Interference, Diffraction and Fourier Theory

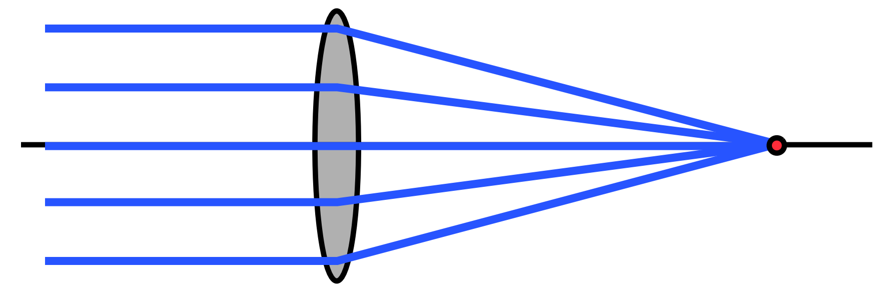
ATI 2014 Lecture 02

Keller and Kenworthy

The three major branches of optics

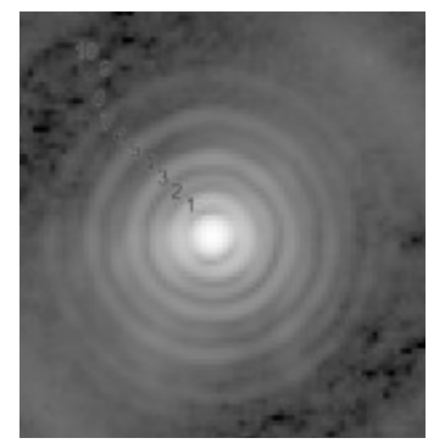
Geometrical Optics

“Light travels as straight rays”



Physical Optics

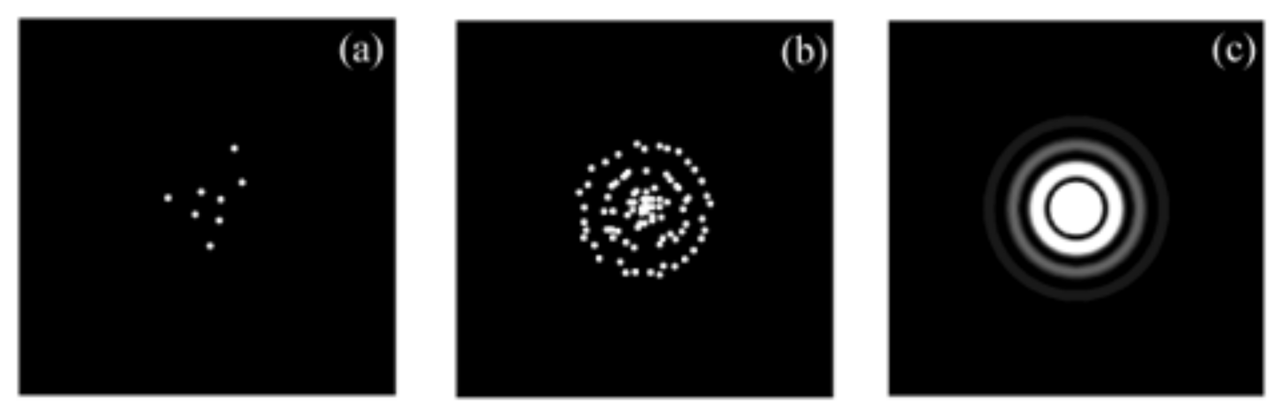
“Light can be described as a series of waves”



Diffraction, interference

Quantum Optics

“Light can be described as discrete particles”



http://skullsinthestars.com/2007/08/31/optics-basics-the-three-major-branches-of-optical-science/

Solving Maxwell's Equations directly

$$\nabla \cdot \vec{D} = 4\pi\rho$$

$$\nabla \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c}\vec{j}$$

$$\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{B} = 0$$

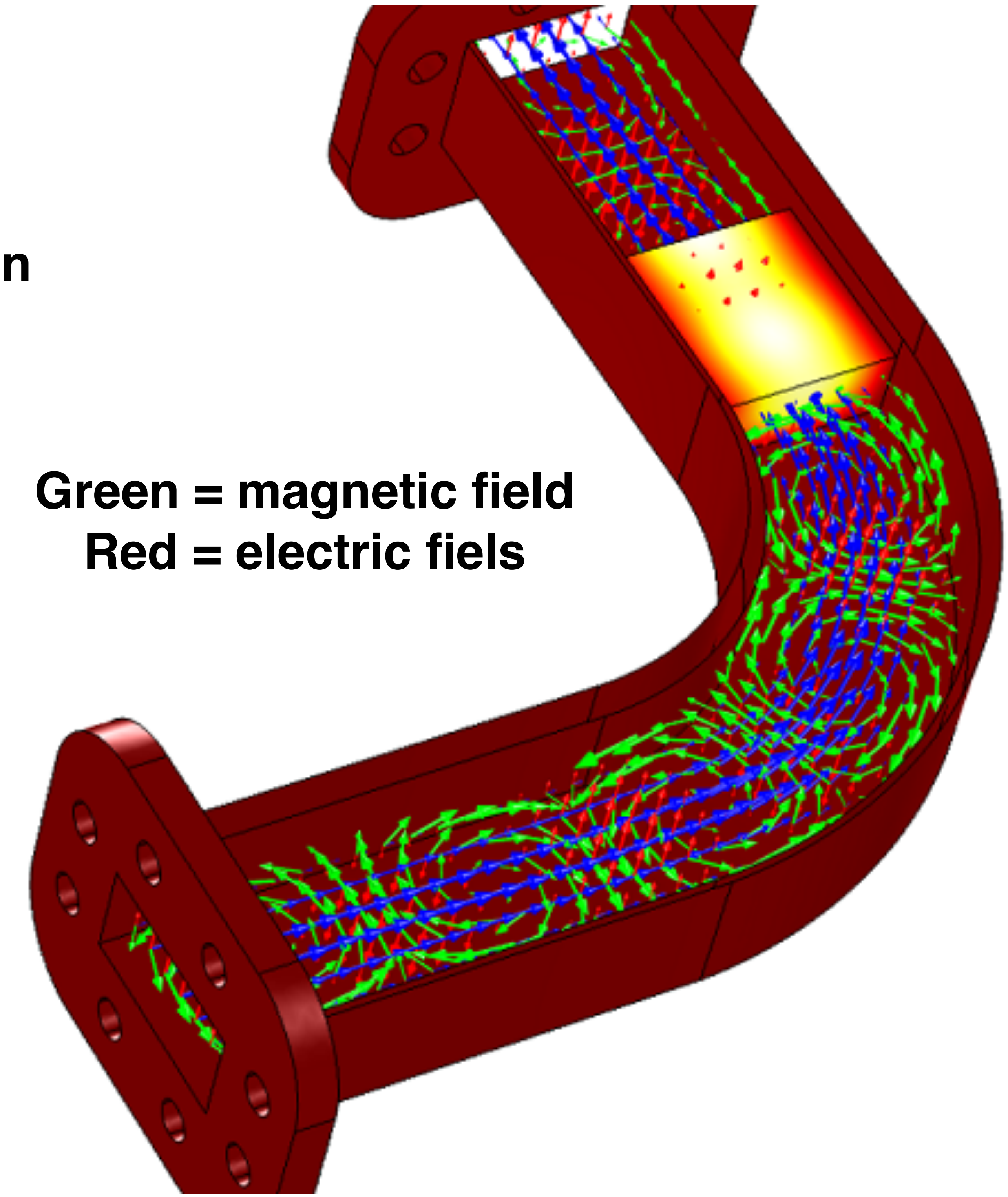
Need to solve four coupled differential equations together with boundary conditions

Numerically, require a 3D mesh with sub-wavelength resolution

Each point has a vector for E and B field

…can be done for optics a few wavelengths in size

Waveguide for microwave radiation

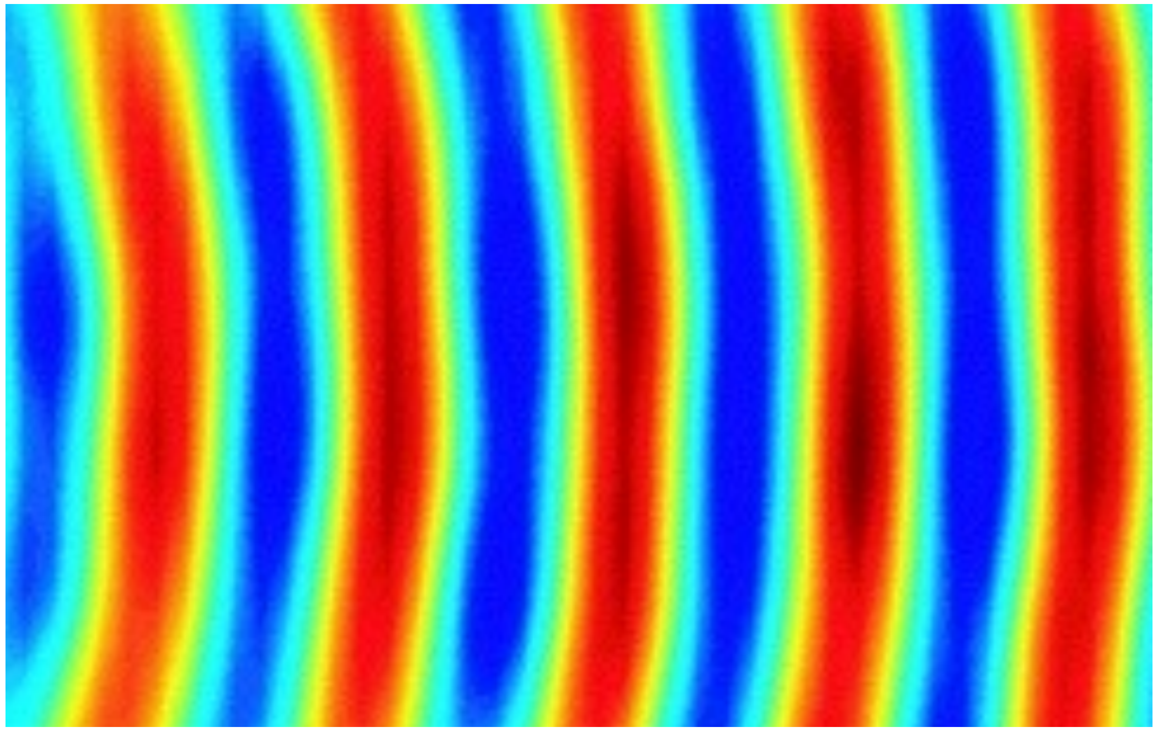


http://www.comsol.com/blogs/quick-intro-modeling-rf-microwave-heating/

Simplified with Wave Equation in Matter

$$\nabla^2 \vec{E} - \frac{\mu\epsilon}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{4\pi\mu\sigma}{c^2}\frac{\partial \vec{E}}{\partial t} = 0$$

Now one equation for E field



Solutions with plane waves

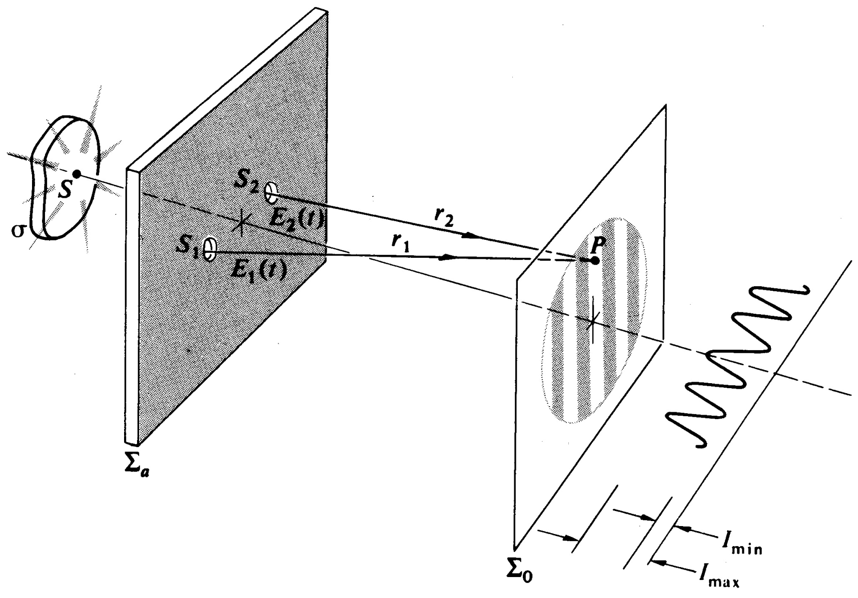
Plane Vector Wave ansatz: $\vec{E} = \vec{E}_0 e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$

- $\vec{k}$ spatially and temporally constant *wave vector*
- $\vec{k}$ normal to surfaces of constant phase
- $|\vec{k}|$ *wave number*
- $\vec{x}$ spatial location
- ω *angular frequency* ($2\pi\times$ frequency)
- t time
- $\vec{E}_0$ a (generally complex) vector independent of time and space
- real electric field vector given by real part of $\vec{E}$

Scalar Wave

- electric field at position $\vec{r}$ at time t is $\tilde{E}(\vec{r}, t)$
- complex notation to easily express amplitude and phase
- real part of complex quantity is the physical quantity

Young's Double Slit Experiment



- monochromatic wave
- infinitely small holes (pinholes)
- source S generates fields $\tilde{E}(\vec{r}_1, t) \equiv \tilde{E}_1(t)$ at S_1 and $\tilde{E}(\vec{r}_2, t) \equiv \tilde{E}_2(t)$ at S_2
- two spherical waves from pinholes interfere on screen
- electrical field at P

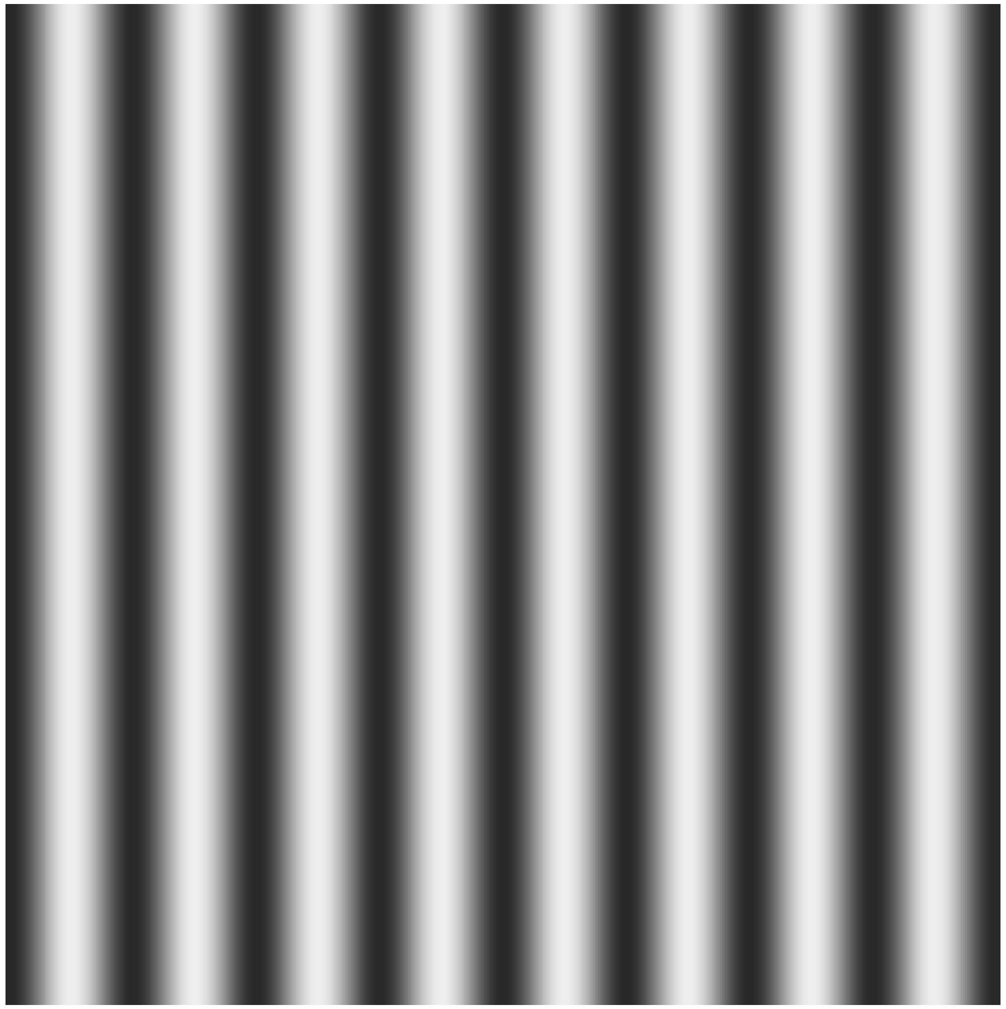
$$\tilde{E}_P(t) \;=\; \tilde{C}_1 \tilde{E}_1(t - t_1) + \tilde{C}_2 \tilde{E}_2(t - t_2)$$

- $t_1 = r_1/c$, $t_2 = r_2/c$
- r_1, r_2: path lengths from S_1, S_2 to P
- propagators $\tilde{C}_{1,2} = \frac{i}{\lambda}$

Two pinholes produce diffraction pattern

no tilt

tilt by 0.5 λ/d



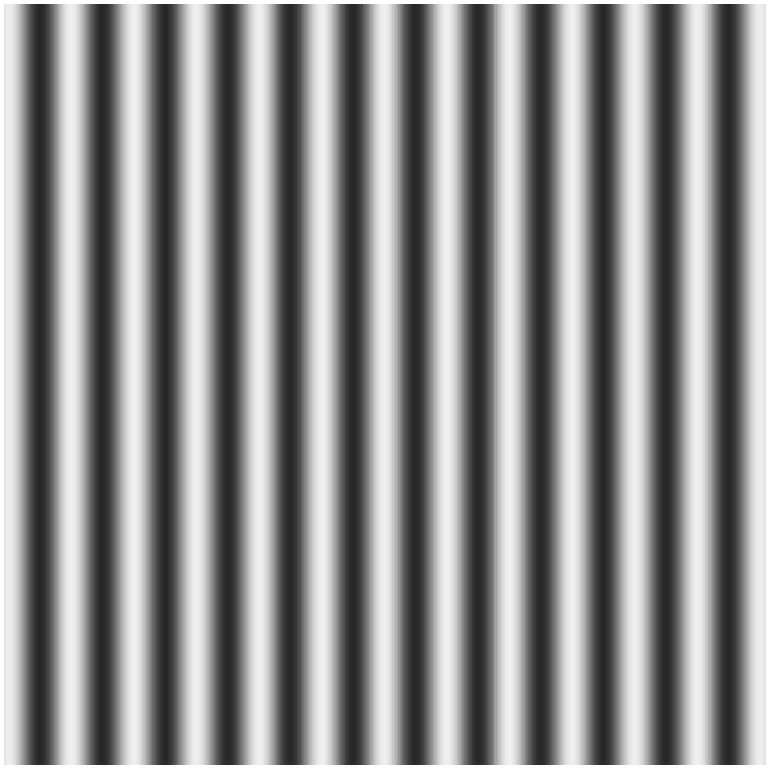
Change in Angle of Incoming Wave

- phase of fringe pattern changes, but not fringe spacing
- tilt of λ/d produces identical fringe pattern

Different wavelengths make envelope

long wavelength

short wavelength



wavelength average

Change in Wavelength

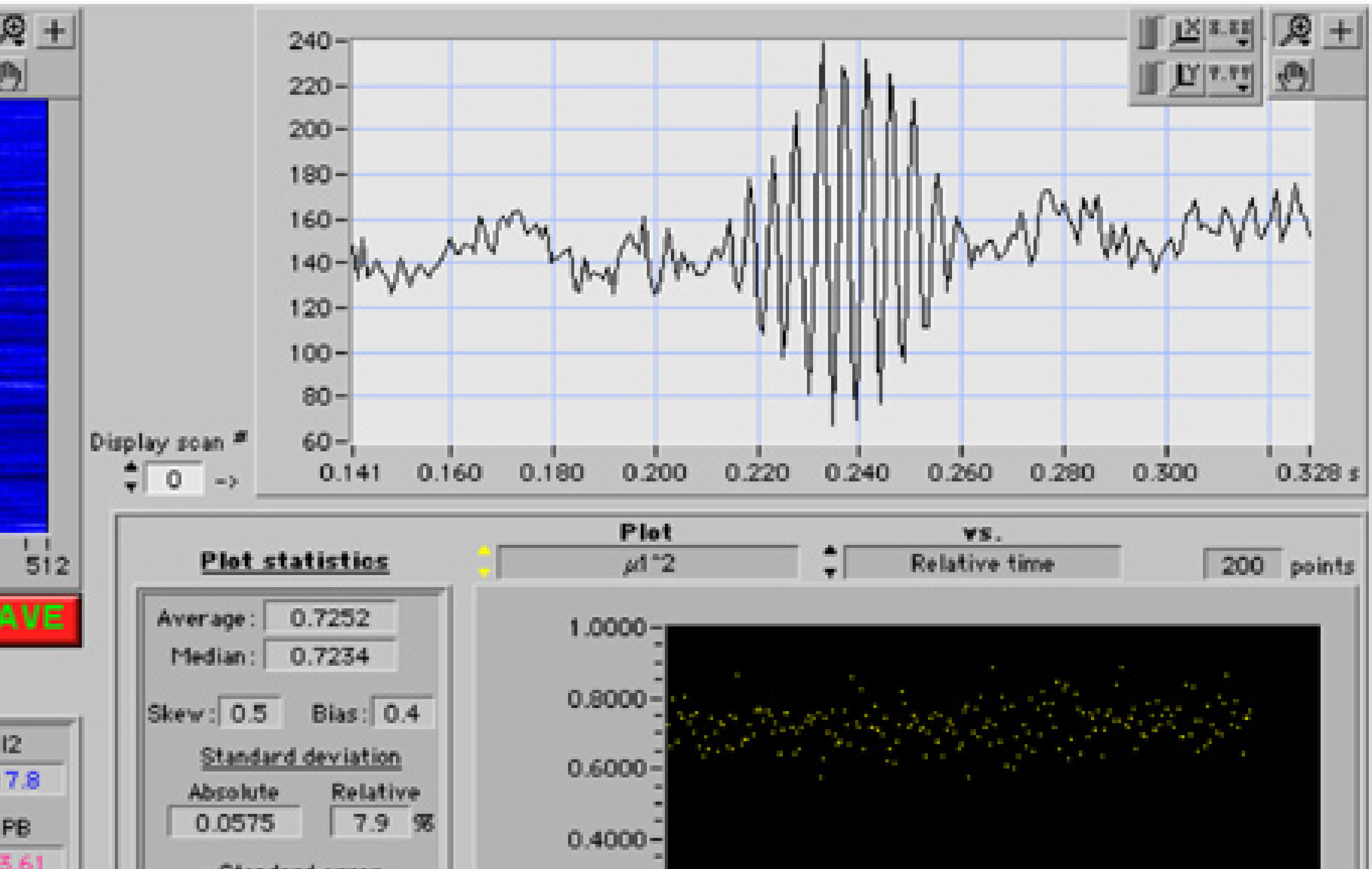
- fringe spacing changes, central fringe broadens
- integral over 0.8 to 1.2 of central wavelength
- integral over wavelength makes fringe envelope

Visibility Function V

$$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$$

$I_{\max}$, $I_{\min}$ are maximum and adjacent minimum in fringe pattern

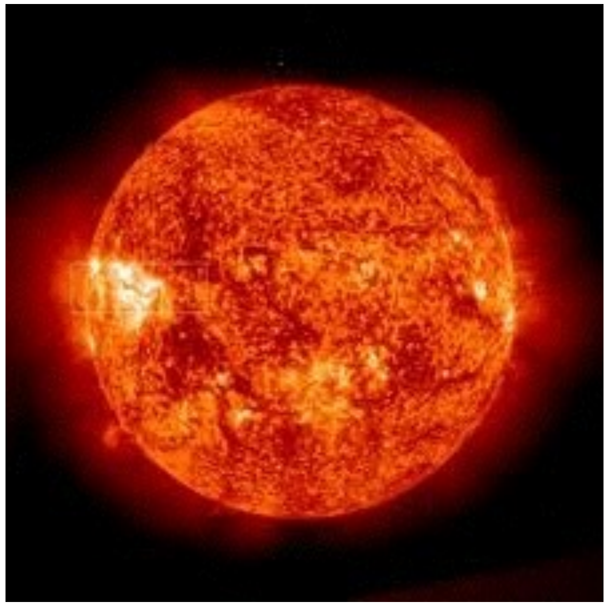
Fringes seen at VLT of the star Sirius



Why do we not see fringes everywhere during the day?

COHERENCE

Real world sources are:

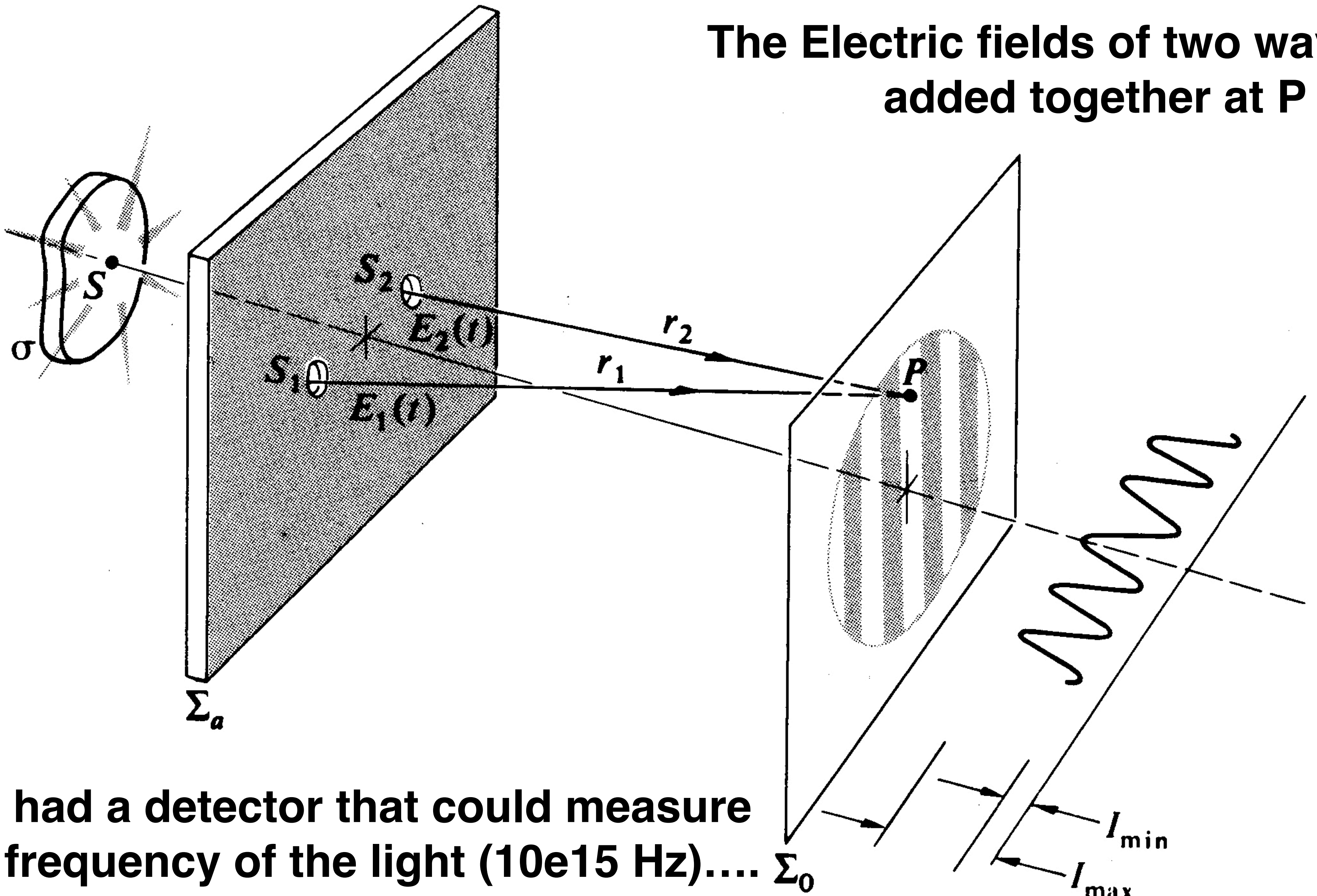


Not point like

Not monochromatic

Why do we not see fringes everywhere during the day?

The Electric fields of two waves are added together at P



If you had a detector that could measure at the frequency of the light (10e15 Hz)…. but we use time averaged detectors

Coherence

Mutual Coherence

- total field in point P

$$\tilde{E}_P(t) \;=\; \tilde{C}_1\tilde{E}_1(t-t_1)+\tilde{C}_2\tilde{E}_2(t-t_2)$$

- irradiance at P, averaged over time (expectation operator **E**)

$$I \;=\; \mathbf{E}|\tilde{E}_P(t)|^2 \;=\; \mathbf{E}\left\{\tilde{E}_P(t)\tilde{E}_P^*(t)\right\}$$

- writing out all the terms

$$\begin{aligned} I = {} & \tilde{C}_1\tilde{C}_1^*\mathbf{E}\left\{\tilde{E}_1(t-t_1)\tilde{E}_1^*(t-t_1)\right\}+\tilde{C}_2\tilde{C}_2^*\mathbf{E}\left\{\tilde{E}_2(t-t_2)\tilde{E}_2^*(t-t_2)\right\} \\ & +\tilde{C}_1\tilde{C}_2^*\mathbf{E}\left\{\tilde{E}_1(t-t_1)\tilde{E}_2^*(t-t_2)\right\}+\tilde{C}_1^*\tilde{C}_2\mathbf{E}\left\{\tilde{E}_1^*(t-t_1)\tilde{E}_2(t-t_2)\right\} \end{aligned}$$

- *stationary* wave field, time average independent of absolute time

$$I_{S_1} = \mathbf{E}\left\{\tilde{E}_1(t)\tilde{E}_1^*(t)\right\},\ I_{S_2} = \mathbf{E}\left\{\tilde{E}_2(t)\tilde{E}_2^*(t)\right\}$$

- irradiance at P is now

$$I = \tilde{C}_1\tilde{C}_1^* I_{S_1} + \tilde{C}_2\tilde{C}_2^* I_{S_2}$$
$$+\tilde{C}_1\tilde{C}_2^*\mathbf{E}\left\{\tilde{E}_1(t-t_1)\tilde{E}_2^*(t-t_2)\right\} + \tilde{C}_1^*\tilde{C}_2\mathbf{E}\left\{\tilde{E}_1^*(t-t_1)\tilde{E}_2(t-t_2)\right\}$$

  - time difference $\tau = t_2 - t_1 \Rightarrow$ last two terms become

$$\tilde{C}_1\tilde{C}_2^*\mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_2^*(t)\right\} + \tilde{C}_1^*\tilde{C}_2\mathbf{E}\left\{\tilde{E}_1^*(t+\tau)\tilde{E}_2(t)\right\}$$

  - equivalent to

$$2\ Re\left[\tilde{C}_1\tilde{C}_2^*\mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_2^*(t)\right\}\right]$$

  - propagators $\tilde{C}$ purely imaginary: $\tilde{C}_1\tilde{C}_2^* = \tilde{C}_1^*\tilde{C}_2 = |\tilde{C}_1||\tilde{C}_2|$
  - cross-term becomes $2|\tilde{C}_1||\tilde{C}_2| Re\left[\mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_2^*(t)\right\}\right]$

- irradiance at P

$$\begin{aligned} I &= \tilde{C}_1 \tilde{C}_1^* I_{S_1} + \tilde{C}_2 \tilde{C}_2^* I_{S_2} \\ &\quad + 2|\tilde{C}_1||\tilde{C}_2| Re \left[\mathbf{E}\left\{ \tilde{E}_1(t+\tau)\tilde{E}_2^*(t) \right\} \right] \end{aligned}$$

- **mutual coherence function** of wave field at S_1 and S_2

$$\tilde{\Gamma}_{12}(\tau) = \mathbf{E}\left\{ \tilde{E}_1(t+\tau)\tilde{E}_2^*(t) \right\}$$

- therefore $I = |\tilde{C}_1|^2 I_{S_1} + |\tilde{C}_2|^2 I_{S_2} + 2|\tilde{C}_1||\tilde{C}_2| \, Re\, \tilde{\Gamma}_{12}(\tau)$
- $I_1 = |\tilde{C}_1|^2 I_{S_1}$, $I_2 = |\tilde{C}_2|^2 I_{S_2}$: irradiances at P from single aperture

$$I = I_1 + I_2 + 2|\tilde{C}_1||\tilde{C}_2| \, Re\, \tilde{\Gamma}_{12}(\tau)$$

Self-Coherence

- $S_1 = S_2 \Rightarrow$ mutual coherence function = autocorrelation

$$\begin{aligned}\tilde{\Gamma}_{11}(\tau) &= \tilde{R}_1(\tau) = \mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_1^*(t)\right\}\\ \tilde{\Gamma}_{22}(\tau) &= \tilde{R}_2(\tau) = \mathbf{E}\left\{\tilde{E}_2(t+\tau)\tilde{E}_2^*(t)\right\}\end{aligned}$$

- autocorrelation functions are also called *self-coherence functions*
- for $\tau = 0$

$$\begin{aligned}I_{S_1} &= \mathbf{E}\left\{\tilde{E}_1(t)\tilde{E}_1^*(t)\right\} = \Gamma_{11}(0) = \mathbf{E}\left\{|\tilde{E}_1(t)|^2\right\}\\ I_{S_2} &= \mathbf{E}\left\{\tilde{E}_2(t)\tilde{E}_2^*(t)\right\} = \Gamma_{22}(0) = \mathbf{E}\left\{|\tilde{E}_2(t)|^2\right\}\end{aligned}$$

- autocorrelation function with zero lag ($\tau = 0$) represent (average) irradiance (power) of wave field at S_1, S_2

Complex Degree of Coherence

- using selfcoherence functions

$$|\tilde{C}_1||\tilde{C}_2| = \frac{\sqrt{I_1}\sqrt{I_2}}{\sqrt{\Gamma_{11}(0)}\sqrt{\Gamma_{22}(0)}}$$

- normalized mutual coherence defines the **complex degree of coherence**

$$\tilde{\gamma}_{12}(\tau) \equiv \frac{\tilde{\Gamma}_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}} = \frac{\mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_2^*(t)\right\}}{\sqrt{\mathbf{E}\left\{|\tilde{E}_1(t)|^2\right\}\mathbf{E}\left\{|\tilde{E}_2(t)|^2\right\}}}$$

- irradiance in point P as *general interference law for a partially coherent radiation field*

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\, Re\, \tilde{\gamma}_{12}(\tau)$$

Spatial and Temporal Coherence

- complex degree of coherence

$$\tilde{\gamma}_{12}(\tau) \equiv \frac{\tilde{\Gamma}_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}} = \frac{\mathbf{E}\left\{\tilde{E}_1(t+\tau)\tilde{E}_2^*(t)\right\}}{\sqrt{\mathbf{E}\left\{|\tilde{E}_1(t)|^2\right\}\mathbf{E}\left\{|\tilde{E}_2(t)|^2\right\}}}$$

- measures both
  - *spatial coherence* at S_1 and S_2
  - *temporal coherence* through time lag τ
- $\tilde{\gamma}_{12}(\tau)$ is a complex variable and can be written as:

$$\tilde{\gamma}_{12}(\tau) = |\tilde{\gamma}_{12}(\tau)|e^{i\psi_{12}(\tau)}$$

- $0 \leq |\tilde{\gamma}_{12}(\tau)| \leq 1$
- phase angle $\psi_{12}(\tau)$ relates to
  - phase angle between fields at S_1 and S_2
  - phase angle difference in P resulting in time lag τ

Coherence of Quasi-Monochromatic Light

- quasi-monochromatic light, mean wavelength $\overline{\lambda}$, frequency $\overline{\nu}$, phase difference ϕ due to optical path difference:

$$\phi = \frac{2\pi}{\overline{\lambda}}(r_2 - r_1) = \frac{2\pi}{\overline{\lambda}}c(t_2 - t_1) = 2\pi\overline{\nu}\tau$$

- with phase angle $\alpha_{12}(\tau)$ between fields at pinholes S_1, S_2

$$\psi_{12}(\tau) = \alpha_{12}(\tau) - \phi$$

- and

$$Re\,\tilde{\gamma}_{12}(\tau) = |\tilde{\gamma}_{12}(\tau)| \cos\left[\alpha_{12}(\tau) - \phi\right]$$

- intensity in P becomes

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\,|\tilde{\gamma}_{12}(\tau)| \cos\left[\alpha_{12}(\tau) - \phi\right]$$

Visibility of Quasi-Monochromatic, Partially Coherent Light

- intensity in P

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\,|\tilde{\gamma}_{12}(\tau)|\cos\left[\alpha_{12}(\tau) - \phi\right]$$

- maximum, minimum I for $\cos(...) = \pm 1$
- visibility V at position P

$$V = \frac{2\sqrt{I_1}\sqrt{I_2}}{I_1 + I_2}|\tilde{\gamma}_{12}(\tau)|$$

- for $I_1 = I_2 = I_0$

$$\begin{aligned} I &= 2I_0\left\{1 + |\tilde{\gamma}_{12}(\tau)|\cos\left[\alpha_{12}(\tau) - \phi\right]\right\} \\ V &= |\tilde{\gamma}_{12}(\tau)| \end{aligned}$$

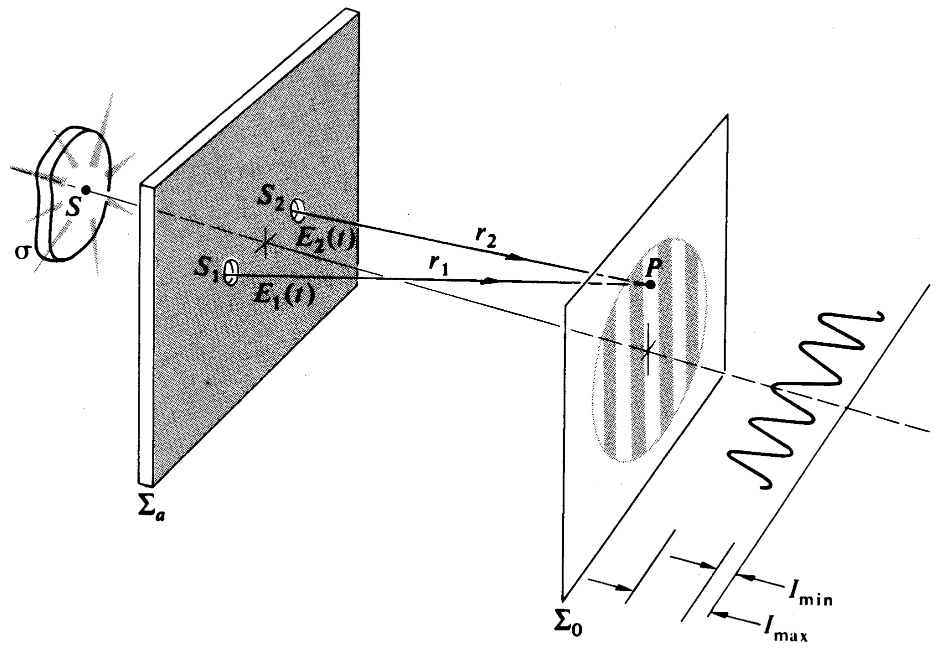
Interpretation of Visibility

- for $I_1 = I_2 = I_0$

$$\begin{aligned} I &= 2I_0\left\{1 + |\tilde{\gamma}_{12}(\tau)| \cos\left[\alpha_{12}(\tau) - \phi\right]\right\} \\ V &= |\tilde{\gamma}_{12}(\tau)| \end{aligned}$$

- *modulus of complex degree of coherence = visibility of fringes*
- modulus can therefore be measured
- shift in location of central fringe (no optical path length difference, $\phi = 0$) is measure of $\alpha_{12}(\tau)$
- measurements of visibility and fringe position yield amplitude and phase of complex degree of coherence

Fringe Pattern

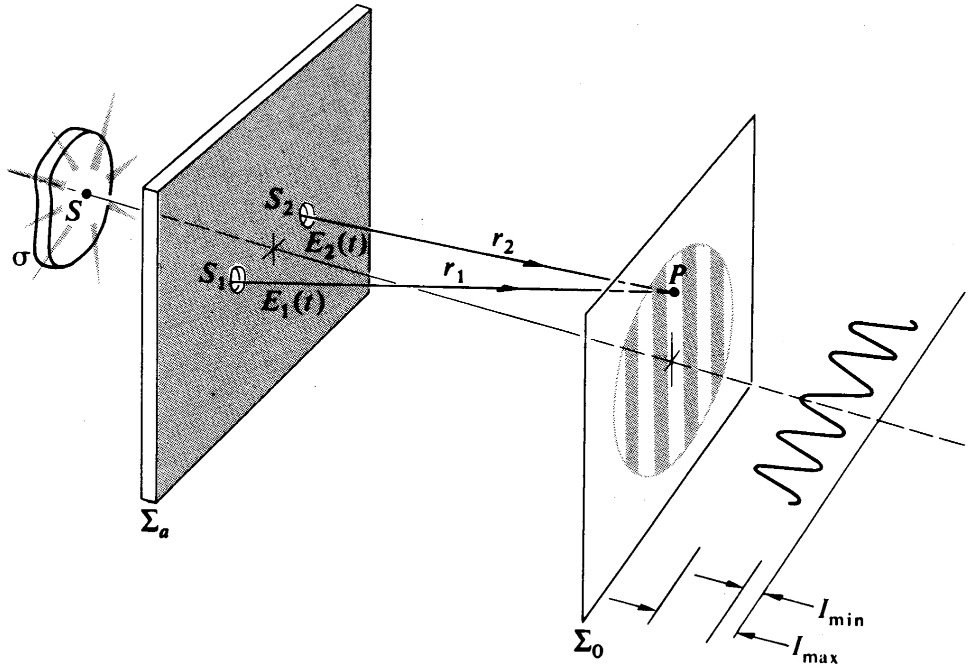


- for $I_1 = I_2 = I_0$

$$I = 2I_0 \left\{1 + |\tilde{\gamma}_{12}(\tau)| \cos\left[\alpha_{12}(\tau) - \phi\right]\right\} \quad V = |\tilde{\gamma}_{12}(\tau)|$$

- source S on central axis, fully coherent waves from two holes

$$I \; = \; 2I_0(1 + \cos\phi) \; = \; 4I_0 \cos^2 \frac{\phi}{2}$$



$$I = 4 I_0 \cos^2 \frac{\phi}{2}$$

$$\phi = \frac{2\pi}{\bar{\lambda}}(r_2 - r_1) = 2\pi\bar{\nu}\tau$$

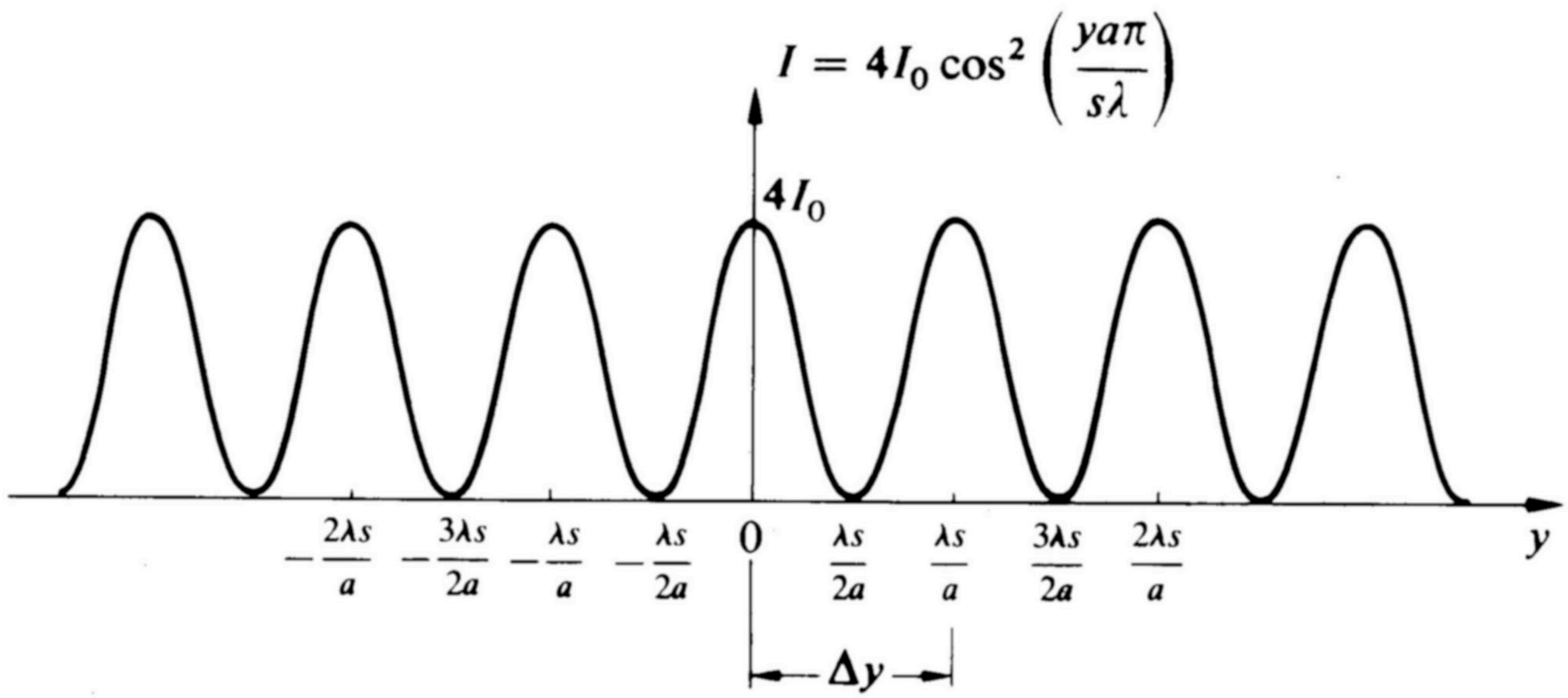
- distance a between pinholes
- distance s to observation plane Σ_O, $s \gg a$
- path difference $(r_2 - r_1)$ in equation for ϕ in good approximation

$$r_2 - r_1 = a\theta = \frac{a}{s}\, y$$

- and therefore

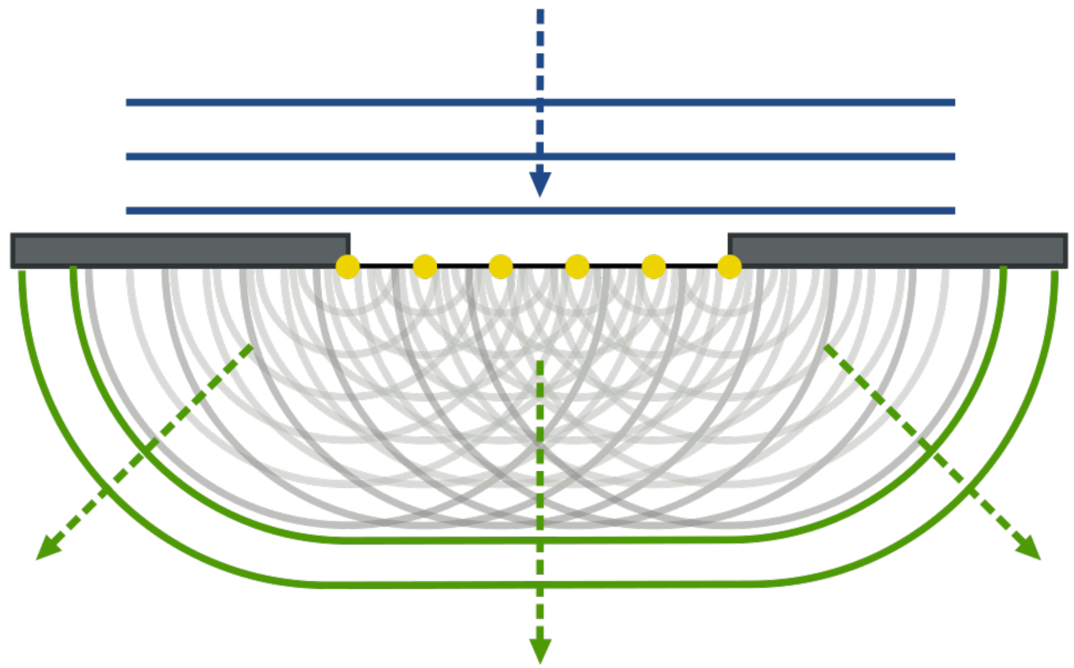
$$I = 4 I_0 \cos^2 \frac{\pi a y}{s\bar{\lambda}}$$

Interference Fringes from Monochromatic Point Source



- irradiance as a function of the y-coordinate of the fringes in observation plane Σ_O
- irradiance vs. distance distribution is Point-Spread Function (PSF) of ideal two-element interferometer

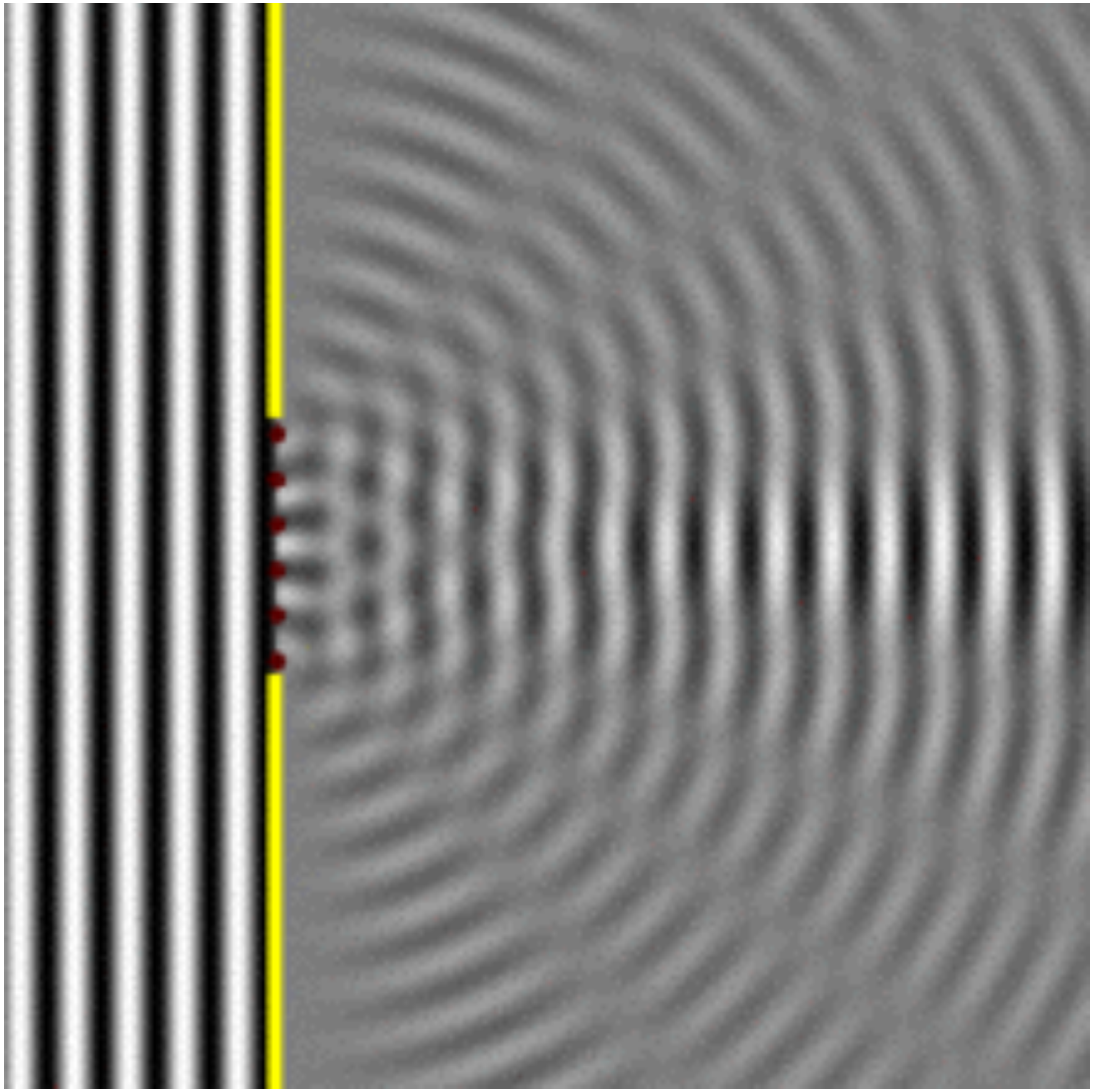
Huygens-Fresnel Principle



en.wikipedia.org/wiki/File:Refraction_on_an_aperture_-_Huygens-Fresnel_principle.svg

- every unobstructed point of a wavefront at a given moment in time serves as a source of spherical, secondary waves with the same frequency as the primary wave
- the amplitude of the optical field at any point beyond is the coherent superposition of all these secondary, spherical waves

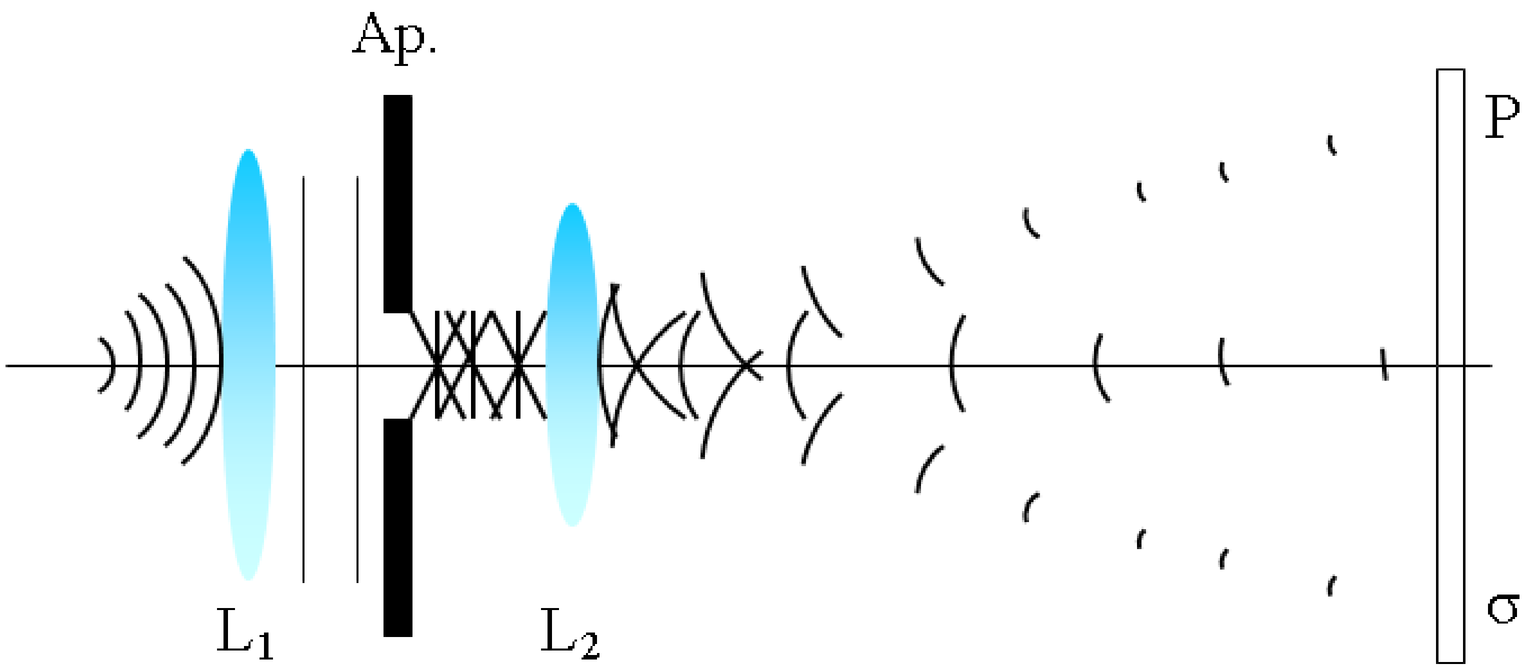
Diffraction



If obstructions are small compared to the wavelength, then waves will spread out

Huygens-Fresnel works for most cases and is much faster than solving Maxwell's Equations!

Fraunhoffer and Fresnel Diffraction

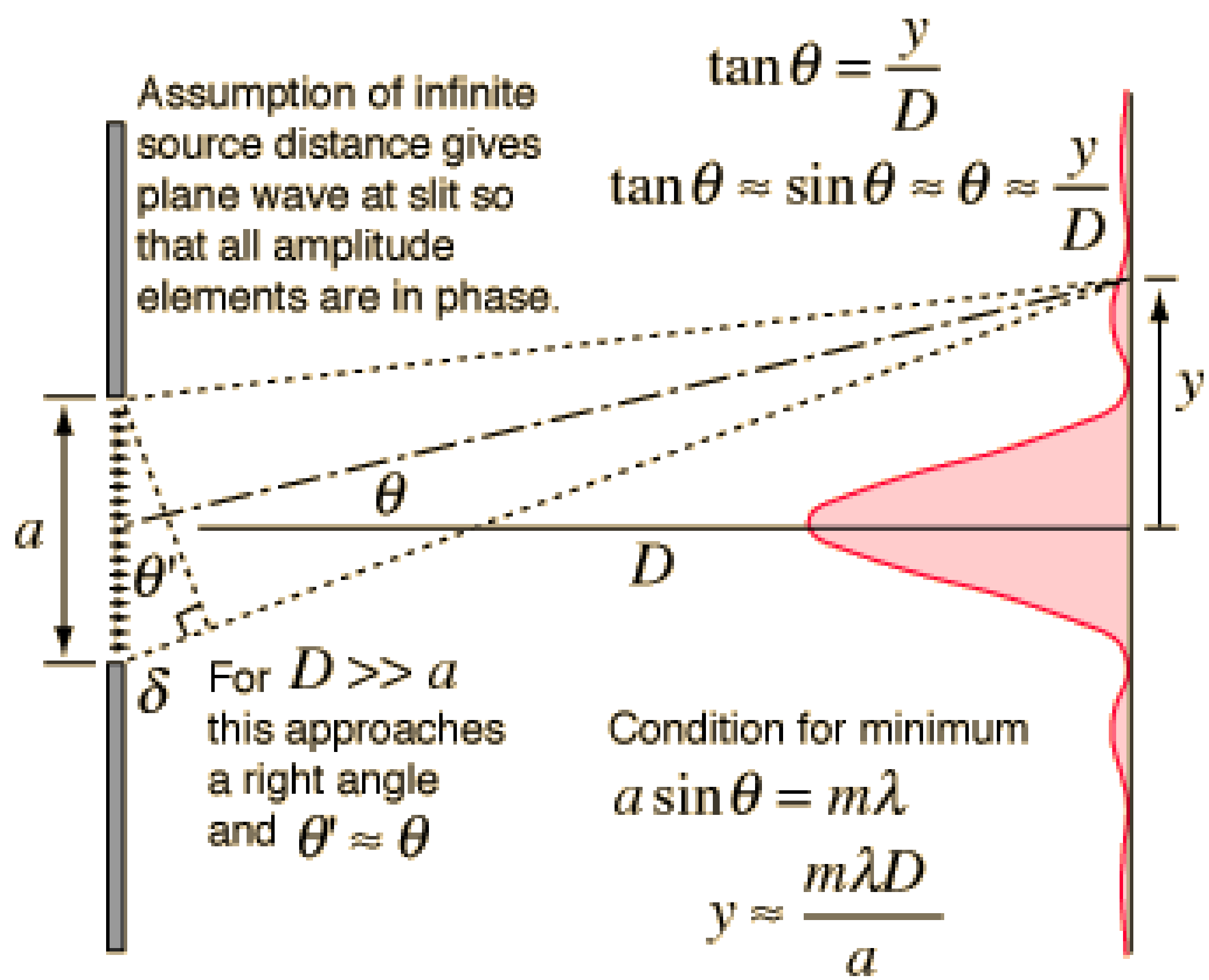


en.wikipedia.org/wiki/File:Fraunhofer_diffraction_pattern_image.PNG

- wave shape changes as it moves away from obstruction
- *Fresnel* (or near-field) diffraction close to obstruction
- em Fraunhofer (or far-field) diffraction far away from obstruction
- rule of thumb: Frauenhofer diffraction for

$$R > a^2/\lambda$$

Single Slit Diffraction



hyperphysics.phy-astr.gsu.edu/hbase/phyopt/sinslit.html

- diffraction of a plane wave on a slit aperture
- Huygens-Fresnel: line of spherical wave sources

Break slit into many small radiating elements

- with E_0 the strength of each slit segment i at point P is

$$E_i(P) = \frac{E_L}{r_i} \sin(k\omega - kr_i)\Delta y_i$$

i	segment index $(1 - M)$
E_L	source strength per unit length
r_i	distance between segment and point P
Δy_i	small segment of slit
D	length of slit

Integrate small elements along slit

- integrate along slit

$$E = E_L \int_{-D/2}^{D/2} \frac{\sin \omega t - kr}{r} dy$$

- express r as a function of y:

$$r = R - y \sin \theta + \frac{y^2}{2R} \cos^2 \theta + ...$$

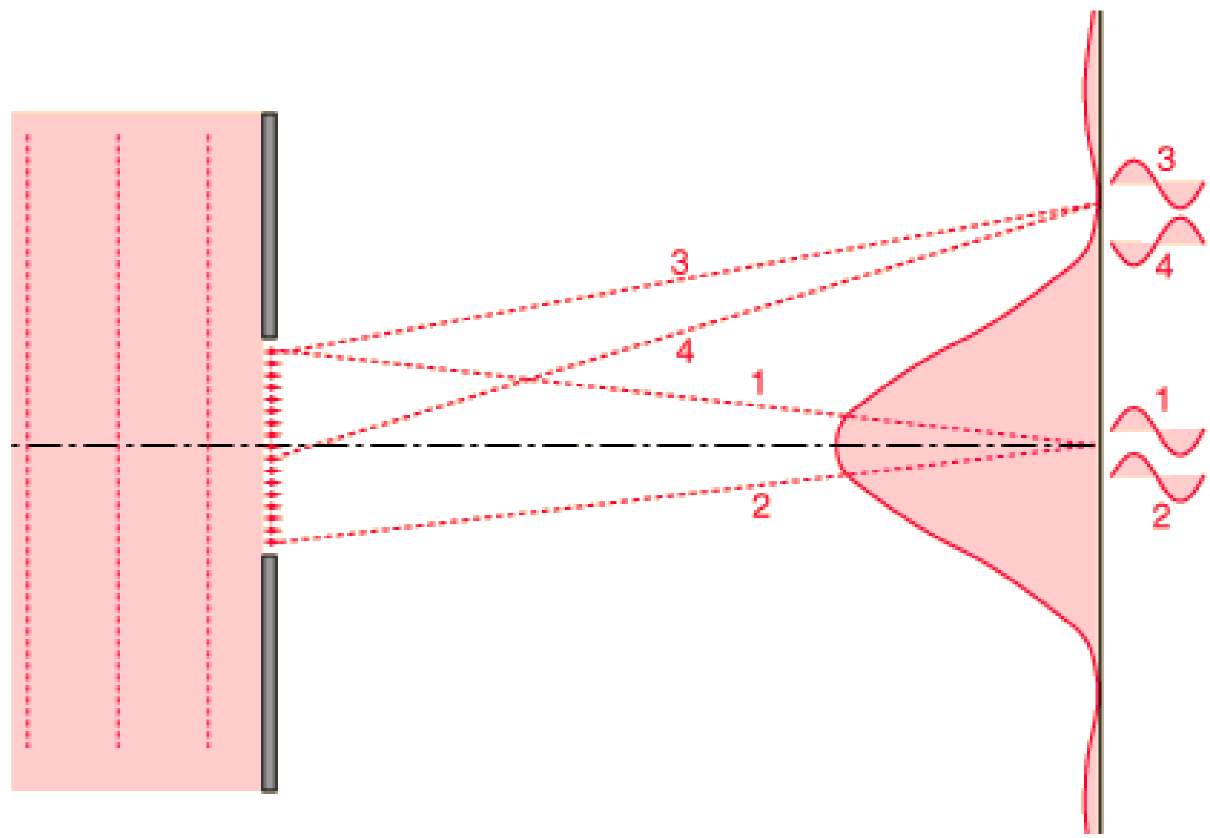
R distance between center of slit and point P

- substituting, integrating and squaring for intensity:

$$I(\theta) = I(0) \left(\frac{\sin \beta}{\beta} \right)^2$$

- $\beta = (kD/2) \sin \theta$

Destructive interference



hyperphysics.phy-astr.gsu.edu/hbase/phyopt/sinslitd.html

- assume infinite distance from aperture for source and observation plane
- equivalent to plane waves coming from aperture into different directions
- first minimum when phase delay at edge is exactly one wave

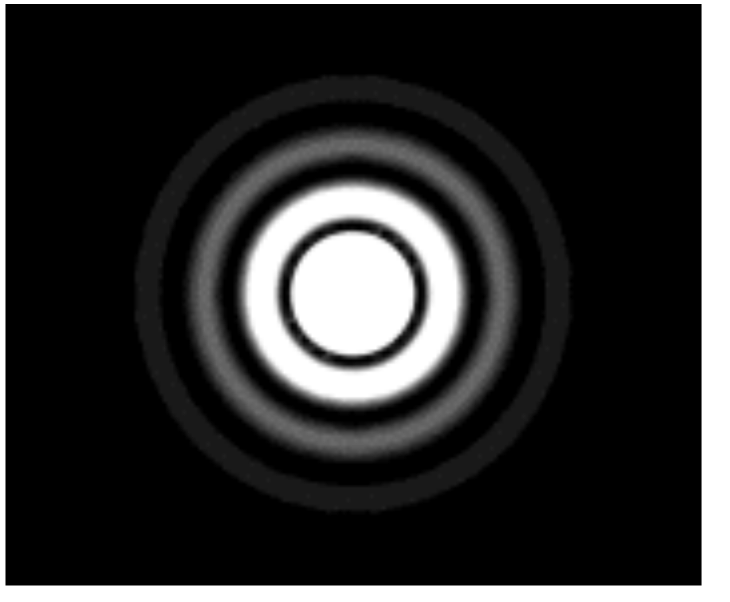
Circular aperture diffraction

- integrate over circular aperture with radius a

$$E = \frac{E_A e^{i(\omega t - kR)}}{R} \int\int_{\text{aperture}} e^{ik(Yy+Zz)/R} dS$$

- using polar coordinates in aperture and plane of observation and Bessel functions

$$I(\theta) = I(0)\left(\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right)^2$$



- J_1 Bessel function of order 1
- Airy function
- first dark ring at $1.22\frac{R\lambda}{2a}$
- images with perfect telescopes are convolution of Airy disk with actual image

…and for any type of aperture

- from before forgetting common phase term and $1/R$ amplitude drop-off

$$E(Y,Z) = \int\int_{\text{aperture}} A(y,z)e^{ik(Yy+Zz)/R}dS$$

- complex aperture function $A(y,z)$ describing non-uniform absorption and phase delays
- finite aperture $\Rightarrow$ change integration boundaries to infinity
- with $k_y = kY/R$ and $k_z = kZ/R$ we obtain

$$E(k_y,k_z) = \int\int_{\text{aperture}} A(y,z)e^{i(k_y y+k_z z)}dy\,dz$$

- field distribution in Fraunhofer diffraction pattern is Fourier transform of field distribution across aperture

The Point Spread Function

- intensity is modulus squared of field distribution $\Rightarrow$ point-spread function
- image of a point source: *Point Spread Function (PSF)*
- image of arbitrary object is a convolution of object with PSF

$$i = o * s$$

i observed image
o true object, constant in time
s point spread function
$*$ convolution

The Optical Transfer Function

- after Fourier transformation:

$$I = O \cdot S$$

  - Fourier transformed

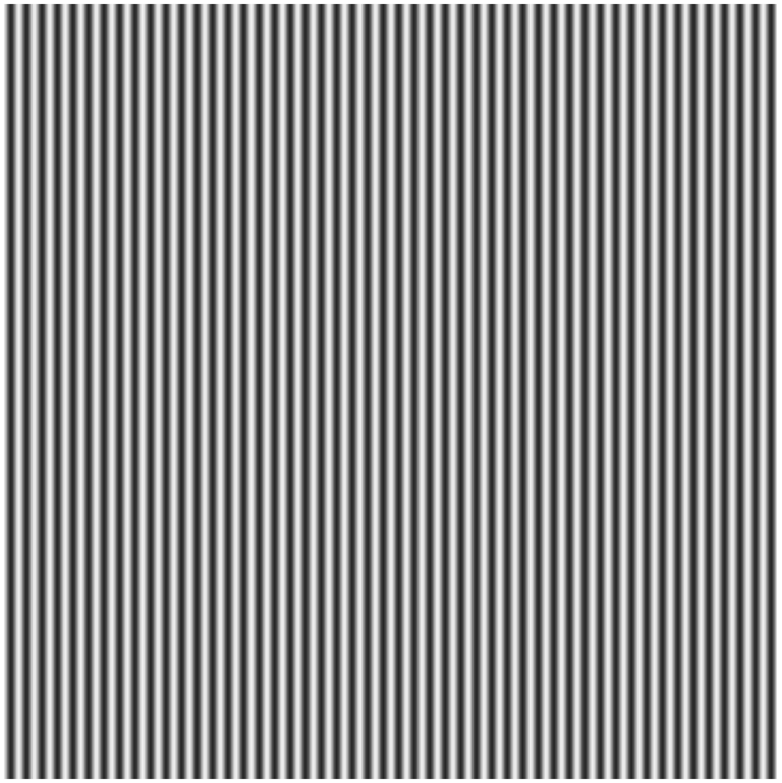
    I Fourier transform of image

    O Fourier transform of object

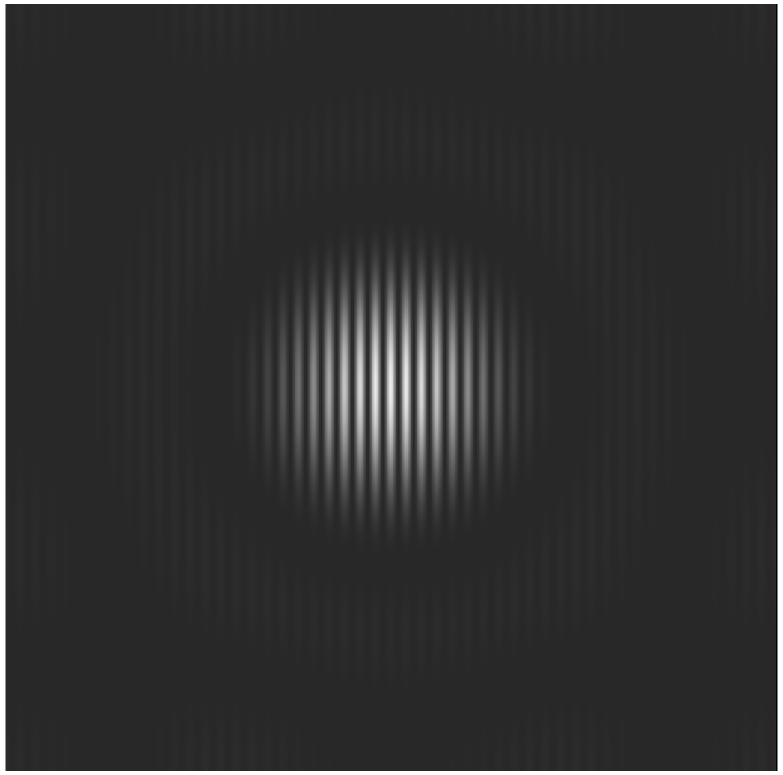
    S Optical Transfer Function (OTF)

- OTF is Fourier transform of PSF and vice versa

2 pinholes



2 small holes



2 large holes

Finite Hole Diameter

- fringe spacing only depends on separation of holes and wavelength
- the smaller the hole, the larger the ’illuminated’ area
- fringe envelope is Airy pattern (diffraction pattern of a single hole)

Interferometer with Finite Apertures

- non-ideal two-element interferometer with finite apertures using *pupil function* concept (Observational Astrophysics 1)
- optical transfer function (OTF)

$$OTF = 2\left(\frac{\lambda}{R}\right)^2\left[\delta(\vec{\zeta}) + \frac{1}{2}\delta\left(\vec{\zeta} - \vec{s}/\lambda\right) + \frac{1}{2}\delta\left(\vec{\zeta} + \vec{s}/\lambda\right)\right]$$

- pair of pinholes transmits three spatial frequencies
  - DC-component $\delta(\vec{0})$
  - two high frequencies related to length of baseline vector $\vec{s}$ at $\pm\vec{s}/\lambda$
- 3 spatial frequencies represent three-point sampling of the **uv-plane** in 2-d spatial frequency space
- complete sampling of uv-plane provides sufficient information to completely reconstruct original brightness distribution

Point-Spread Function (PSF)

- PSF is Fourier Transform of OTF

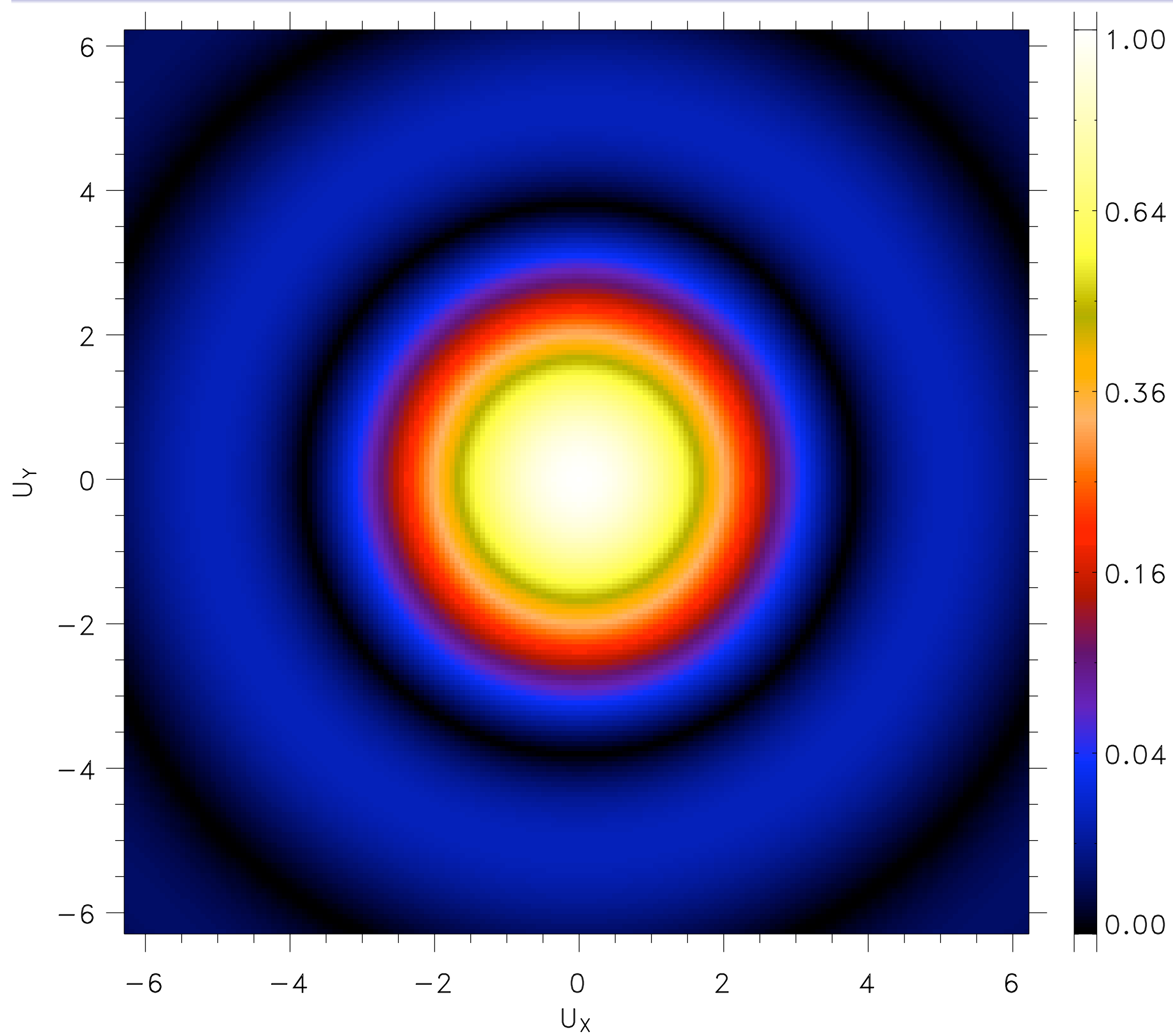
$$
\begin{aligned}
\delta(\vec{\zeta}) &\Leftrightarrow 1 \\
\delta\left(\vec{\zeta} - \vec{s}/\lambda\right) &\Leftrightarrow e^{i2\pi\vec{\theta}\cdot\vec{s}/\lambda} \\
\delta\left(\vec{\zeta} + \vec{s}/\lambda\right) &\Leftrightarrow e^{-i2\pi\vec{\theta}\cdot\vec{s}/\lambda}
\end{aligned}
$$

- Point-Spread Function of 2-element interferometer

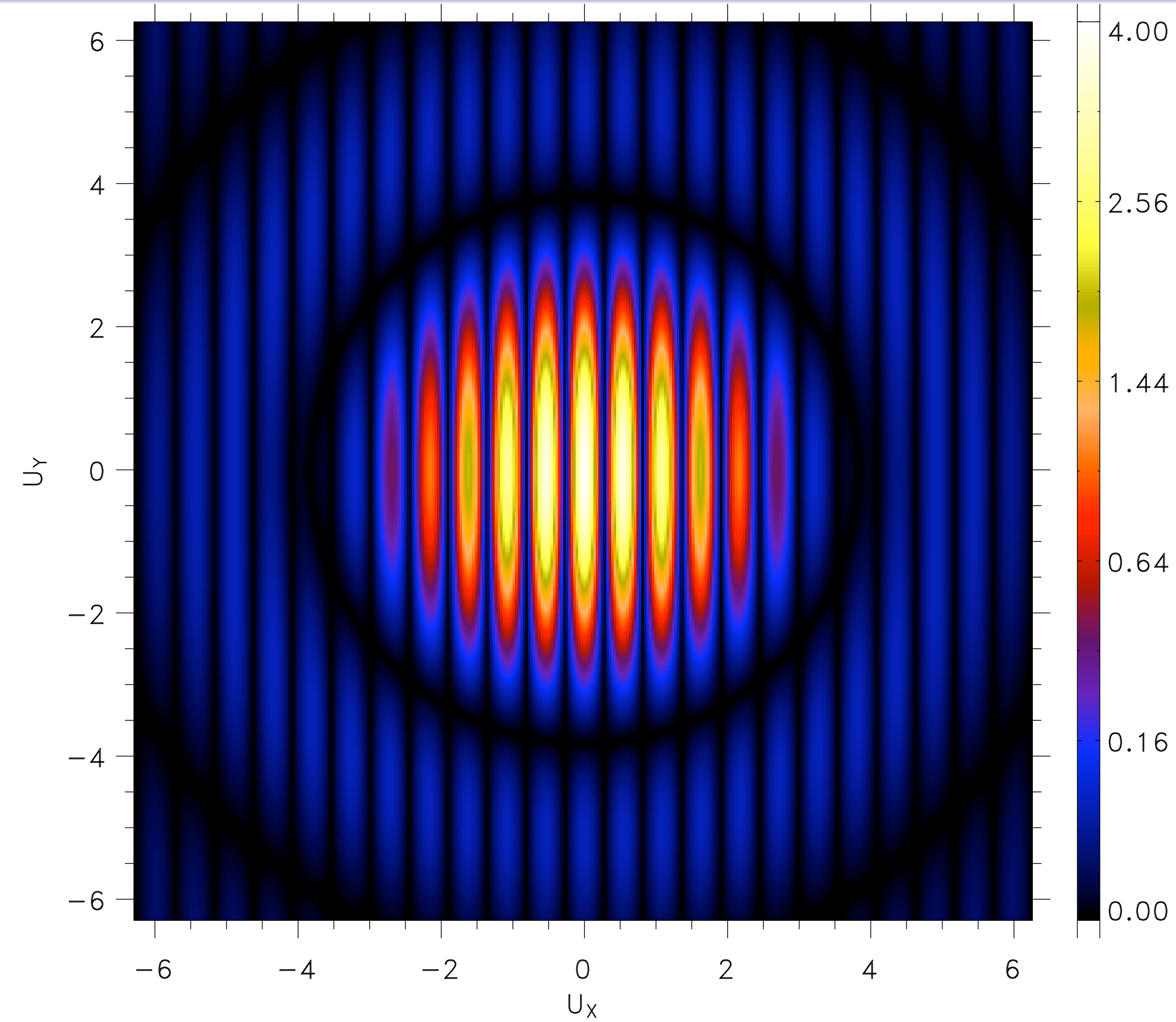
$$
\left(\frac{\lambda}{R}\right)^2 \left[2(1 + \cos 2\pi\vec{\theta}\cdot\vec{s}/\lambda)\right] = 4\left(\frac{\lambda}{R}\right)^2 \cos^2 \pi\vec{\theta}\cdot\vec{s}/\lambda
$$

- $\vec{\theta}$: 2-d angular coordinate vector
- attenuation factor $(\lambda/R)^2$ from spherical expansion

2-d Brightness Distribution



PSF of single circular aperture



PSF of two-element interferometer, aperture diameter $d = 25$ m, length of baseline vector $|\vec{s}| = 144$ m

- double beam interference fringes showing modulation effect of diffraction by aperture of a single pinhole

Why do we see Airy patterns of stars?

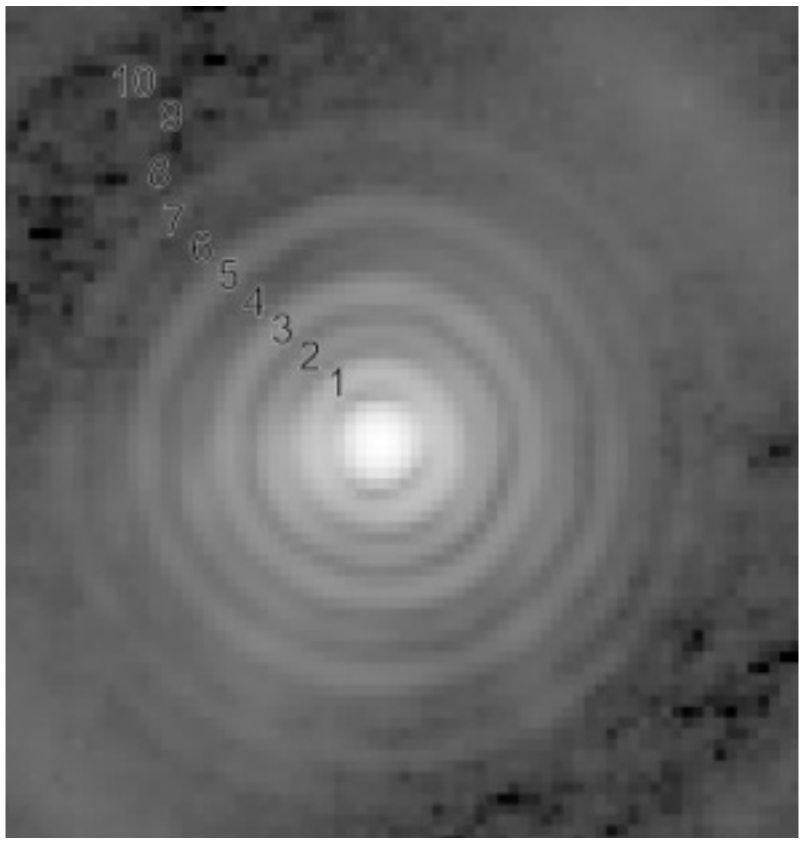
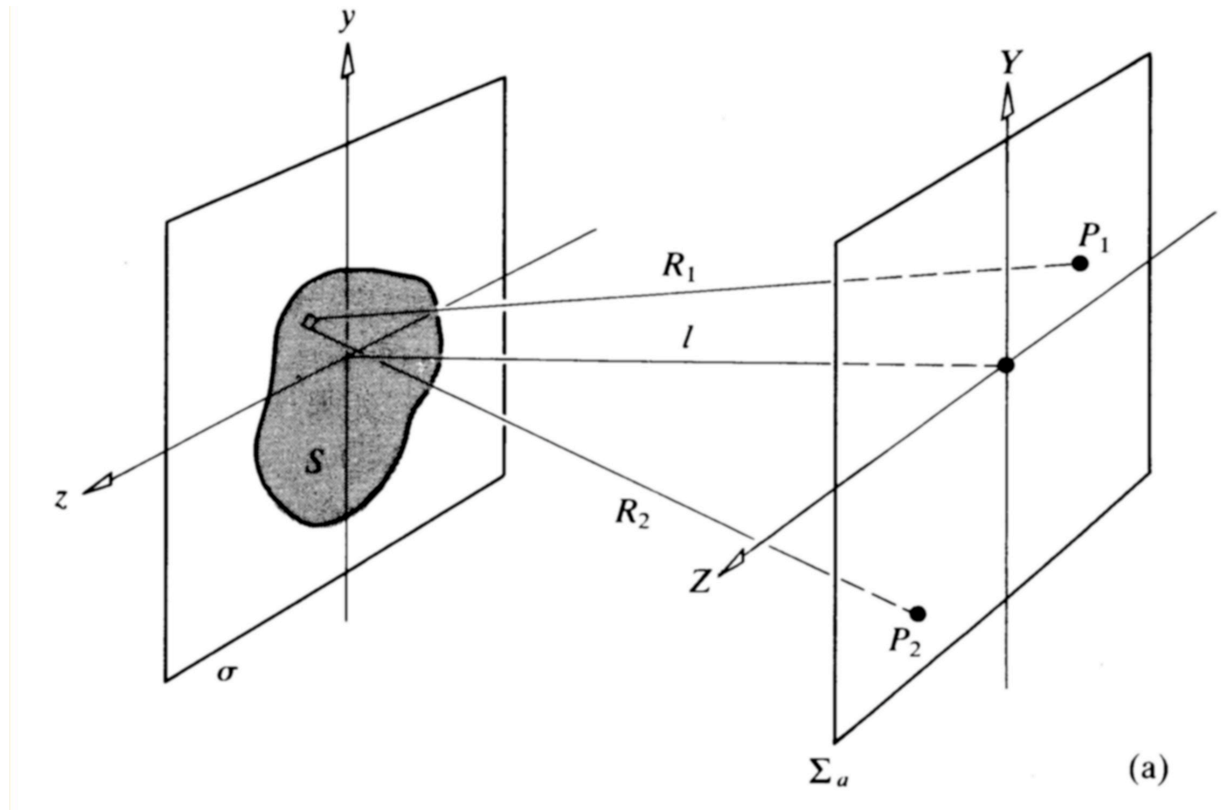


Image of Vega taken with
8.4m diameter telescope in Arizona

van Cittert-Zernike Theorem



- relates brightness distribution of extended source and phase correlation between two points in radiation field
- extended source S incoherent, quasi-monochromatic
- positions P_1 and P_2 in observers plane Σ

$$\tilde{E}_1(t)\tilde{E}_2^*(t) = \mathbf{E}\{\tilde{E}_1(t)\tilde{E}_2^*(t)\} = \tilde{\Gamma}_{12}(0)$$

The Solution

- $I(\vec{\Omega})$ is intensity distribution of extended source as function of unit direction vector $\vec{\Omega}$ as seen from observation plane Σ
- $\tilde{\Gamma}(\vec{r})$ is coherence function in Σ-plane
- vector $\vec{r}$ represents arbitrary baseline
- van Cittert-Zernike theorem

$$\tilde{\Gamma}(\vec{r}) = \int\int_{\text{source}} I(\vec{\Omega}) e^{\frac{2\pi i \vec{\Omega}.\vec{r}}{\lambda}} d\vec{\Omega}$$

$$I(\vec{\Omega}) = \lambda^{-2} \int\int_{\Sigma\text{-plane}} \tilde{\Gamma}(\vec{r}) e^{-\frac{2\pi i \vec{\Omega}.\vec{r}}{\lambda}} d\vec{r}$$

- $\tilde{\Gamma}(\vec{r})$ and $I(\vec{\Omega})$ are linked through Fourier transform, except for scaling with wavelength λ
- "true" Fourier transform with *conjugate variables* $\vec{\Omega}$, $\vec{r}/\lambda$,
- Fourier pair: $I(\vec{\Omega}) \quad \Leftrightarrow \quad \tilde{\Gamma}(\vec{r}/\lambda)$