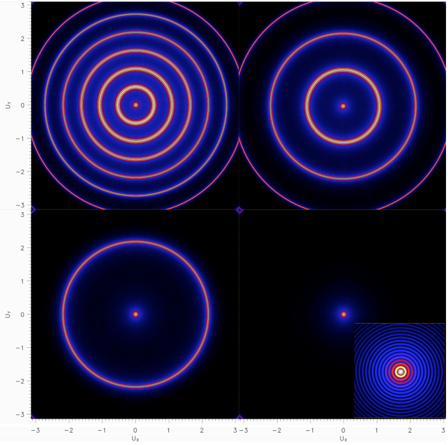
$$PSF_{ERAS} = \left(\frac{\lambda}{R}\right)^2 \left[\frac{1}{4}\pi\,(d/\lambda)^2\right]^2 \left[\frac{2J_1(u)}{u}\right]^2 \frac{\sin^2 N(u\triangle L/D)}{\sin^2(u\triangle L/D)}$$

  with $u = \pi\theta D/\lambda$ and θ, the radially symmetric, diffraction angle
- central peak: similar to Airy function with spatial resolution

$$\triangle\theta = \frac{\lambda}{2L_{max}} radians$$

  with $2L_{max}$ the maximum diameter of the array in the YZ-plane
- concentric grating lobes: angular distances of annuli from central peak follow from the location of principal maxima given by modulation term $\sin^2 N(u\triangle L/D)/\sin^2(u\triangle L/D)$

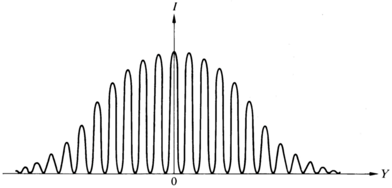
PSF (continued)



- for an N-element array with increment $\triangle L$, these angular positions are given by:

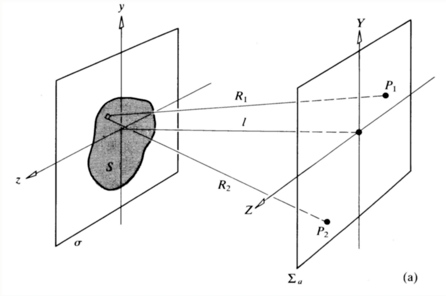
$$\theta_{grating} = \frac{\lambda}{\triangle L}, 2\frac{\lambda}{\triangle L}, \ldots.., (N-1)\frac{\lambda}{\triangle L}$$

Modulation Effect of Aperture



- typical one-dimensional cross-section along $u_y = 0$ of the central part of the interferogram
- visibilities are equal to one, because $I_{min} = 0$
- $|\tilde{\gamma}_{12}(\tau)| = 1$ for all values of τ and any pair of spatial points, if and only if the radiation field is *strictly monochromatic*

The Problem



- relates brightness distribution of extended source and phase correlation between two points in radiation field
- extended source S incoherent, quasi-monochromatic
- positions P_1 and P_2 in observers plane Σ

$$\tilde{E}_1(t)\tilde{E}_2^*(t) = \mathbf{E}\{\tilde{E}_1(t)\tilde{E}_2^*(t)\} = \tilde{\Gamma}_{12}(0)$$

The Solution

- $I(\vec{\Omega})$ is intensity distribution of extended source as function of unit direction vector $\vec{\Omega}$ as seen from observation plane Σ
- $\tilde{\Gamma}(\vec{r})$ is coherence function in Σ-plane
- vector $\vec{r}$ represents arbitrary baseline
- van Cittert-Zernike theorem

$$\tilde{\Gamma}(\vec{r}) = \int\int_{\text{source}} I(\vec{\Omega}) e^{\frac{2\pi i \vec{\Omega}.\vec{r}}{\lambda}} d\vec{\Omega}$$

$$I(\vec{\Omega}) = \lambda^{-2} \int\int_{\Sigma\text{-plane}} \tilde{\Gamma}(\vec{r}) e^{-\frac{2\pi i \vec{\Omega}.\vec{r}}{\lambda}} d\vec{r}$$

- $\tilde{\Gamma}(\vec{r})$ and $I(\vec{\Omega})$ are linked through Fourier transform, except for scaling with wavelength λ
- "true" Fourier transform with *conjugate variables* $\vec{\Omega}$, $\vec{r}/\lambda$,
- Fourier pair: $I(\vec{\Omega}) \quad \Leftrightarrow \quad \tilde{\Gamma}(\vec{r}/\lambda)$

Aperture Synthesis

Overview

- positions 1, 2 in observation plane Σ not pointlike, but finite aperture with diameter D
- single aperture has diffraction-sized beam of λ/D
- Van Cittert-Zernike relation needs to be "weighted" with telescope element (single dish) transfer function $H(\vec{\Omega})$
- circular dish antenna: $H(\vec{\Omega})$ is *Airy brightness function*
- The Van Cittert-Zernike relations now become:

$$\tilde{\Gamma}'(\vec{r}) = \int\int_{\text{source}} I(\vec{\Omega})H(\vec{\Omega})e^{\frac{2\pi i\vec{\Omega}.\vec{r}}{\lambda}}d\vec{\Omega}$$

$$I(\vec{\Omega})H(\vec{\Omega}) = \lambda^{-2}\int\int_{\Sigma\text{-plane}} \tilde{\Gamma}'(\vec{r})e^{-\frac{2\pi i\vec{\Omega}.\vec{r}}{\lambda}}d\vec{r}$$

- field of view scales with λ/D; if λ decreases, synthesis resolution improves but field-of-view reduces proportionally!

Overview (continued)

- aperture synthesis: incoming beams from antenna dish 1 and antenna dish 2 are fed into a *correlator (multiplier)* producing as output $\tilde{E}_1(t)\tilde{E}_2^*(t)$
- output subsequently fed into *integrator/averager* producing

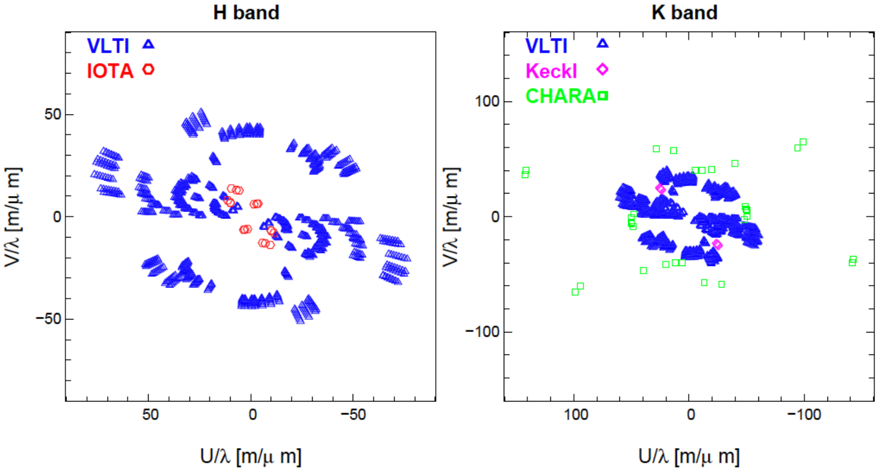
$$\mathbf{E}\left\{\tilde{E}_1(t)\tilde{E}_2^*(t)\right\} = \tilde{\Gamma}'(\vec{r})$$

- applying Fourier transform and correcting for beam profile of single dish $H(\vec{\Omega}) \Rightarrow$ reconstruct source brightness distribution $I(\vec{\Omega})$
- limited to measuring image details within *single pixel* of an individual telescope element, dish

Earth-Rotation Aperture Synthesis

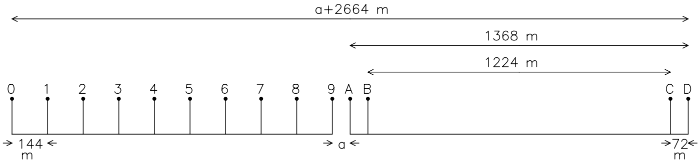
- due to rotation of Earth, baseline vectors $k \cdot \vec{s}/\lambda$ of N-element array scan the YZ-plane if X-axis is lined up with North polar axis
- principal maxima or 'grating lobes' in PSF are concentric annuli around central source peak at angular distances $k \cdot \lambda/|\vec{s}\,|$
- if circular scans in YZ-plane are too widely spaced ($|\vec{s}\,|$ is larger than single dish diameter), the Nyquist criterion is not respected and undersampling of spatial frequency uv-plane (=YZ-plane) occurs
- consequently, grating lobes will show up within the field of view defined by the single-dish beam profile
- can be avoided by decreasing sampling distance $|\vec{s}\,|$

UV Plane Coverage



Renard et al. 2010

Westerbork Synthesis Radio Telescope (WSRT)



- 14 parabolic antennae, diameters $D = 25$ m
- lined up along East-West direction over ≈ 2750 m
- 10 antennae have fixed mutual distance of 144 m
- 4 antennae can be moved collectively with respect to fixed array
- 14 antennae comprise 40 simultaneously operating interferometers
- array is rotated in plane containing Westerbork perpendicular to Earth's rotation axis
- limited to sources near the North polar axis
- standard distance *a* between 9 and A equals 72 meters

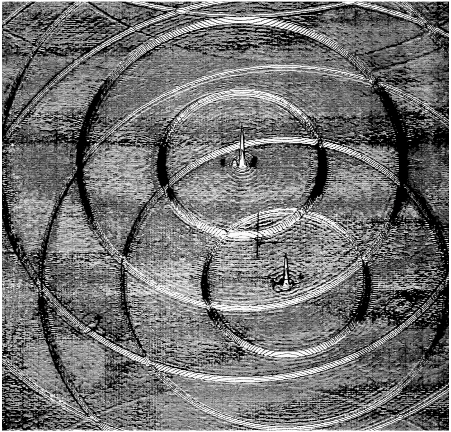
WSRT (continued)

- after 12 hours, 38 concentric semi-circles with radii ranging from $L_{min} = 72$ meters to $L_{max} = 2736$ meters in increments of $\triangle L = 72$ meters
- correlators integrate over 10 s, sampling of semi-circles every 1/24 degrees
- other half can be found by mirroring the first half since $I(\vec{\Omega})$ is a real function

General Case

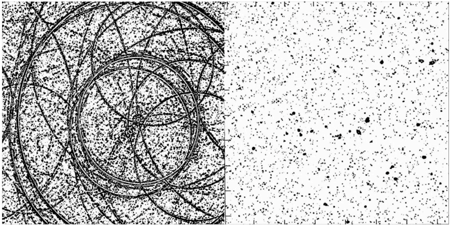
- extended source in arbitrary direction
- during Earth's rotation, antenna beams kept pointed at source
- tip of baseline vector describes a trajectory
- maintain maximum coherence by delaying one antenna signal with respect to the other antenna within fraction of a wavelength
- source at angle ϕ_0 to Earth's rotation axis
- circles in uv-plane change into ellipses and coherence function is sampled on ellipses rather than on circles
- major axes of these ellipses remain equal to the physical length of the WSRT baselines, minor axes are shortened by $\cos\phi_0$
- PSF becomes elliptical $PSF = \frac{\alpha\lambda}{2L_{max}\cos\phi_0}$
- source in equatorial plane: no resolution in one direction
- baselines need North-South components (e.g. VLA)

Imaging



- brightness distribution $I(\vec{\Omega})$ by inverse Fourier transform
- reconstructed $\hat{I}(\vec{\Omega})$ needs to be corrected for single dish response function $H(\vec{\Omega})$

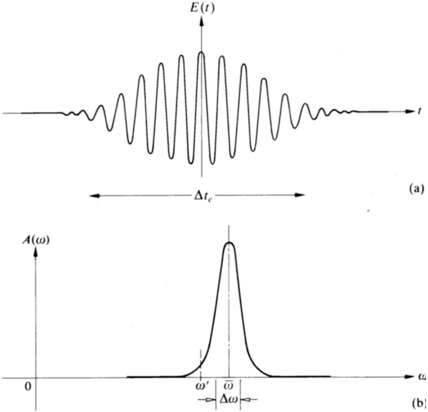
CLEAN



- undersampling of uv-plane, grating lobes within field of view
- decrease distance between antennas 9 and A during second half of rotation for 36 meter increment coverage
- four half rotations in 48 hours can increase coverage to 18 meter increments $\Rightarrow$ complete uv coverage
- incomplete coverage of uv-plane $\Rightarrow$ coherence function $\tilde{\Gamma}(\vec{r})$ are zero in some places $\Rightarrow$ erroneous results
- apply CLEAN method for improving dirty radio maps

Temporal and Spatial Coherence

Temporal Coherence



Gaussian shaped line profile of quasi-monochromatic source and shape of associated wave packet

- temporal coherence characterised by coherence time τ_c
- τ_c due to finite bandwidth of source
- quasi-monochromatic source

$$\tau_c \approx \frac{1}{\triangle\nu}$$

$\triangle\nu$: frequency band width

Temporal Coherence (continued)

- power spectrum $S(\nu)$, autocorrelation $R(\tau)$ related by Fourier transforms

$$S(\nu) = \int_{-\infty}^{+\infty} R(\tau) e^{-2\pi i \nu \tau} d\tau \quad R(\tau) = \int_{-\infty}^{+\infty} S(\nu) e^{2\pi i \nu \tau} d\nu$$

- example: Gaussian-shaped spectral profile

$$S(\nu) \sim e^{-\left(\frac{\nu}{\triangle\nu}\right)^2} \Longleftrightarrow R(\tau) \sim e^{-\left(\frac{\tau}{\tau_c}\right)^2}$$

- corresponding wave packet has Gaussian autocorrelation function with characteristic width τ_c
- correlator produces temporal coherence function $R(\tau)$
- Fourier transform yields spectral distribution $S(\nu)$

Coherence Length

- *coherence length*

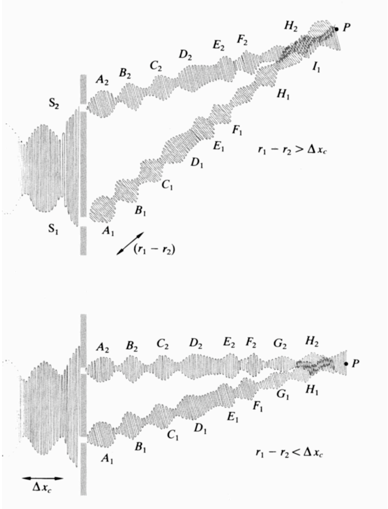
$$l_c = c\tau_c$$

- coherence length in wavelength domain

$$l_c = \frac{\lambda^2}{\triangle\lambda}$$

- quasi-monochromatic wave propagating along a line
  - two positions P_1 and P_2 on this line of propagation at distance R_{12}
  - if $R_{12} \ll l_c$, there will be strong correlation between the EM-fields at P_1 and P_2, interference effects will be possible
  - if $R_{12} \gg l_c$, no interference effects are possible

Coherence Length (continued)



influence of coherence length on interference pattern of two diffracted coherent thermal sources S_1, S_2

- waves in Young's interference experiment
- diffracted beams from coherent sources S_1 and S_2 cause interference pattern
- large path differences $\Rightarrow$ interference contrast reduced

Spatial Coherence

- spatial coherence relates to spatial extent of source
- for $\tau \ll \tau_c$

$$\tilde{\gamma}_{12}(\tau) = \tilde{\gamma}_{12}(0)e^{2\pi i\nu_0\tau}$$

  - $|\tilde{\gamma}_{12}(\tau)| = |\tilde{\gamma}_{12}(0)|$
  - fixed phase difference $\alpha_{12}(\tau) = 2\pi\nu_0\tau$
  - ν_0: average frequency of wave
- frequency bandwidth of radiation source suffiently narrow: comparison between two points with respect to spatial coherence occurs at times differing by $\triangle t \ll \tau_c$

Etendue of Coherence

- circular source of uniform intensity with angular diameter θ_s
- source brightness distribution described as circular two-dimensional window function

$$I(\vec{\Omega}) = \Pi\left(\frac{\theta}{\theta_s}\right)$$

- complex degree of coherence in observation plane Σ at two positions: position 1 at origin, position 2 at distance ρ from origin
- applying van Cittert-Zernike theorem

$$\Pi\left(\frac{\theta}{\theta_s}\right) \Leftrightarrow \tilde{\Gamma}(\rho/\lambda) = \frac{(\theta_s/2)J_1(\pi\theta_s\rho/\lambda)}{\rho/\lambda}$$

- J_1: Bessel function of first kind

Etendue of Coherence (continued)

- normalisation to source brightness $(\pi\theta_s^2)/4$

$$\tilde{\gamma}(\rho) = \frac{2J_1(\pi\theta_s\rho/\lambda)}{\pi\theta_s\rho/\lambda}$$

- modulus of complex degree of coherence

$$|\tilde{\gamma}(\rho)| = \left|\frac{2J_1(u)}{u}\right|$$

  with $u = \pi\theta_s\rho/\lambda$
- defines extent of coherence in observation plane Σ
- for $u = 2$, $|\tilde{\gamma}(\rho)| = J_1(2) = 0.577$
- coherence remains significant for $u \leq 2$, or

$$\rho \leq 2\lambda/(\pi\theta_s)$$

Etendue of Coherence (continued)

- area S in Σ over which coherence remains significant

$$\pi\rho^2 = 4\lambda^2/(\pi\theta_s^2)$$

- $\pi\theta_s^2/4$ equals solid angle Ω_{source} of source
- significant coherence if

$$\epsilon = S\Omega_{\text{source}} \leq \lambda^2$$

- condition $\epsilon = S\Omega_{\text{source}} = \lambda^2$ is called the *Etendue of Coherence*
- needs to be fulfilled if coherent detection is required

Etendue of Coherence: Examples

- red giant, radius $r_0 = 1.5$ x10^{11} meter at 10 parsec distance, $\theta_s = 10^{-6}$ radians
- at $\lambda = 0.5\mu$m, coherence radius ρ, on earth, on screen normal to incident beam is $\rho = 2\lambda/(\pi\theta_s) = 32$ cm
- at $\lambda = 25\mu$m, radius ρ is increased fifty fold to ≈ 15 m
- in radio domain at$\lambda = 6$ cm, $\rho \approx 35$ km

Good Coherence

- good coherence means visibility of 0.88 or better
- uniform circular source: occurs for $u = 1$, that is when $\rho = 0.32\lambda/\theta$
- narrow-bandwidth uniform radiation source at distance R

$$\rho = 0.32(\lambda R)/D$$

- example: red filter over 1-mm-diameter, disk-shaped flashlight at 20 m away: $\rho = 3.8$ mm
- set of apertures spaced at about 4 mm or less should produce clear fringes
- we always assume that comparison between two points occurs at times differing by a $\triangle t \ll \tau_c$
- if necessary, additional frequency filtering required to reduce spectral bandwidth of source signal

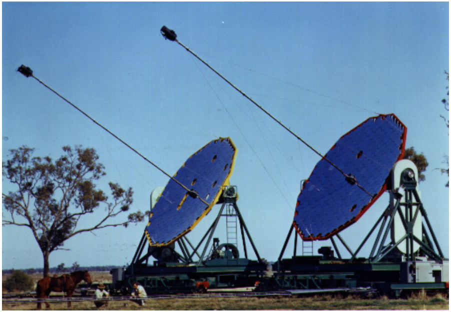
Bandwidth Restrictions

- coherence length of source needs to be larger than maximum path length difference at longest baseline
- imposes maximum frequency bandwidth for observations
- largest angle of incidence equals half the field of view, i.e. $\lambda/2D$
- coherence length compliant with largest baseline $L_{coh} \gg \frac{\lambda}{2D} L_{max}$
- frequency bandwidth requirement

$$\frac{\triangle\nu}{\nu_0} \ll \frac{2D}{L_{max}}$$

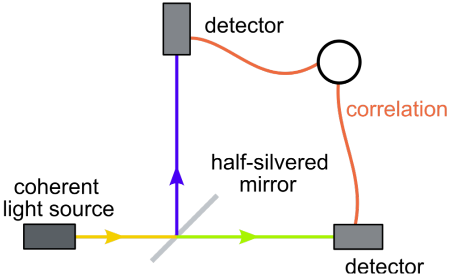
- WRST: $2D/L_{max} \approx 1/50$, at 21 cm ($\approx$ 1400 MHz), $\triangle\nu \ll 28$ MHz, coherence length $>$ 10 m
- in practise: $\triangle\nu \approx 10$ MHz
- increase bandwidth by division into frequency subbands
- subband maps scaled with λ and added

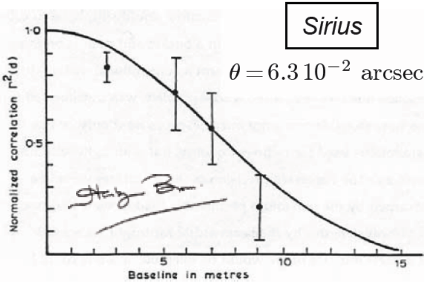
Hanburry-Brown and Twiss Effect



- Robert Hanbury Brown and Richard Q. Twiss (1956): *A test of a new type of stellar interferometer on Sirius*
- intensity correlation can be explained classically and with quantum mechanichs (photon bunching)

Classical Derivation





- looking at plane waves with two detectors and phase difference ϕ

$$I_1 = E_1^2 \sin^2 \omega t \qquad I_2 = E_1^2 \sin^2(\omega t + \phi)$$

- intensity correlation

$$< I_1 I_2 >= E^4/4 \left(1 + \cos^2 2\phi\right)$$