Lecture 5: Polarization

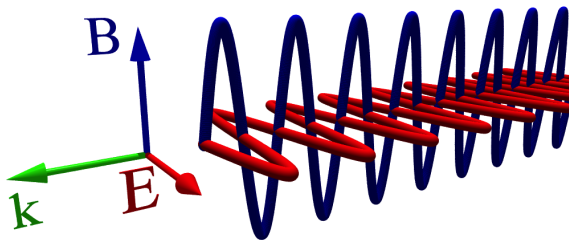## Outline

1. Polarized Light in the Universe
2. Brewster Angle and Total Internal Reflection
3. Descriptions of Polarized Light
4. Polarizers
5. Retarders

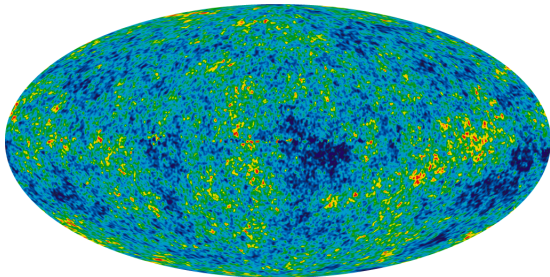*Polarization* indicates *anisotropy* ⇒ not all directions are equal

Typical anisotropies introduced by

- geometry (not everything is spherically symmetric)
- temperature gradients
- density gradients
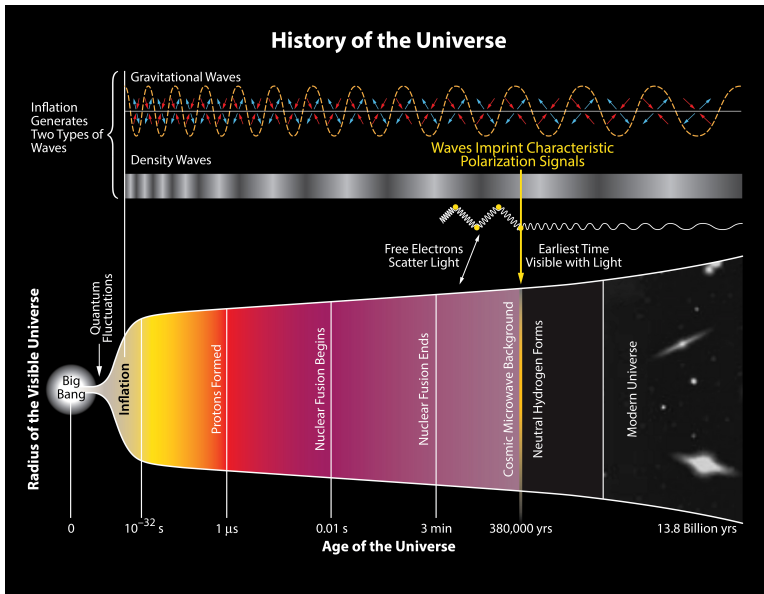- magnetic fields
- electrical fields

Polarized Light from the Big Bang

- Cosmic Microwave Background (CMB) is red-shifted radiation from Big Bang 14×10^9 years ago
- age, geometry, density of universe from CMB intensity pattern
- first 0.1 seconds from polarization pattern of CMB
- inflation $\Rightarrow$ gravitational waves $\Rightarrow$ polarization signals
- polarization expected at (or below) 10^{-6} of intensity

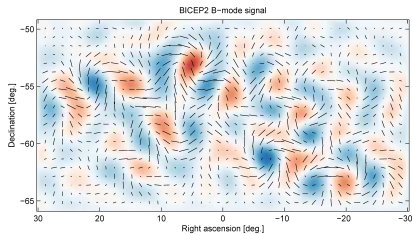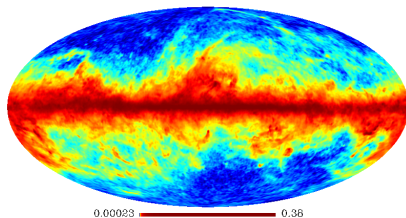## 13.7 billion year old temperature fluctuations from WMAP

Watch Out for the Dust

- achieving sensitivity at the 10^{-6} level is difficult
- polarization due to galactic dust grains being aligned by galactic magnetic field

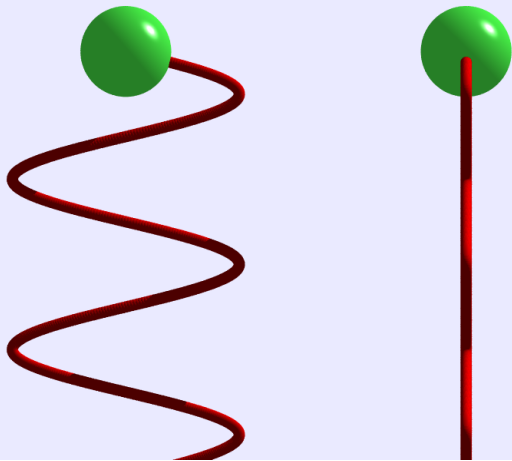## BICEP2 results and Planck Dust Polarization Map

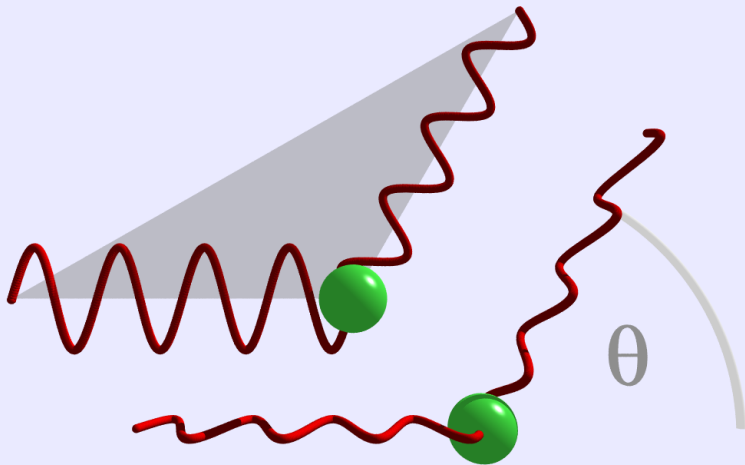Miller et al., 1991, ApJ 378, 47-64

T Tauri in intensity
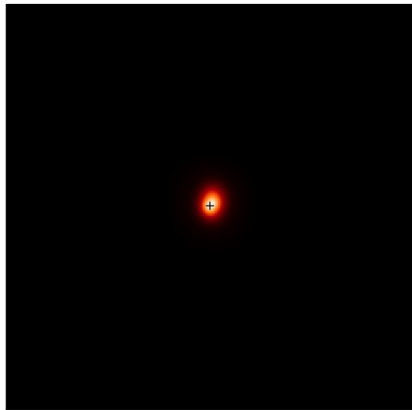
MWC147 in intensity
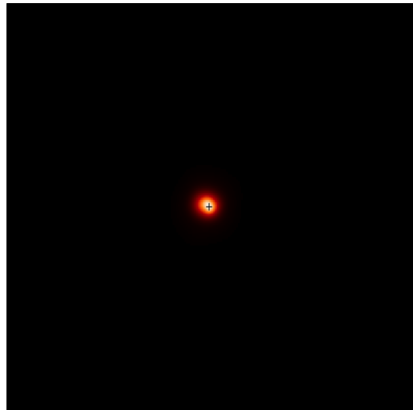
T Tauri in Linear Polarization
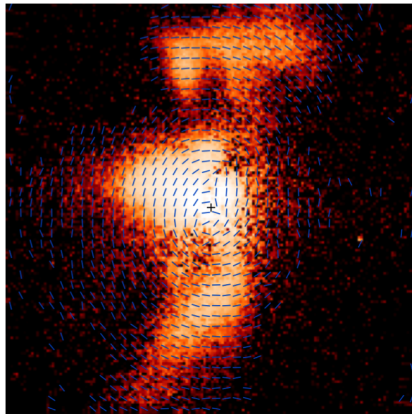
MWC147 in Linear Polarization
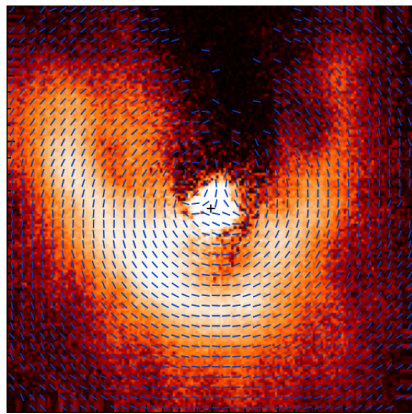
R Cr B with HST in intensity
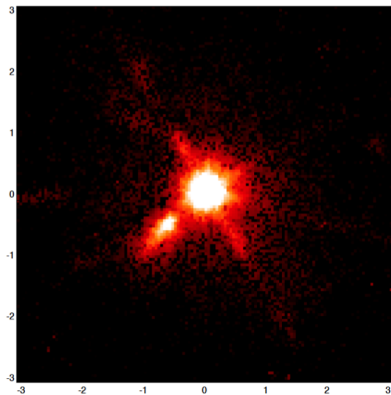
R Cr B with ExPo in lin. pol.
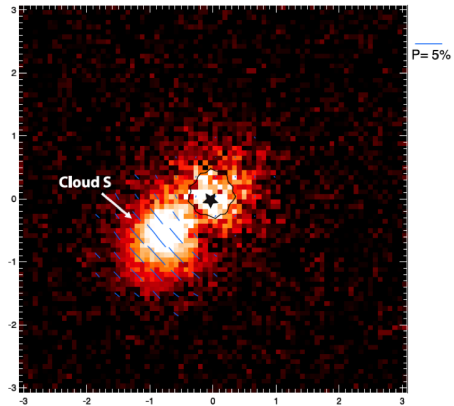
Solar Magnetic Field Maps from Longitudinal Zeeman Effect
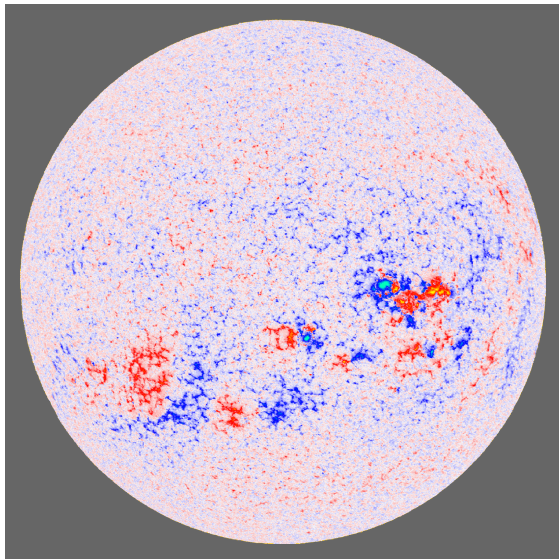
Planetary Scattered Light
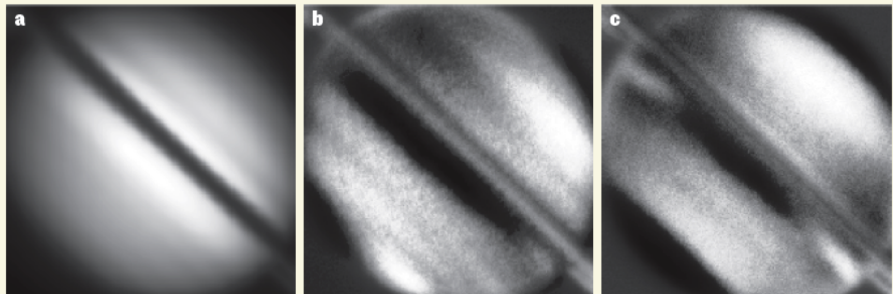
- solar-system planets show scattering polarization
- much depends on cloud height
- can be used to study exoplanets
- ExPo, SPHERE, EPICS at Leiden Observatory

Polarization

- Plane Vector Wave ansatz $\vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}$
- spatially, temporally constant vector $\vec{E}_0$ lays in plane perpendicular to propagation direction $\vec{k}$
- represent $\vec{E}_0$ in 2-D basis, unit vectors $\vec{e}_1$ and $\vec{e}_2$, both perpendicular to $\vec{k}$

$$\vec{E}_0 = E_1 \vec{e}_1 + E_2 \vec{e}_2.$$

  E_1, E_2: arbitrary complex scalars
- damped plane-wave solution with given ω, $\vec{k}$ has 4 degrees of freedom (two complex scalars)
- additional property is called *polarization*
- many ways to represent these four quantities
- if E_1 and E_2 have identical phases, $\vec{E}$ oscillates in fixed plane

Summary of Fresnel Equations

- electric field amplitude transmission $t_{s,p}$, reflection $r_{s,p}$

$$t_s = \frac{2\tilde{n}_1 \cos\theta_i}{\tilde{n}_1 \cos\theta_i + \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}}$$

$$t_p = \frac{2\tilde{n}_1 \tilde{n}_2 \cos\theta_i}{\tilde{n}_2^2 \cos\theta_i + \tilde{n}_1 \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}}$$

$$r_s = \frac{\tilde{n}_1 \cos\theta_i - \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}}{\tilde{n}_1 \cos\theta_i + \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}}$$

$$r_p = \frac{\tilde{n}_2^2 \cos\theta_i - \tilde{n}_1 \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}}{\tilde{n}_2^2 \cos\theta_i + \tilde{n}_1 \sqrt{\tilde{n}_2^2 - \tilde{n}_1^2 \sin^2\theta_i}}$$

- reflectivity: $R = |r_{s,p}|^2$, transmissivity: $T = \frac{|\tilde{n}_2| \cos\theta_t}{|\tilde{n}_1| \cos\theta_i} |t_{s,p}|^2$

Brewster Angle
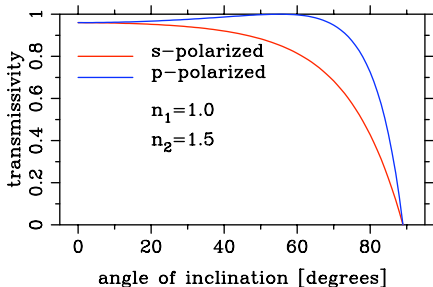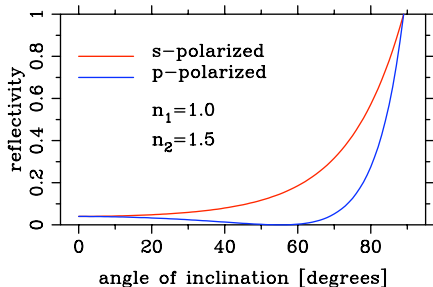
- $r_p = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} = 0$ when $\theta_i + \theta_t = \frac{\pi}{2}$
- corresponds to *Brewster angle* of incidence of $\tan\theta_B = \frac{n_2}{n_1}$
- occurs when reflected wave is perpendicular to transmitted wave
- reflected light is completely s-polarized
- transmitted light is moderately polarized

Total Internal Reflection (TIR)
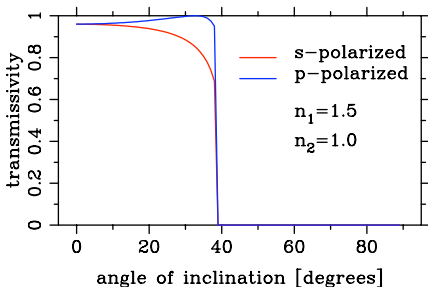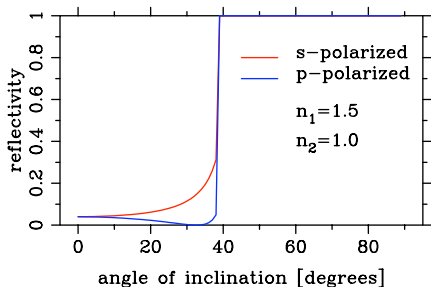
- Snell's law: $\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i$
- wave from high-index medium into lower index medium (e.g. glass to air): $n_1/n_2 > 1$
- right-hand side > 1 for $\sin\theta_i > \frac{n_2}{n_1}$
- all light is reflected in high-index medium $\Rightarrow$ *total internal reflection*
- transmitted wave has complex phase angle $\Rightarrow$ damped wave along interface

Phase Change on Total Internal Reflection (TIR)

- TIR induces phase change that depends on polarization
- complex ratios:
  $r_{s,p} = |r_{s,p}|e^{i\delta_{s,p}}$
- phase change $\delta = \delta_s - \delta_p$

$$\tan\frac{\delta}{2} = \frac{\cos\theta_i\sqrt{\sin^2\theta_i - \left(\frac{n_2}{n_1}\right)^2}}{\sin^2\theta_i}$$

- relation valid between critical angle and grazing incidence
- for critical angle and grazing incidence, phase difference is zero

Polarization Ellipse
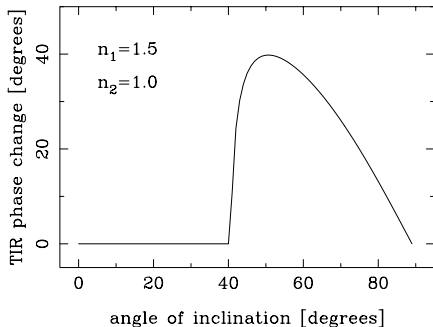
Polarization Ellipse
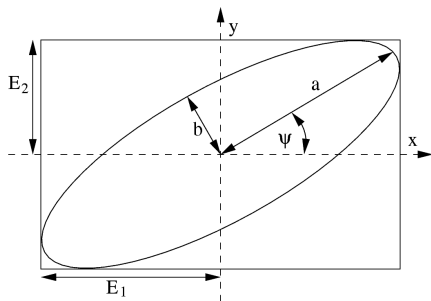
Polarization

$$\vec{E}(t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

$$\vec{E}_0 = E_1 e^{i\delta_1} \vec{e}_x + E_2 e^{i\delta_2} \vec{e}_y$$

- wave vector in *z*-direction
- $\vec{e}_x$, $\vec{e}_y$: unit vectors in *x*, *y*
- E_1, E_2: (real) amplitudes
- $\delta_{1,2}$: (real) phases

Polarization Description

- 2 complex scalars not the most useful description
- at given $\vec{x}$, time evolution of $\vec{E}$ described by *polarization ellipse*
- ellipse described by axes *a*, *b*, orientation ψ

Jones Formalism
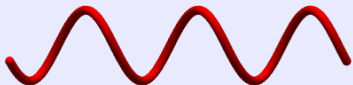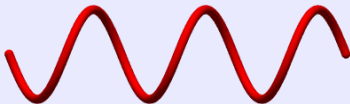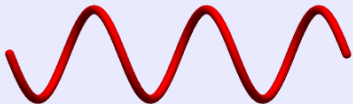
Jones Vectors

$$\vec{E}_0 = E_x \vec{e}_x + E_y \vec{e}_y$$

- beam in z-direction
- $\vec{e}_x$, $\vec{e}_y$ unit vectors in x, y-direction
- complex scalars $E_{x,y}$
- Jones vector

$$\vec{e} = \left(\begin{array}{c} E_x \\ E_y \end{array} \right)$$

- phase difference between E_x, E_y multiple of π, electric field vector oscillates in a fixed plane $\Rightarrow$ *linear polarization*
- phase difference $\pm\frac{\pi}{2}$ $\Rightarrow$ *circular polarization*

Summing and Measuring Jones Vectors

$$\vec{E}_0 = E_x \vec{e}_x + E_y \vec{e}_y$$

$$\vec{e} = \left(\begin{array}{c} E_x \\ E_y \end{array} \right)$$

- Maxwell's equations linear $\Rightarrow$ sum of two solutions again a solution
- Jones vector of sum of two waves $=$ sum of Jones vectors of individual waves if wave vectors $\vec{k}$ the same
- addition of Jones vectors: *coherent* superposition of waves
- elements of Jones vectors are not observed directly
- observables always depend on products of elements of Jones vectors, i.e. intensity

$$I = \vec{e} \cdot \vec{e}^* = e_x e_x^* + e_y e_y^*$$

Jones matrices

- influence of medium on polarization described by 2×2 complex *Jones matrix* J

$$\vec{e}' = J\vec{e} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \vec{e}$$

- assumes that medium not affected by polarization state
- different media 1 to N in order of wave direction $\Rightarrow$ combined influence described by

$$J = J_N J_{N-1} \cdots J_2 J_1$$

- order of matrices in product is crucial
- Jones calculus describes coherent superposition of polarized light

Linear Polarization

- horizontal: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

- vertical: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- $45°$: $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Circular Polarization

- left: $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$

- right: $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Notes on Jones Formalism

- Jones formalism operates on amplitudes, not intensities
- coherent superposition important for coherent light (lasers, interference effects)
- Jones formalism describes 100% polarized light

Stokes and Mueller Formalisms

Stokes Vector

- formalism to describe polarization of quasi-monochromatic light
- directly related to measurable intensities
- Stokes vector fulfills these requirements

$$\vec{I} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} E_x E_x^* + E_y E_y^* \\ E_x E_x^* - E_y E_y^* \\ E_x E_y^* + E_y E_x^* \\ i\left(E_x E_y^* - E_y E_x^*\right) \end{pmatrix} = \begin{pmatrix} E_1^2 + E_2^2 \\ E_1^2 - E_2^2 \\ 2E_1 E_2 \cos\delta \\ 2E_1 E_2 \sin\delta \end{pmatrix}$$

Jones vector elements $E_{x,y}$, real amplitudes $E_{1,2}$, phase difference $\delta = \delta_2 - \delta_1$

- $I^2 \geq Q^2 + U^2 + V^2$
- can describe unpolarized ($Q = U = V = 0$) light
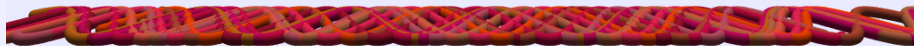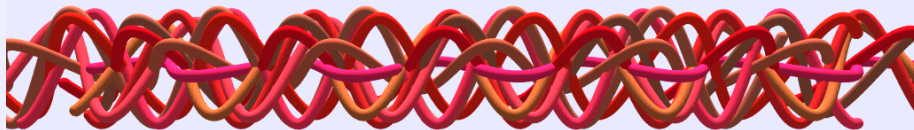
Stokes Vector Interpretation

$$\vec{I} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} \text{intensity} \\ \text{linear } 0° - \text{linear } 90° \\ \text{linear } 45° - \text{linear } 135° \\ \text{circular left} - \text{right} \end{pmatrix}$$

- *degree of polarization*

$$P = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}$$

  1 for fully polarized light, 0 for unpolarized light
- summing of Stokes vectors = *incoherent* adding of quasi-monochromatic light waves

Linear Polarization

- horizontal: $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

- vertical: $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

- 45°: $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

Circular Polarization

- left: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

- right: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$

Mueller Matrices

- 4×4 real Mueller matrices describe (linear) transformation between Stokes vectors when passing through or reflecting from media

$$\vec{I}' = M\vec{I},$$

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}$$

- N optical elements, combined Mueller matrix is

$$M' = M_N M_{N-1} \cdots M_2 M_1$$

Rotating Mueller Matrices

- optical element with Mueller matrix M
- Mueller matrix of the same element rotated by θ around the beam given by

$$\mathsf{M}(\theta) = \mathsf{R}(-\theta)\mathsf{M}\mathsf{R}(\theta)$$

with

$$\mathsf{R}\left(\theta\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
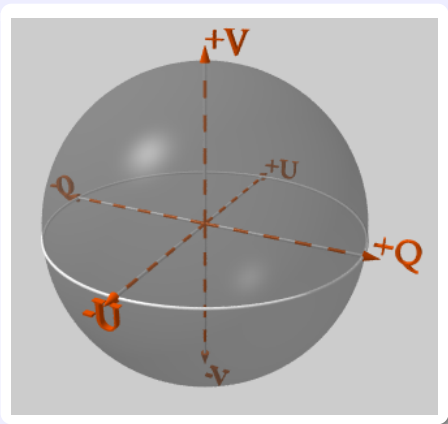
Vertical Linear Polarizer

$$M_{pol}(\theta) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Horizontal Linear Polarizer

$$M_{pol}(\theta) = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
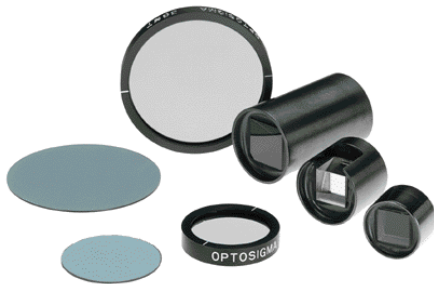
Mueller Matrix for Ideal Linear Polarizer at Angle θ

$$M_{pol}(\theta) = \frac{1}{2} \begin{pmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \sin 2\theta \cos 2\theta & 0 \\ \sin 2\theta & \sin 2\theta \cos 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Relation to Stokes Vector

- fully polarized light:
  $I^2 = Q^2 + U^2 + V^2$
- for $I^2 = 1$: sphere in Q, U, V coordinate system
- point on Poincaré sphere represents particular state of polarization
- graphical representation of fully polarized light

- polarizer: optical element that produces polarized light from unpolarized input light
- linear, circular, or in general elliptical polarizer, depending on type of transmitted polarization
- linear polarizers by far the most common
- large variety of polarizers

Jones Matrix for Linear Polarizers

- Jones matrix for linear polarizer:

$$J_p = \left(\begin{array}{cc} p_x & 0 \\ 0 & p_y \end{array} \right)$$

- $0 \leq p_x \leq 1$ and $0 \leq p_y \leq 1$, real: transmission factors for x, y-components of electric field: $E_x' = p_x E_x, \quad E_y' = p_y E_y$
- $p_x = 1$, $p_y = 0$: linear polarizer in $+Q$ direction
- $p_x = 0$, $p_y = 1$: linear polarizer in $-Q$ direction
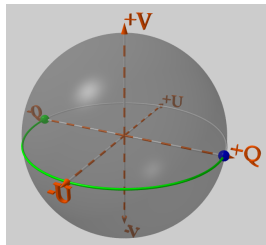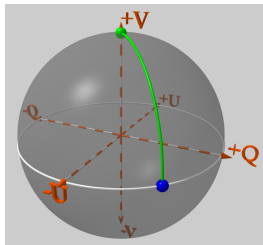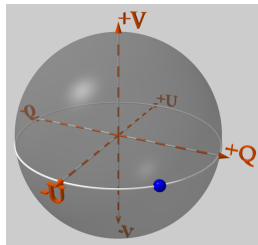- $p_x = p_y$: neutral density filter

Mueller Matrix for Linear Polarizers

$$M_p = \frac{1}{2} \left(\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Mueller Matrix for Ideal Linear Polarizer at Angle θ

$$M_{\rm pol}\left(\theta\right) = \frac{1}{2} \begin{pmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \sin 2\theta \cos 2\theta & 0 \\ \sin 2\theta & \sin 2\theta \cos 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Poincare Sphere

- polarizer is a point on the Poincaré sphere
- transmitted intensity: $\cos^2(l/2)$, l is arch length of great circle between incoming polarization and polarizer on Poincaré sphere

Wire Grid Polarizers
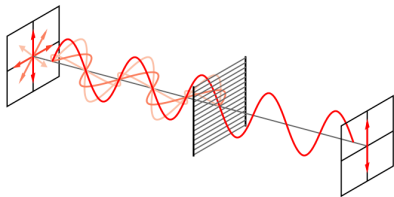
- parallel conducting wires, spacing $d \lesssim \lambda$ act as polarizer
- electric field parallel to wires induces electrical currents in wires
- induced electrical current reflects polarization parallel to wires
- polarization perpendicular to wires is transmitted
- rule of thumb:
  - $d < \lambda/2 \Rightarrow$ strong polarization
  - $d \gg \lambda \Rightarrow$ high transmission of both polarization states (weak polarization)
- mostly used in infrared

Polaroid-type Polarizers

LONG POLYMER MOLECULES

SHEET OF POLYVINYL ALCOHOL

STRETCHED SHEET OF POLYVINYL ALCOHOL
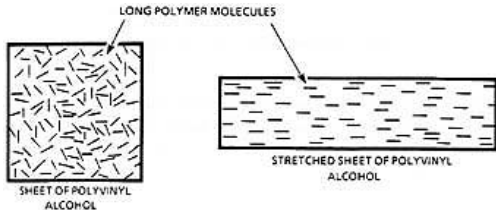
- developed by Edwin Land in 1938 ⇒ Polaroid
- sheet polarizers: stretched polyvynil alcohol (PVA) sheet, laminated to sheet of cellulose acetate butyrate, treated with iodine
- PVA-iodine complex analogous to short, conducting wire
- cheap, can be manufactured in large sizes

Crystal-Based Polarizers
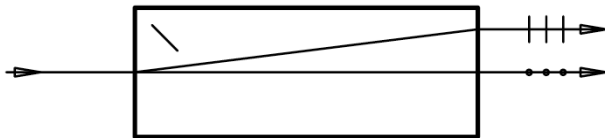
- crystals are basis of highest-quality polarizers
- precise arrangement of atom/molecules and anisotropy of index of refraction separate incoming beam into two beams with precisely orthogonal linear polarization states
- work well over large wavelength range
- many different configurations
- calcite most often used in crystal-based polarizers because of large birefringence, low absorption in visible
- many other suitable materials
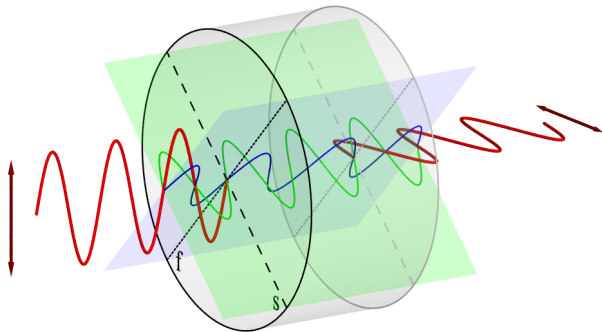
- retards (delays) phase of one electric field component with respect to orthogonal component
- anisotropic material (crystal) has index of refraction that depends on polarization

Retarder Properties

- does not change intensity or degree of polarization
- characterized by two (not identical, not trivial) Stokes vectors of incoming light that are not changed by retarder ⇒ *eigenvectors* of retarder
- depending on polarization described by eigenvectors, retarder is
  - *linear retarder*
  - *circular retarder*
  - *elliptical retarder*
- linear, circular retarders are special cases of elliptical retarders
- circular retarders sometimes called *rotators* since they rotate the orientation of linearly polarized light
- linear retarders by far the most common type of retarder

Jones Matrix for Linear Retarders

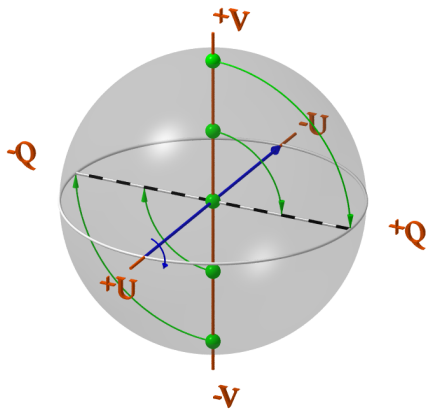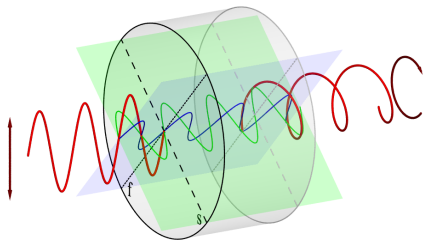- linear retarder with fast axis at $0°$ characterized by Jones matrix

$$\mathsf{J}_r\left(\delta\right) = \left(\begin{array}{cc} e^{i\delta} & 0 \\ 0 & 1 \end{array}\right), \quad \mathsf{J}_r\left(\delta\right) = \left(\begin{array}{cc} e^{i\frac{\delta}{2}} & 0 \\ 0 & e^{-i\frac{\delta}{2}} \end{array}\right)$$

- δ: phase shift between two linear polarization components (in radians)
- absolute phase does not matter

Mueller Matrix for Linear Retarder

$$\mathsf{M}_r = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\delta & -\sin\delta \\ 0 & 0 & \sin\delta & \cos\delta \end{array}\right)$$

- retarder eigenvector (fast axis) in Poincaré sphere
- points on sphere are rotated around retarder axis by amount of retardation

Variable Retarders

- sensitive polarimeters requires retarders whose properties (retardance, fast axis orientation) can be varied quickly (*modulated*)
- retardance changes (change of birefringence):
    - liquid crystals
    - Faraday, Kerr, Pockels cells
    - piezo-elastic modulators (PEM)
- fast axis orientation changes (change of *c*-axis direction):
    - rotating fixed retarder
    - ferro-electric liquid crystals (FLC)

Liquid Crystals

Crystalline Solid — Liquid Crystal — Isotropic Liquid (Temperature)
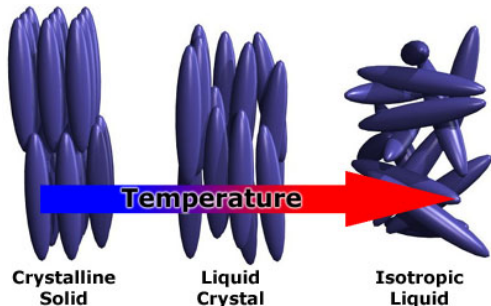
- liquid crystals: fluids with elongated molecules
- at high temperatures: liquid crystal is isotropic
- at lower temperature: molecules become ordered in orientation and sometimes also space in one or more dimensions
- liquid crystals can line up parallel or perpendicular to external electrical field

Liquid Crystal Retarders
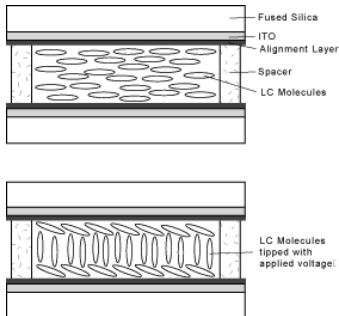
- dielectric constant anisotropy often large $\Rightarrow$ very responsive to changes in applied electric field
- birefringence δn can be very large (larger than typical crystal birefringence)
- liquid crystal layer only a few μm thick
- birefringence shows strong temperature dependence