

Outline

- 1 Interference
- 2 Coherence
- 3 Two-Element Interferometer
- 4 Fraunhofer and Fresnel Diffraction
- 5 Transfer Functions
- 6 Van Cittert-Zernike Theorem

Very Large Telescope Interferometer (VLTI)

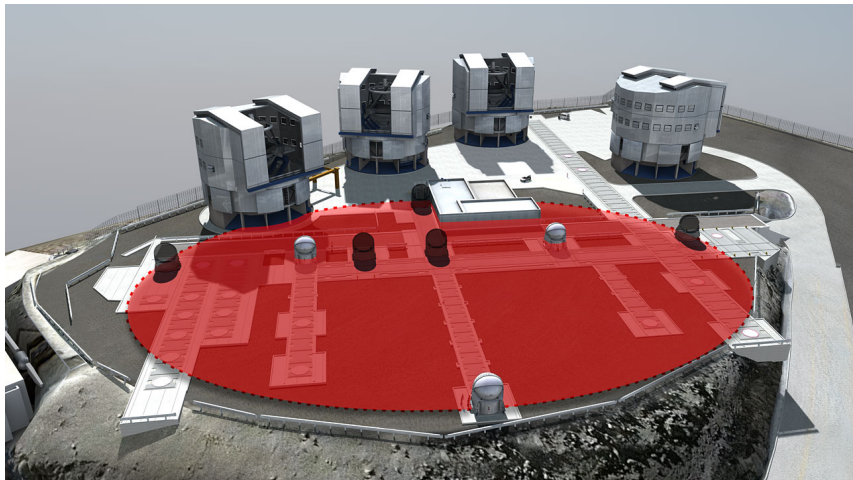


Image credit: ESO

T Leporis: a Mira-like Star with the VLTI in the NIR

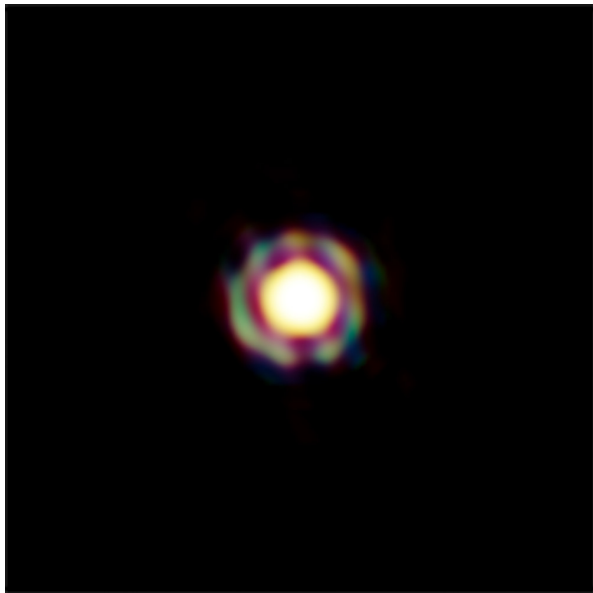


Image credit: ESO/J.-B. Le Bouquin et al.

3 Ways to Understanding Optics

- 1 **Geometrical Optics:** Light can be described as rays
- 2 **Physical Optics:** Light can be described as waves
- 3 **Quantum Optics:** Light can be described as discrete particles

Plane-Wave Solutions

Plane Vector Wave ansatz: $\vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}$

$\vec{k}$ spatially and temporally constant *wave vector*

$\vec{k}$ normal to surfaces of constant phase

$|\vec{k}|$ *wave number*

$\vec{x}$ spatial location

ω *angular frequency* ($2\pi \times$ frequency)

t time

$\vec{E}_0$ a (generally complex) vector independent of time and space

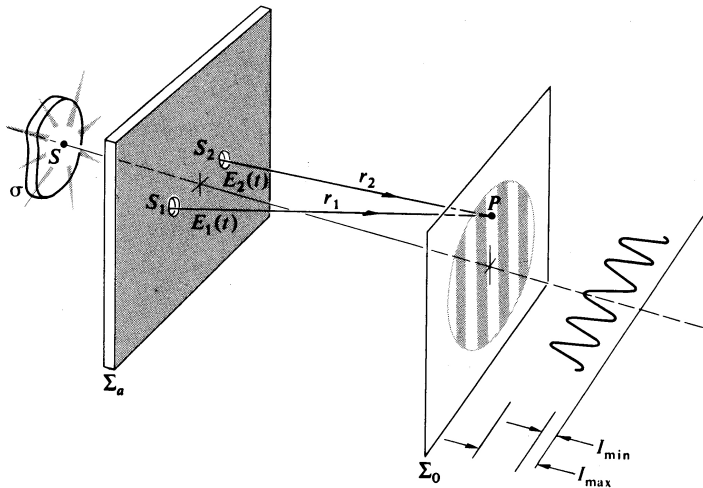
- real electric field vector given by real part of $\vec{E}$

Scalar Wave

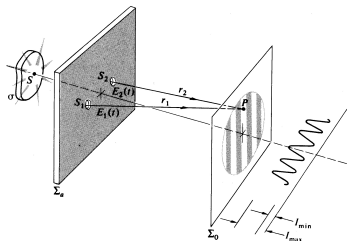
- electric field at position $\vec{r}$ at time t is $\tilde{E}(\vec{r}, t)$
- complex notation to easily express amplitude and phase
- real part of complex quantity is the physical quantity

Interference

Young's Double Slit Experiment



Young's Double Slit Experiment

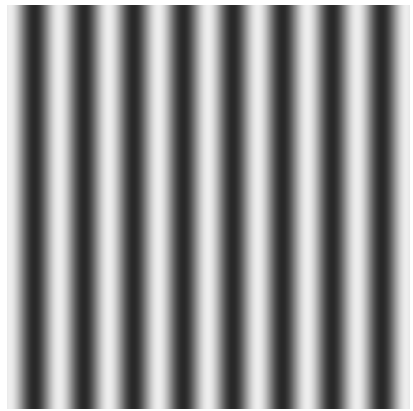


- monochromatic wave
- infinitely small holes (pinholes)
- source S generates fields $\tilde{E}(\vec{r}_1, t) \equiv \tilde{E}_1(t)$ at S_1 and $\tilde{E}(\vec{r}_2, t) \equiv \tilde{E}_2(t)$ at S_2
- two spherical waves from pinholes interfere on screen
- electrical field at P (without propagators i/λ)

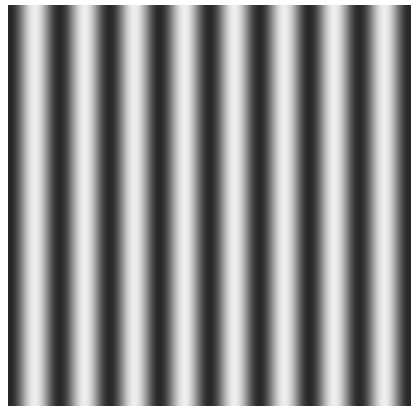
$$\tilde{E}_P(t) = \tilde{E}_1(t - t_1) + \tilde{E}_2(t - t_2)$$

- $t_1 = r_1/c$, $t_2 = r_2/c$
- r_1, r_2 : path lengths from S_1, S_2 to P

no tilt



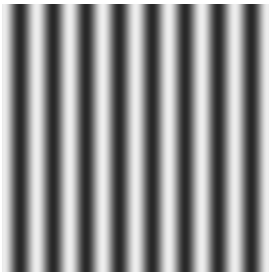
tilt by $0.5 \lambda/d$



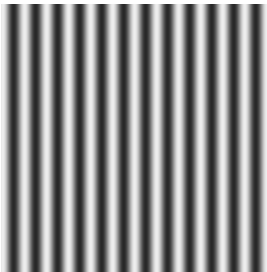
Change in Angle of Incoming Wave

- phase of fringe pattern changes, but not fringe spacing
- tilt of λ/d produces identical fringe pattern

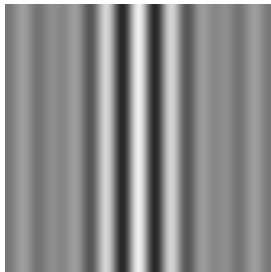
long wavelength



short wavelength



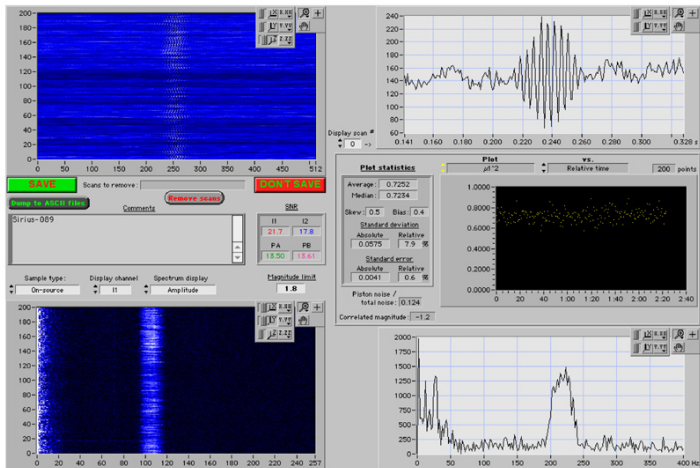
wavelength average



Change in Wavelength

- fringe spacing changes, central fringe broadens
- integral over 0.8 to 1.2 of central wavelength
- integral over wavelength makes fringe envelope

First Fringes from VLT Interferometer

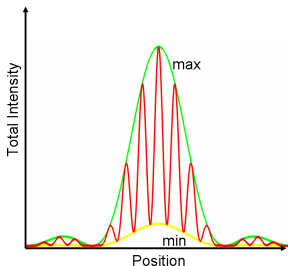


'First Fringes' from Sirius with VLTI

ESO PR Photo 10a/01 (18 March 2001)

© European Southern Observatory





en.wikipedia.org/wiki/Interferometric_visibility

- “quality” of fringes described by **Visibility function**

$$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$$

- $I_{\max}$, $I_{\min}$ are maximum and adjacent minimum in fringe pattern
- first introduced by Michelson

Why Not Fringes Everywhere?

- extended sources are not phase locked
- sources are not monochromatic

Mutual Coherence

- total field in point P (neglecting propagators)

$$\tilde{E}_P(t) = \tilde{E}_1(t - t_1) + \tilde{E}_2(t - t_2)$$

- irradiance at P , averaged over time (expectation operator $\mathbf{E}$)

$$I = \mathbf{E} |\tilde{E}_P(t)|^2 = \mathbf{E} \left\{ \tilde{E}_P(t) \tilde{E}_P^*(t) \right\}$$

- writing out all the terms

$$\begin{aligned} I = & \mathbf{E} \left\{ \tilde{E}_1(t - t_1) \tilde{E}_1^*(t - t_1) \right\} + \mathbf{E} \left\{ \tilde{E}_2(t - t_2) \tilde{E}_2^*(t - t_2) \right\} \\ & + \mathbf{E} \left\{ \tilde{E}_1(t - t_1) \tilde{E}_2^*(t - t_2) \right\} + \mathbf{E} \left\{ \tilde{E}_1^*(t - t_1) \tilde{E}_2(t - t_2) \right\} \end{aligned}$$

Mutual Coherence (continued)

- as before

$$I = \mathbf{E} \left\{ \tilde{E}_1(t - t_1) \tilde{E}_1^*(t - t_1) \right\} + \mathbf{E} \left\{ \tilde{E}_2(t - t_2) \tilde{E}_2^*(t - t_2) \right\} \\ + \mathbf{E} \left\{ \tilde{E}_1(t - t_1) \tilde{E}_2^*(t - t_2) \right\} + \mathbf{E} \left\{ \tilde{E}_1^*(t - t_1) \tilde{E}_2(t - t_2) \right\}$$

- *stationary* wave field, time average independent of absolute time

$$I_{S_1} = \mathbf{E} \left\{ \tilde{E}_1(t) \tilde{E}_1^*(t) \right\}, \quad I_{S_2} = \mathbf{E} \left\{ \tilde{E}_2(t) \tilde{E}_2^*(t) \right\}$$

- irradiance at P is now

$$I = I_{S_1} + I_{S_2} \\ + \mathbf{E} \left\{ \tilde{E}_1(t - t_1) \tilde{E}_2^*(t - t_2) \right\} + \mathbf{E} \left\{ \tilde{E}_1^*(t - t_1) \tilde{E}_2(t - t_2) \right\}$$

Mutual Coherence (continued)

- as before

$$I = I_{S_1} + I_{S_2} + \mathbf{E} \left\{ \tilde{E}_1(t - t_1) \tilde{E}_2^*(t - t_2) \right\} + \mathbf{E} \left\{ \tilde{E}_1^*(t - t_1) \tilde{E}_2(t - t_2) \right\}$$

- time difference $\tau = t_2 - t_1 \Rightarrow$ last two terms become

$$\mathbf{E} \left\{ \tilde{E}_1(t + \tau) \tilde{E}_2^*(t) \right\} + \mathbf{E} \left\{ \tilde{E}_1^*(t + \tau) \tilde{E}_2(t) \right\}$$

- equivalent to

$$2 \operatorname{Re} \left[\mathbf{E} \left\{ \tilde{E}_1(t + \tau) \tilde{E}_2^*(t) \right\} \right]$$

- cross-term becomes $2 \operatorname{Re} \left[\mathbf{E} \left\{ \tilde{E}_1(t + \tau) \tilde{E}_2^*(t) \right\} \right]$

Mutual Coherence (continued)

- irradiance at P

$$I = I_{S_1} + I_{S_2} + 2 \operatorname{Re} \left[\mathbf{E} \left\{ \tilde{E}_1(t + \tau) \tilde{E}_2^*(t) \right\} \right]$$

- **mutual coherence function** of wave field at S_1 and S_2

$$\tilde{\Gamma}_{12}(\tau) = \mathbf{E} \left\{ \tilde{E}_1(t + \tau) \tilde{E}_2^*(t) \right\}$$

- therefore $I = I_{S_1} + I_{S_2} + 2 \operatorname{Re} \tilde{\Gamma}_{12}(\tau)$
- $I_1 = I_{S_1}$, $I_2 = I_{S_2}$: irradiances at P from single aperture

$$I = I_1 + I_2 + 2 \operatorname{Re} \tilde{\Gamma}_{12}(\tau)$$

Self-Coherence

- $S_1 = S_2 \Rightarrow$ mutual coherence function = autocorrelation

$$\tilde{\Gamma}_{11}(\tau) = \tilde{R}_1(\tau) = \mathbf{E} \left\{ \tilde{E}_1(t + \tau) \tilde{E}_1^*(t) \right\}$$

$$\tilde{\Gamma}_{22}(\tau) = \tilde{R}_2(\tau) = \mathbf{E} \left\{ \tilde{E}_2(t + \tau) \tilde{E}_2^*(t) \right\}$$

- autocorrelation functions are also called *self-coherence functions*
- for $\tau = 0$

$$I_{S_1} = \mathbf{E} \left\{ \tilde{E}_1(t) \tilde{E}_1^*(t) \right\} = \Gamma_{11}(0) = \mathbf{E} \left\{ |\tilde{E}_1(t)|^2 \right\}$$

$$I_{S_2} = \mathbf{E} \left\{ \tilde{E}_2(t) \tilde{E}_2^*(t) \right\} = \Gamma_{22}(0) = \mathbf{E} \left\{ |\tilde{E}_2(t)|^2 \right\}$$

- autocorrelation function with zero lag ($\tau = 0$) represent (average) irradiance (power) of wave field at S_1 , S_2

Complex Degree of Coherence

- normalized mutual coherence defines the **complex degree of coherence**

$$\tilde{\gamma}_{12}(\tau) \equiv \frac{\tilde{\Gamma}_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}} = \frac{\mathbf{E} \left\{ \tilde{E}_1(t+\tau) \tilde{E}_2^*(t) \right\}}{\sqrt{\mathbf{E} \left\{ |\tilde{E}_1(t)|^2 \right\} \mathbf{E} \left\{ |\tilde{E}_2(t)|^2 \right\}}}$$

- irradiance in point P as *general interference law for a partially coherent radiation field*

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \operatorname{Re} \tilde{\gamma}_{12}(\tau)$$

Spatial and Temporal Coherence

- complex degree of coherence

$$\tilde{\gamma}_{12}(\tau) \equiv \frac{\tilde{\Gamma}_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}} = \frac{\mathbf{E} \left\{ \tilde{E}_1(t+\tau) \tilde{E}_2^*(t) \right\}}{\sqrt{\mathbf{E} \left\{ |\tilde{E}_1(t)|^2 \right\} \mathbf{E} \left\{ |\tilde{E}_2(t)|^2 \right\}}}$$

- measures both
 - *spatial coherence* at S_1 and S_2
 - *temporal coherence* through time lag τ
- $\tilde{\gamma}_{12}(\tau)$ is a complex variable and can be written as:

$$\tilde{\gamma}_{12}(\tau) = |\tilde{\gamma}_{12}(\tau)| e^{i\psi_{12}(\tau)}$$

- $0 \leq |\tilde{\gamma}_{12}(\tau)| \leq 1$
- phase angle $\psi_{12}(\tau)$ relates to
 - phase angle between fields at S_1 and S_2
 - phase angle difference in P resulting in time lag τ

Fringes in Fiber-fed High-Resolution Spectrograph

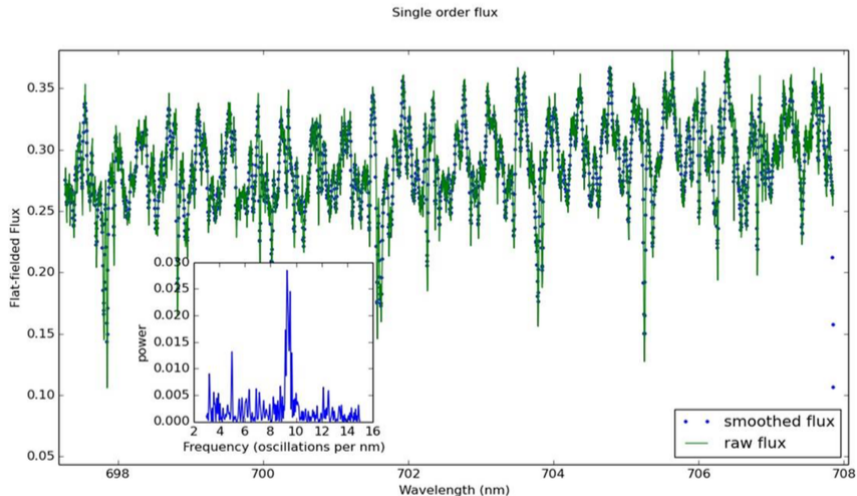
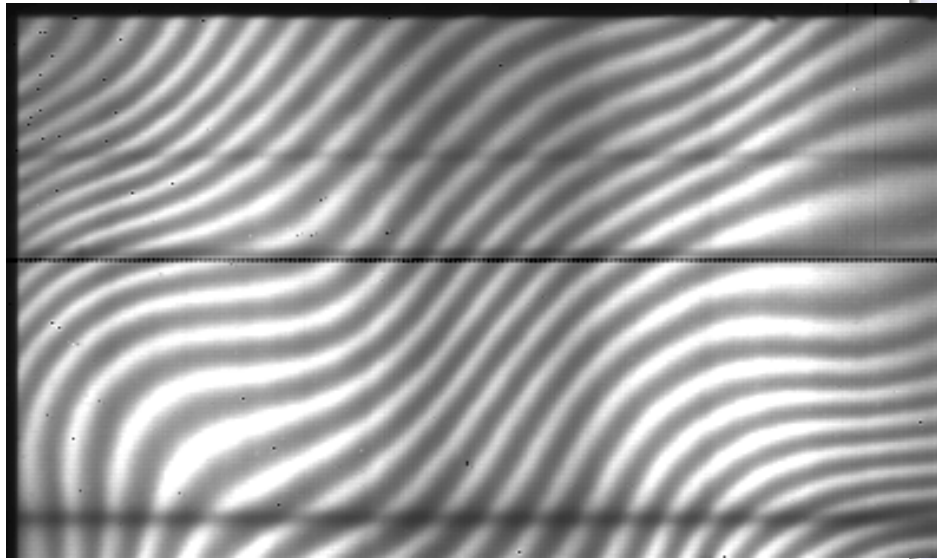


Figure 7 from Rains et al. 2016, SPIE 9908, 990876

Detector Fringes in HyViSi Detector



Coherence of Quasi-Monochromatic Light

- quasi-monochromatic light, mean wavelength $\bar{\lambda}$, frequency $\bar{\nu}$, phase difference ϕ due to optical path difference:

$$\phi = \frac{2\pi}{\bar{\lambda}}(r_2 - r_1) = \frac{2\pi}{\bar{\lambda}}c(t_2 - t_1) = 2\pi\bar{\nu}\tau$$

- with phase angle $\alpha_{12}(\tau)$ between fields at pinholes S_1, S_2

$$\psi_{12}(\tau) = \alpha_{12}(\tau) - \phi$$

- and

$$\text{Re } \tilde{\gamma}_{12}(\tau) = |\tilde{\gamma}_{12}(\tau)| \cos [\alpha_{12}(\tau) - \phi]$$

- intensity in P becomes

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} |\tilde{\gamma}_{12}(\tau)| \cos [\alpha_{12}(\tau) - \phi]$$

Visibility of Quasi-Monochromatic, Partially Coherent Light

- intensity in P

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} |\tilde{\gamma}_{12}(\tau)| \cos[\alpha_{12}(\tau) - \phi]$$

- maximum, minimum I for $\cos(\dots) = \pm 1$
- visibility V at position P

$$V = \frac{2\sqrt{I_1}\sqrt{I_2}}{I_1 + I_2} |\tilde{\gamma}_{12}(\tau)|$$

- for $I_1 = I_2 = I_0$

$$\begin{aligned} I &= 2I_0 \{1 + |\tilde{\gamma}_{12}(\tau)| \cos[\alpha_{12}(\tau) - \phi]\} \\ V &= |\tilde{\gamma}_{12}(\tau)| \end{aligned}$$

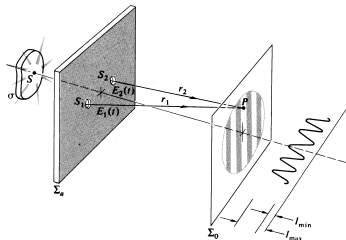
Interpretation of Visibility

- for $I_1 = I_2 = I_0$

$$\begin{aligned} I &= 2I_0 \{1 + |\tilde{\gamma}_{12}(\tau)| \cos [\alpha_{12}(\tau) - \phi]\} \\ V &= |\tilde{\gamma}_{12}(\tau)| \end{aligned}$$

- *modulus of complex degree of coherence = visibility of fringes*
- modulus can therefore be measured
- shift in location of central fringe (no optical path length difference, $\phi = 0$) is measure of $\alpha_{12}(\tau)$
- measurements of visibility and fringe position yield amplitude and phase of complex degree of coherence

Fringe Pattern



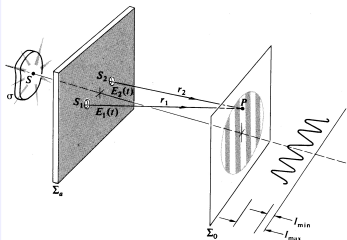
- for $I_1 = I_2 = I_0$

$$I = 2I_0 \{1 + |\tilde{\gamma}_{12}(\tau)| \cos [\alpha_{12}(\tau) - \phi]\} \quad V = |\tilde{\gamma}_{12}(\tau)|$$

- source S on central axis, fully coherent waves from two holes

$$I = 2I_0(1 + \cos \phi) = 4I_0 \cos^2 \frac{\phi}{2}$$

Fringe Pattern (continued)



$$I = 4I_0 \cos^2 \frac{\phi}{2}$$

$$\phi = \frac{2\pi}{\lambda}(r_2 - r_1) = 2\pi \bar{\nu} \tau$$

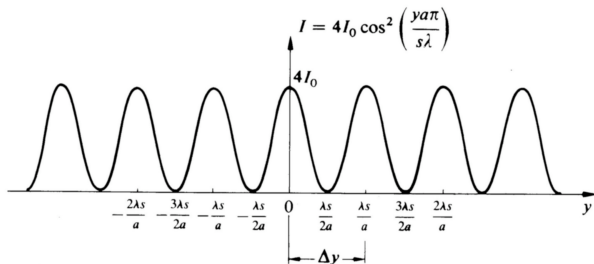
- distance a between pinholes
- distance s to observation plane Σ_O , $s \gg a$
- path difference $(r_2 - r_1)$ in equation for ϕ in good approximation

$$r_2 - r_1 = a\theta = \frac{a}{s} y$$

- and therefore

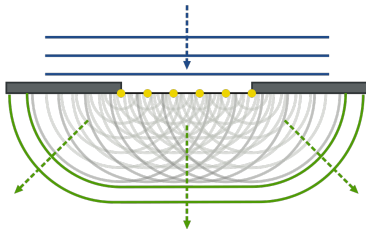
$$I = 4I_0 \cos^2 \frac{\pi a y}{s \lambda}$$

Interference Fringes from Monochromatic Point Source



- irradiance as a function of the y -coordinate of the fringes in observation plane Σ_O
- irradiance vs. distance distribution is Point-Spread Function (PSF) of ideal two-element interferometer

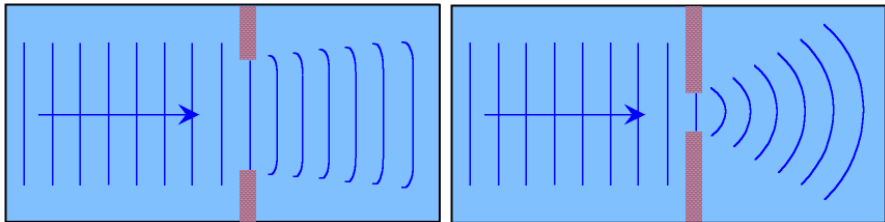
Huygens-Fresnel Principle



en.wikipedia.org/wiki/File:Refraction_on_an_aperture_-_Huygens-Fresnel_principle.svg

- every unobstructed point of a wavefront at a given moment in time serves as a source of spherical, secondary waves with the same frequency as the primary wave
- the amplitude of the optical field at any point beyond is the coherent superposition of all these secondary, spherical waves

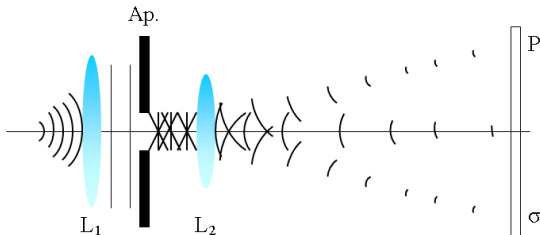
Diffraction



www.smkbud4.edu.my/Data/sites/vschool/phy/wave/diffraction.htm

- if obstructing structures are small compared to the wavelength, waves will spread out $\Rightarrow$ diffraction
- really need to solve wave equation with boundary constraints $\Rightarrow$ rigorous solution for only a few special cases
- various numerical ways to solve such problems (e.g. Rigorous Coupled Wave Analysis)
- Huygens-Fresnel is useful for most applications

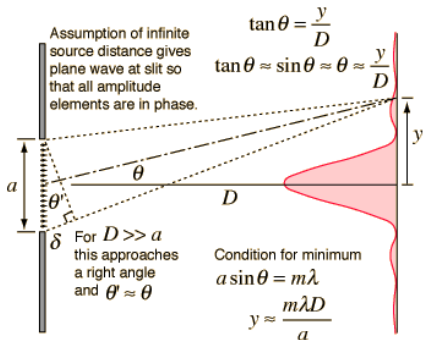
Fraunhofer and Fresnel Diffraction



en.wikipedia.org/wiki/File:Fraunhofer_diffraction_pattern_image.PNG

- wave shape changes as it moves away from obstruction
- *Fresnel* (or near-field) diffraction close to obstruction
- *Fraunhofer* (or far-field) diffraction far away from obstruction
- rule of thumb: Fraunhofer diffraction for $R > a^2/\lambda$
 - a greatest width of obscuration
 - λ wavelength
 - R greater of distance between source/detector and obscuration

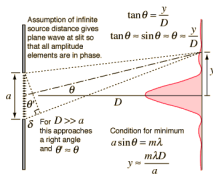
Slit Diffraction



hyperphysics.phy-astr.gsu.edu/hbase/phyopt/sinslit.html

- diffraction of a plane wave on a slit aperture
- Huygens-Fresnel: line of spherical wave sources

Slit Diffraction



hyperphysics.phy-astr.gsu.edu/hbase/phyopt/sinslit.html

- with E_0 the strength of each slit segment i at point P is

$$E_i(P) = \frac{E_L}{r_i} \sin(k\omega - kr_i) \Delta y_i$$

- i segment index ($1 - M$)
- E_L source strength per unit length
- r_i distance between segment and point P
- Δy_i small segment of slit
- D length of slit

Fraunhofer Diffraction at Single Slit

- integrate along slit

$$E = E_L \int_{-D/2}^{D/2} \frac{\sin \omega t - kr}{r} dy$$

- express r as a function of y :

$$r = R - y \sin \theta + \frac{y^2}{2R} \cos^2 \theta + \dots$$

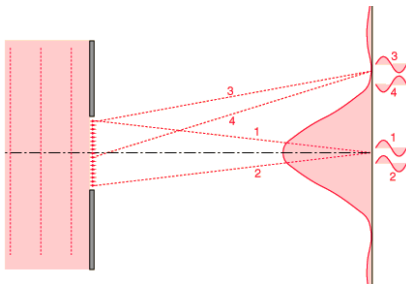
R distance between center of slit and point P

- substituting, integrating and squaring for intensity:

$$I(\theta) = I(0) \left(\frac{\sin \beta}{\beta} \right)^2$$

- $\beta = (kD/2) \sin \theta$

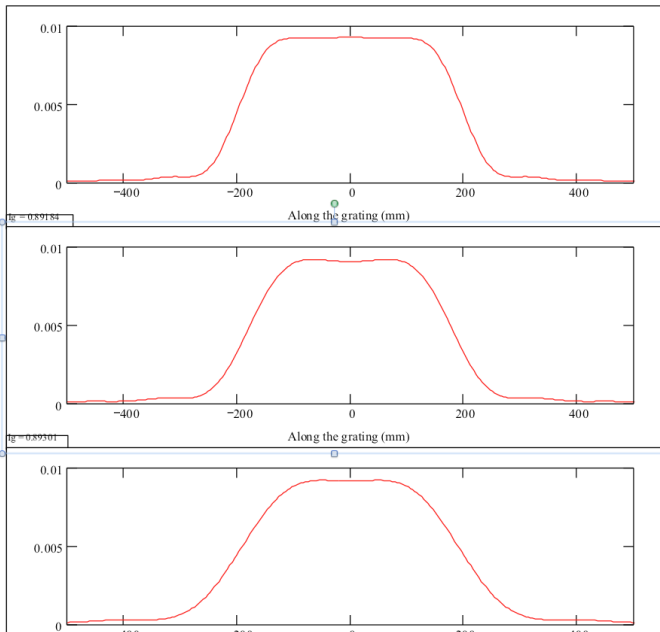
Interpretation of Single Slit Diffraction



hyperphysics.phy-astr.gsu.edu/hbase/phyopt/sinslitd.html

- assume infinite distance from aperture for source and observation plane
- equivalent to plane waves coming from aperture into different directions
- first minimum when phase delay at edge is exactly one wave

Diffraction Effects in High-Resolution Spectrographs



$\lambda = 630.2 \text{ nm}$, grating
angle = 65.09°

$\lambda = 854.2 \text{ nm}$, grating
angle = 62.745°

$\lambda = 1083 \text{ nm}$, grating
angle = 65.377°

Fraunhofer Diffraction at Circular Aperture

- integrate over circular aperture with radius a

$$E = \frac{E_A e^{i(\omega t - kR)}}{R} \int \int_{\text{aperture}} e^{ik(Yy + Zz)/R} dS$$

- using polar coordinates in aperture and plane of observation and Bessel functions

$$I(\theta) = I(0) \left(\frac{2J_1(ka \sin \theta)}{ka \sin \theta} \right)^2$$

- J_1 Bessel function of order 1
- Airy function
- first dark ring at $1.22 \frac{R\lambda}{2a}$
- images with perfect telescopes are convolution of Airy disk with actual image

Arbitrary Diffracting Aperture

- from before forgetting common phase term and $1/R$ amplitude drop-off

$$E(Y, Z) = \int \int_{\text{aperture}} A(y, z) e^{ik(Yy + Zz)/R} dS$$

- complex aperture function $A(y, z)$ describing non-uniform absorption and phase delays
- finite aperture $\Rightarrow$ change integration boundaries to infinity
- with $k_y = kY/R$ and $k_z = kZ/R$ we obtain

$$E(k_y, k_z) = \int \int_{\text{aperture}} A(y, z) e^{i(k_y y + k_z z)} dy dz$$

- field distribution in Fraunhofer diffraction pattern is Fourier transform of field distribution across aperture

Introduction

- linear black box system
- measure response to delta function input (transfer function)
- express output as convolution between input signal and transfer function

Point-Spread Function

- intensity is modulus squared of field distribution $\Rightarrow$ point-spread function
- image of a point source: *Point Spread Function (PSF)*
- image of arbitrary object is a convolution of object with PSF

$$i = o * s$$

- i observed image
- o true object, constant in time
- s point spread function
- $*$ convolution

Optical Transfer Function

- after Fourier transformation:

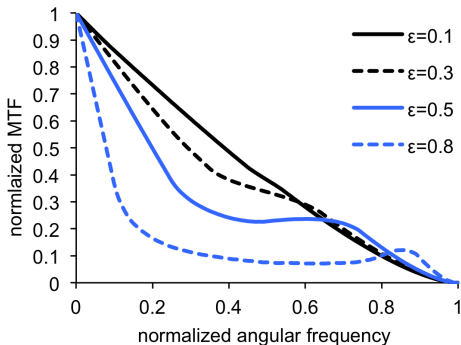
$$I = O \cdot S$$

- Fourier transformed
 - / Fourier transform of image
 - O Fourier transform of object
 - S Optical Transfer Function (OTF)
- OTF is Fourier transform of PSF and vice versa

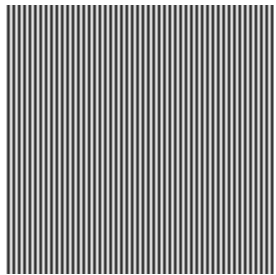
Modulation Transfer Function (MTF)

- is the absolute value of the optical transfer function
- describes the amplitude reduction of a sinusoidal source
- is the autocorrelation of the aperture function A
- $OTF = FT^{-1}(PSF) = FT^{-1}(|FT(A)|^2) = FT^{-1}(FT(A) \cdot FT(A)^*) = A * A$

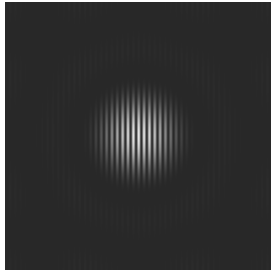
MTF of Circular Telescope with Central Obscuration ϵ



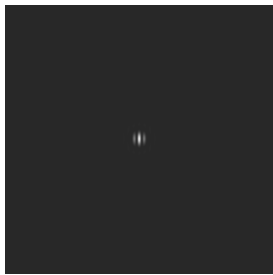
2 pinholes



2 small holes



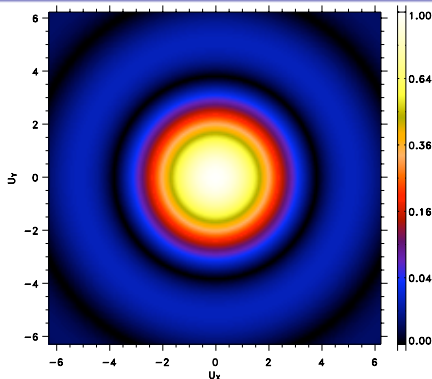
2 large holes



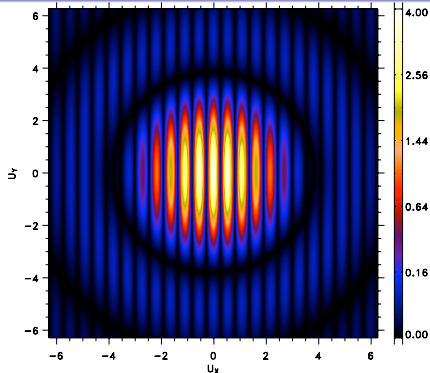
Finite Hole Diameter

- fringe spacing only depends on separation of holes and wavelength
- the smaller the hole, the larger the 'illuminated' area
- fringe envelope is Airy pattern (diffraction pattern of a single hole)

2-d Brightness Distribution



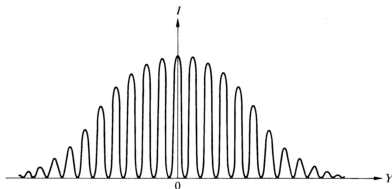
PSF of single circular aperture



PSF of two-element
interferometer, aperture diameter
 $d = 25$ m, length of baseline
vector $|\vec{s}| = 144$ m

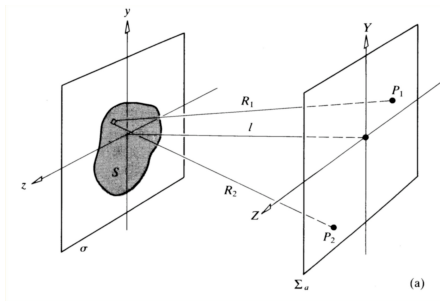
- double beam interference fringes showing modulation effect of diffraction by aperture of a single pinhole

Modulation Effect of Aperture



- typical one-dimensional cross-section through central part of interferogram
- visibilities are equal to one, because $I_{min} = 0$
- $|\tilde{\gamma}_{12}(\tau)| = 1$ for all values of τ and any pair of spatial points, if and only if the radiation field is *strictly monochromatic*

The Problem



- relates brightness distribution of extended source and phase correlation between two points in radiation field
- extended source S incoherent, quasi-monochromatic
- positions P_1 and P_2 in observers plane Σ

$$\tilde{E}_1(t)\tilde{E}_2^*(t) = \mathbf{E}\{\tilde{E}_1(t)\tilde{E}_2^*(t)\} = \tilde{r}_{12}(0)$$

The Solution

- $I(\vec{\Omega})$ is intensity distribution of extended source as function of unit direction vector $\vec{\Omega}$ as seen from observation plane Σ
- $\tilde{\Gamma}(\vec{r})$ is coherence function in Σ -plane
- vector $\vec{r}$ represents arbitrary baseline
- van Cittert-Zernike theorem

$$\tilde{\Gamma}(\vec{r}) = \int \int_{\text{source}} I(\vec{\Omega}) e^{\frac{2\pi i \vec{\Omega} \cdot \vec{r}}{\lambda}} d\vec{\Omega}$$

$$I(\vec{\Omega}) = \lambda^{-2} \int \int_{\Sigma\text{-plane}} \tilde{\Gamma}(\vec{r}) e^{-\frac{2\pi i \vec{\Omega} \cdot \vec{r}}{\lambda}} d\vec{r}$$

- $\tilde{\Gamma}(\vec{r})$ and $I(\vec{\Omega})$ are linked through Fourier transform, except for scaling with wavelength λ
- "true" Fourier transform with *conjugate variables* $\vec{\Omega}$, $\vec{r}/\lambda$,
- Fourier pair: $I(\vec{\Omega}) \Leftrightarrow \tilde{\Gamma}(\vec{r}/\lambda)$