Astronomical Observing Techniques

Lecture 5: Monsieur Fourier and his Elegant Transform

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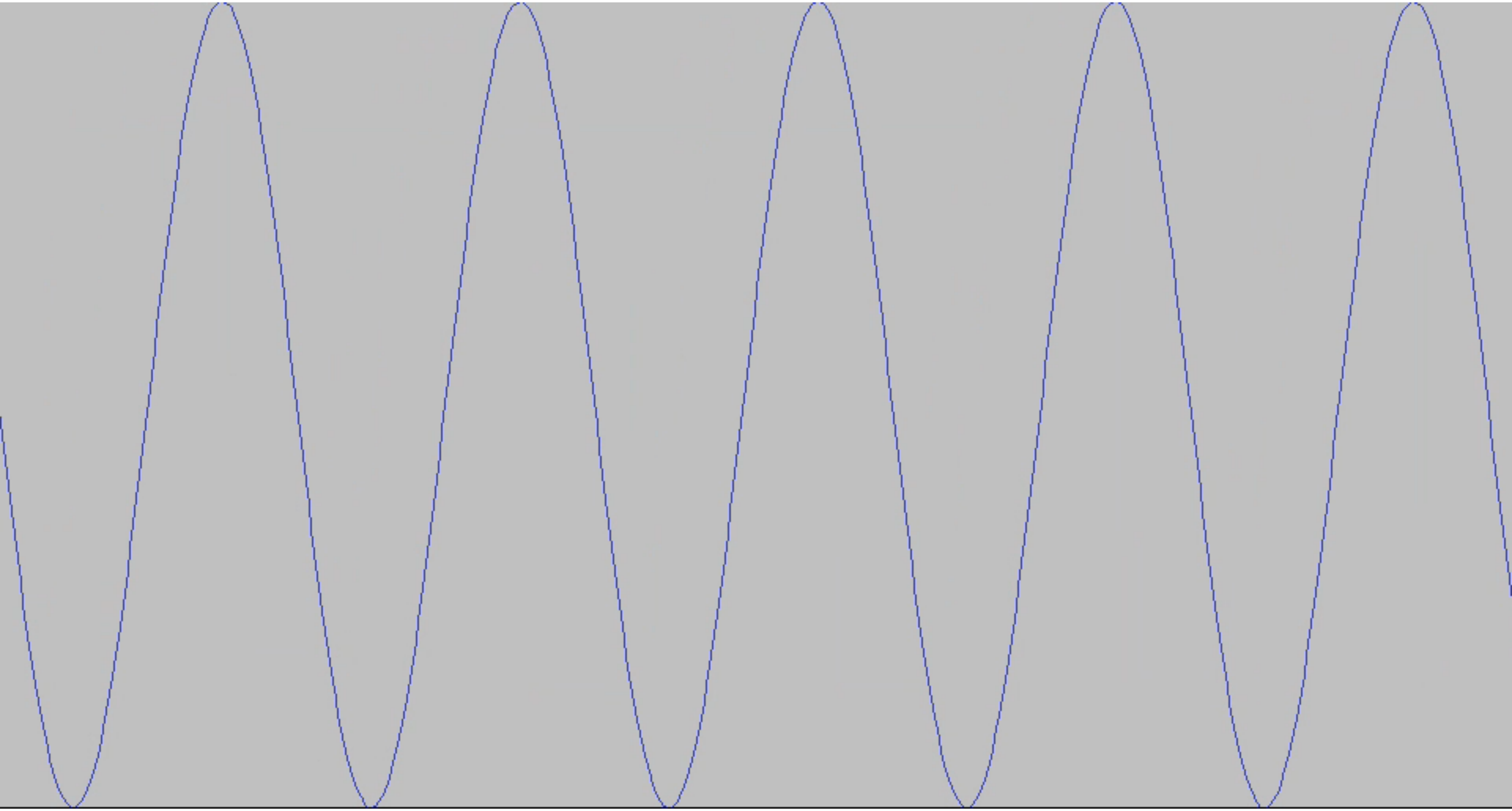
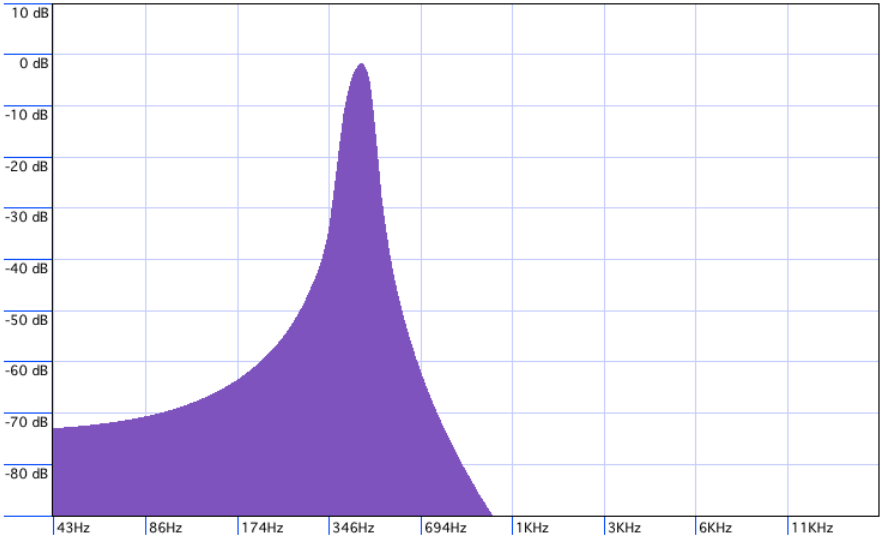
Outline

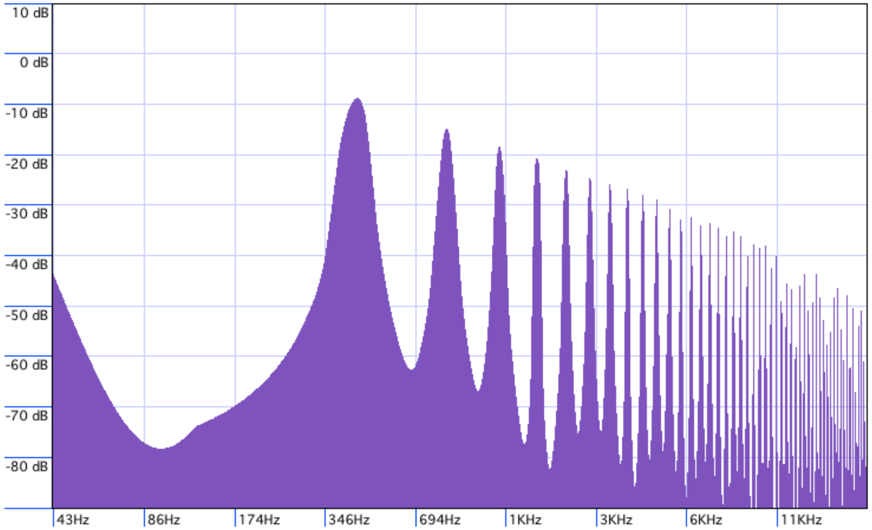
1. Fourier Transform
2. Fourier Transform Examples
3. Fourier Series of Periodic Functions
4. Telescope ⇔ PSF
5. Important Theorems

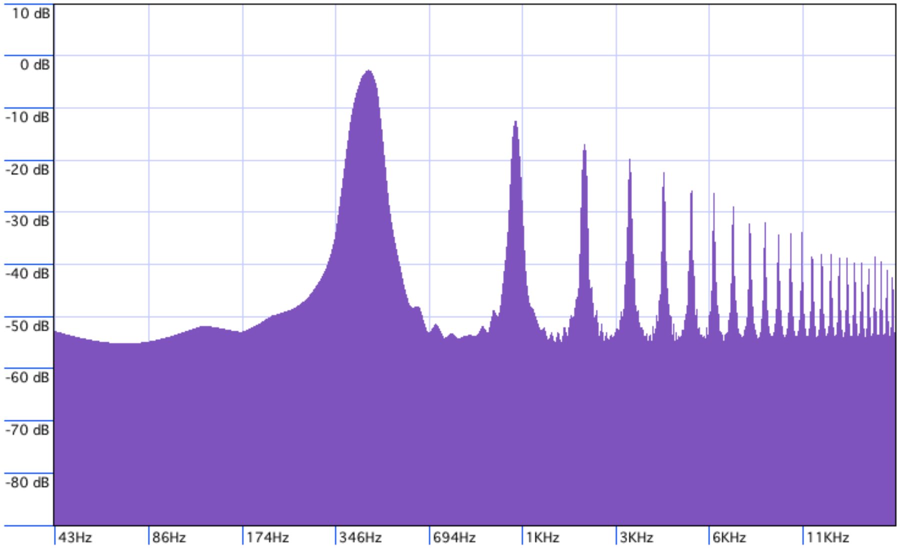
Hear the Difference



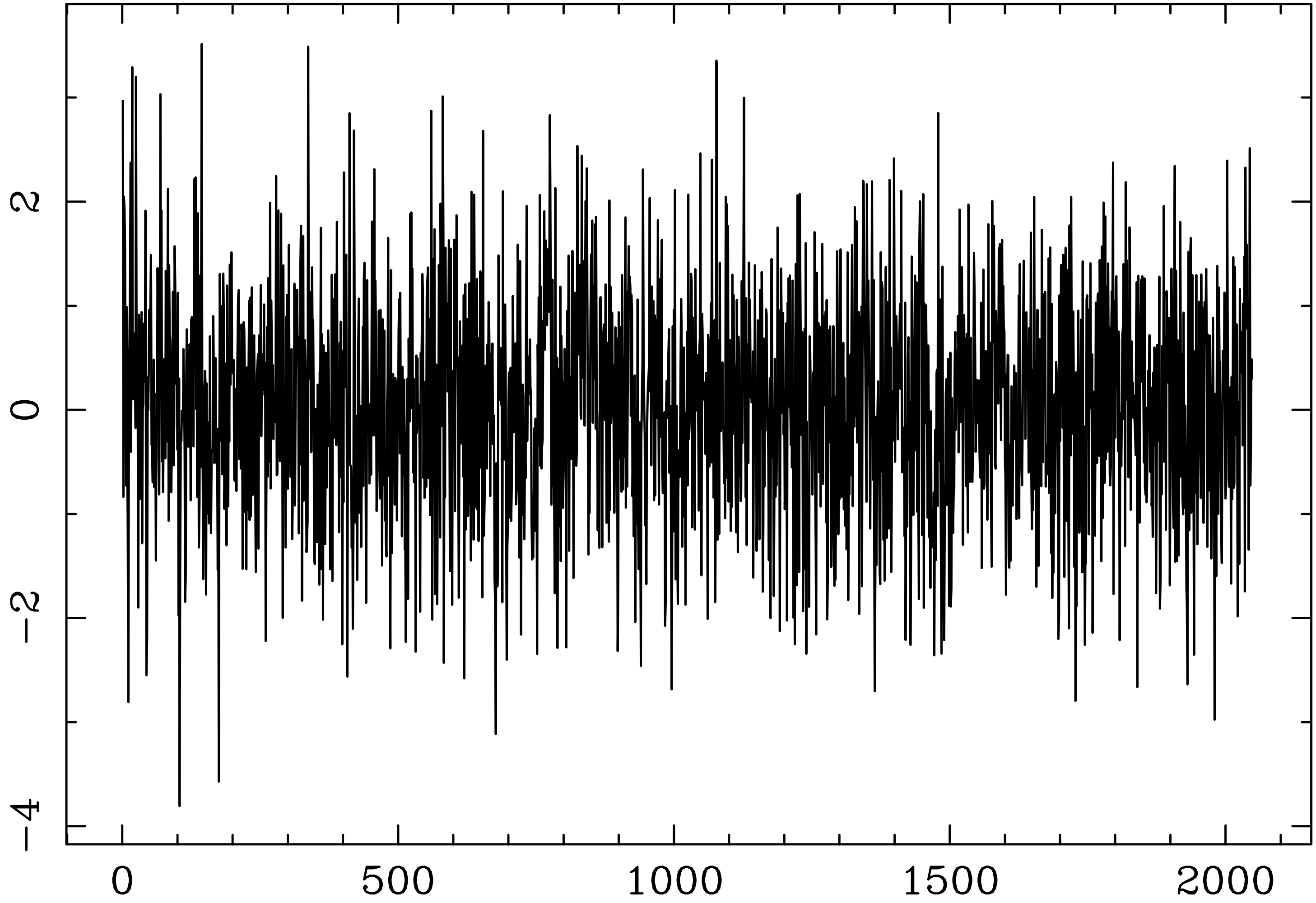
See the Difference



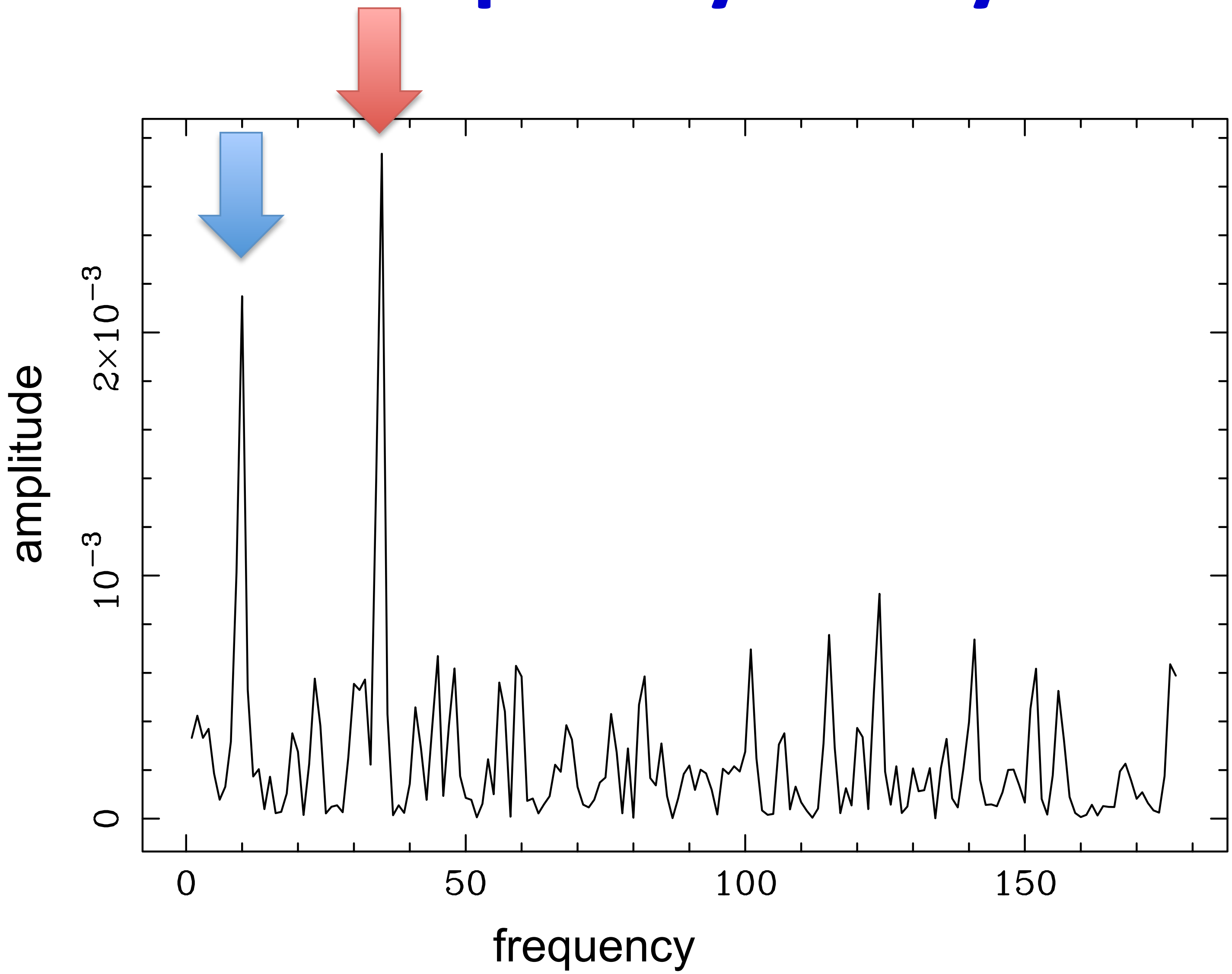




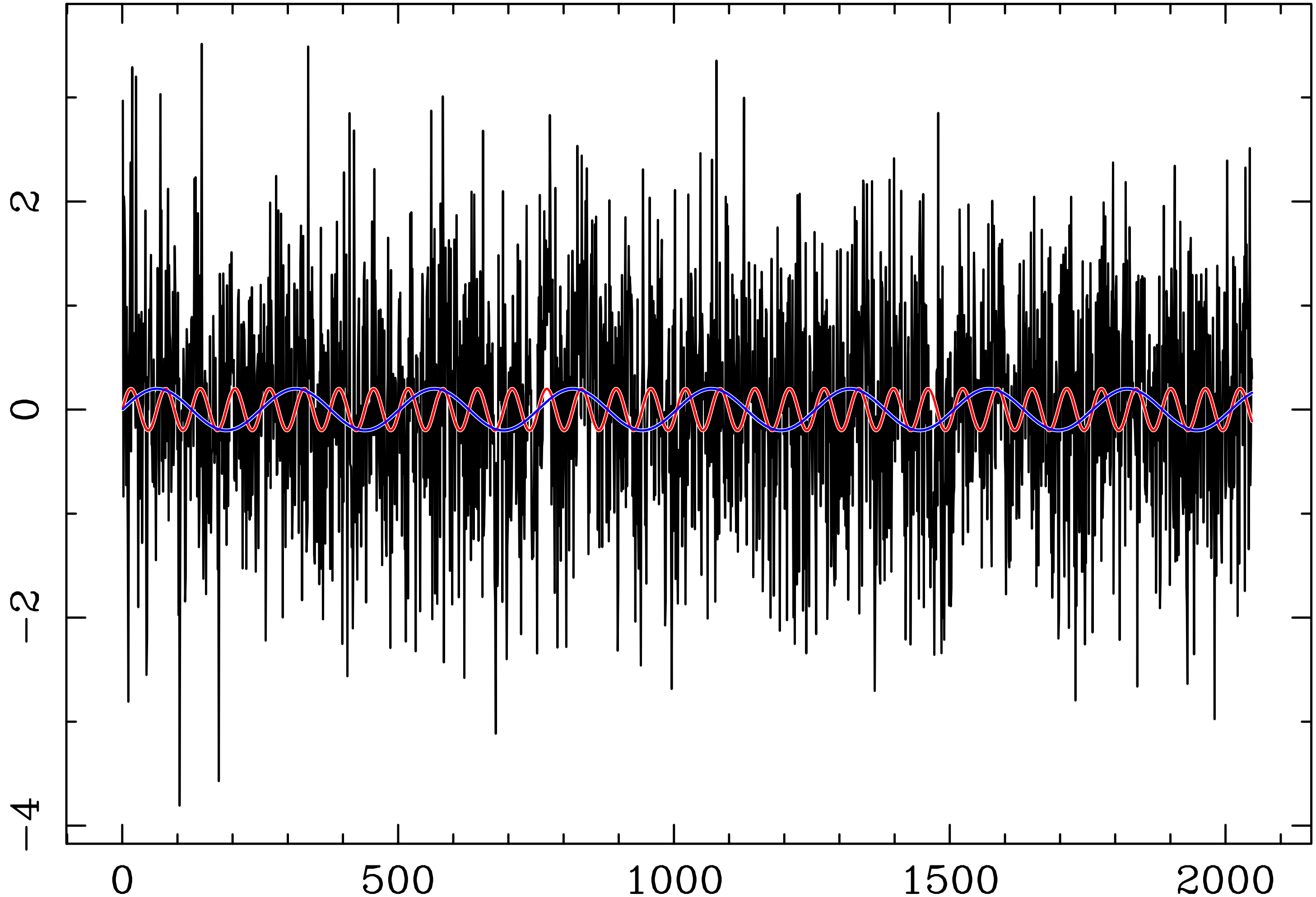
Find the Signal



Frequency Analysis



See the Periodic Signal



Fourier Transformation

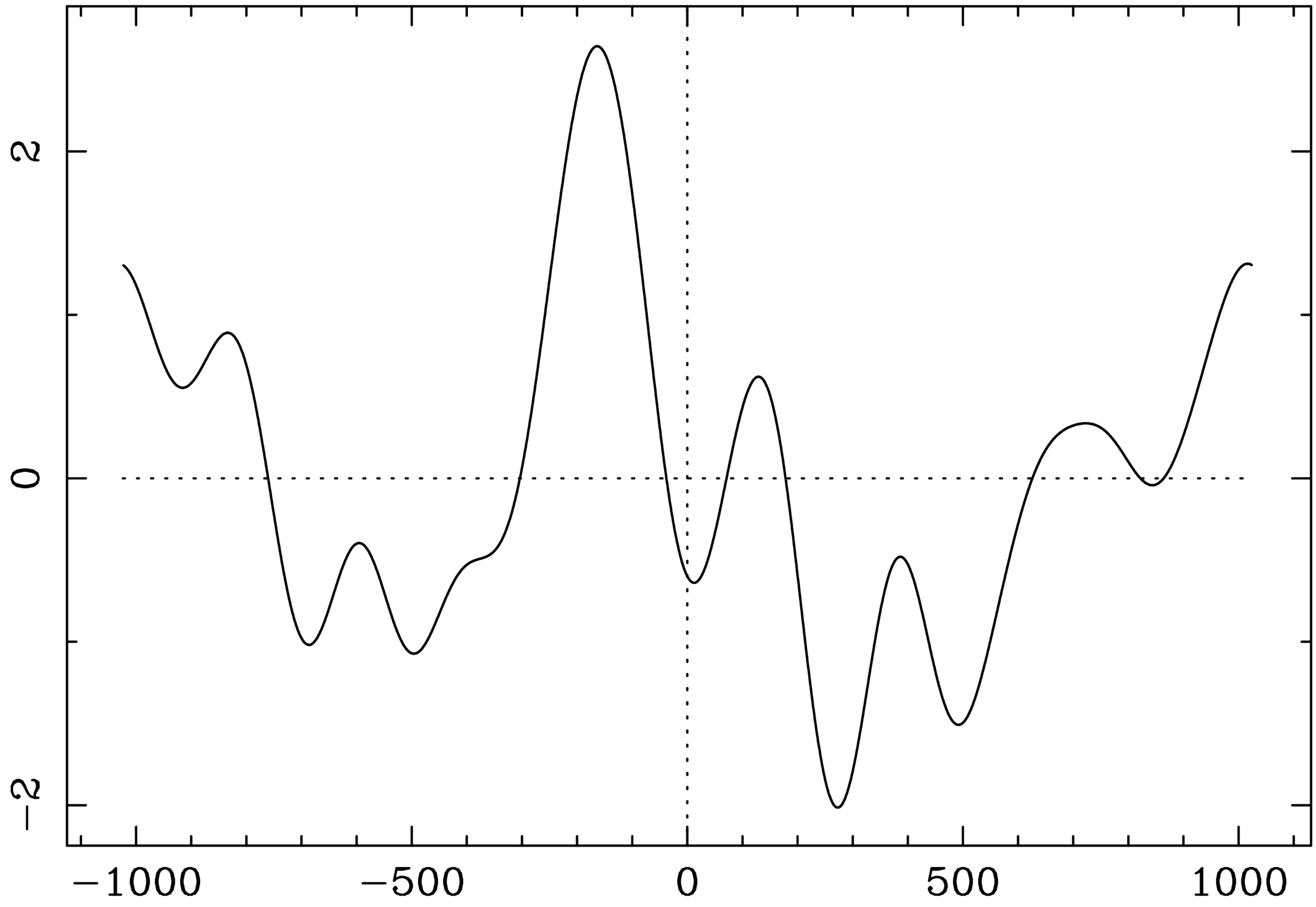
Functions *f(x)* and *F(s)* are Fourier pairs

$$F(s)=\int_{-\infty}^{+\infty} f(x)\cdot e^{-i2\pi xs}dx$$

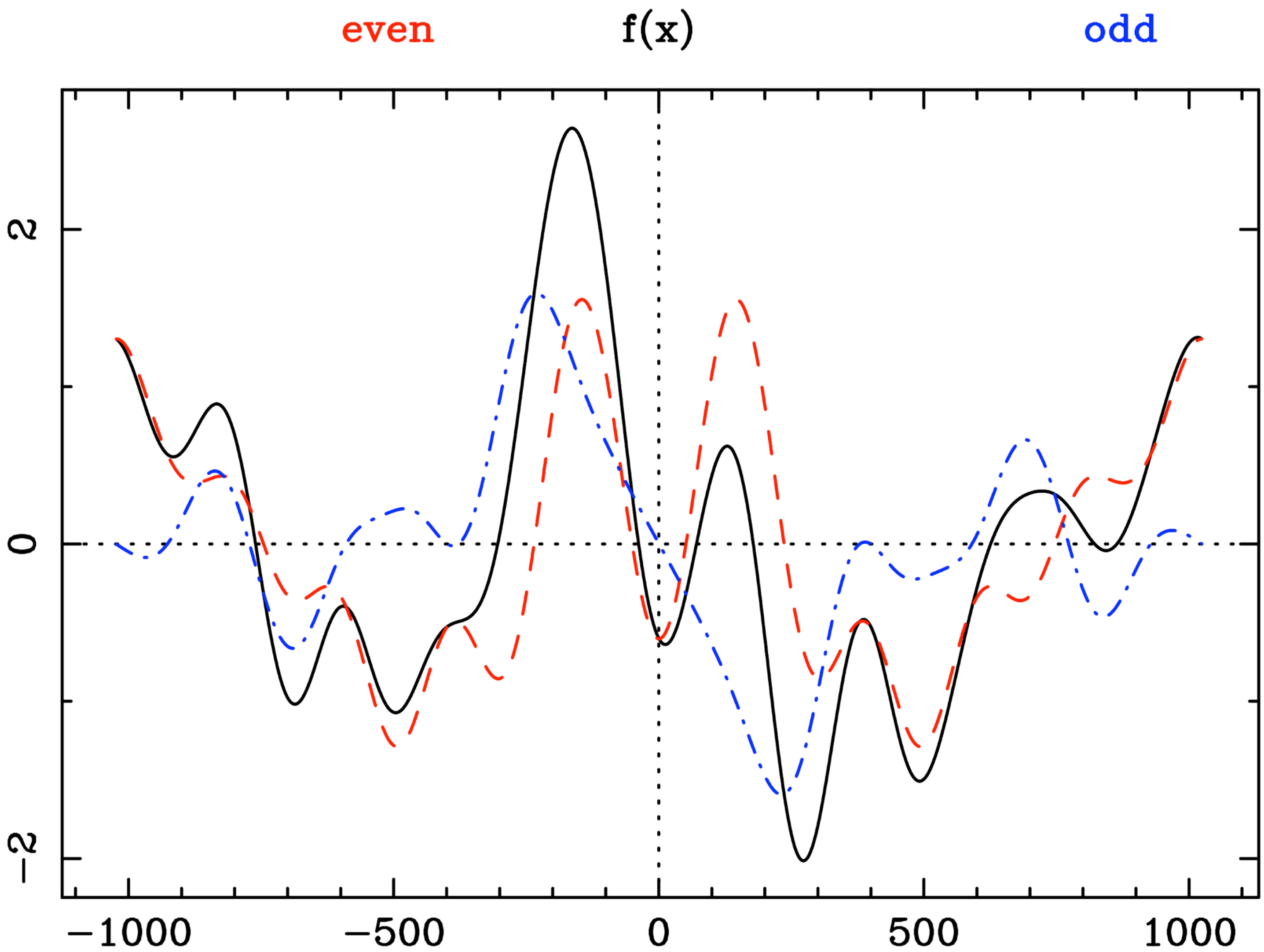
$$f(x)=\int_{-\infty}^{+\infty} F(s)\cdot e^{i2\pi xs}ds$$

- *x*, *s* can be scalar or vector (*xs* becomes scalar product)
- Fourier transform is reciprocal (exponent sign changes)
- exponent sign and normalization are not well defined

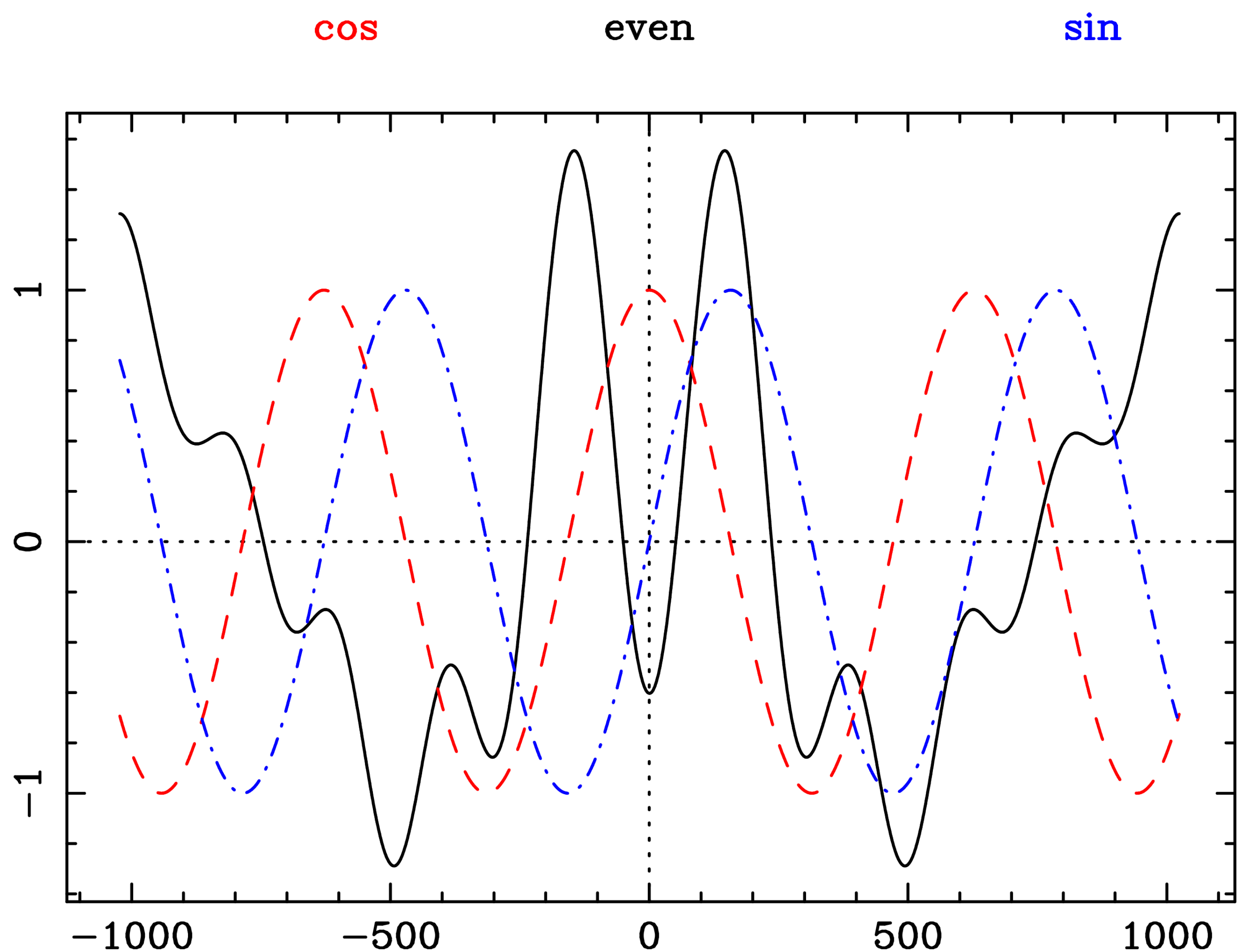
Arbitrary Function



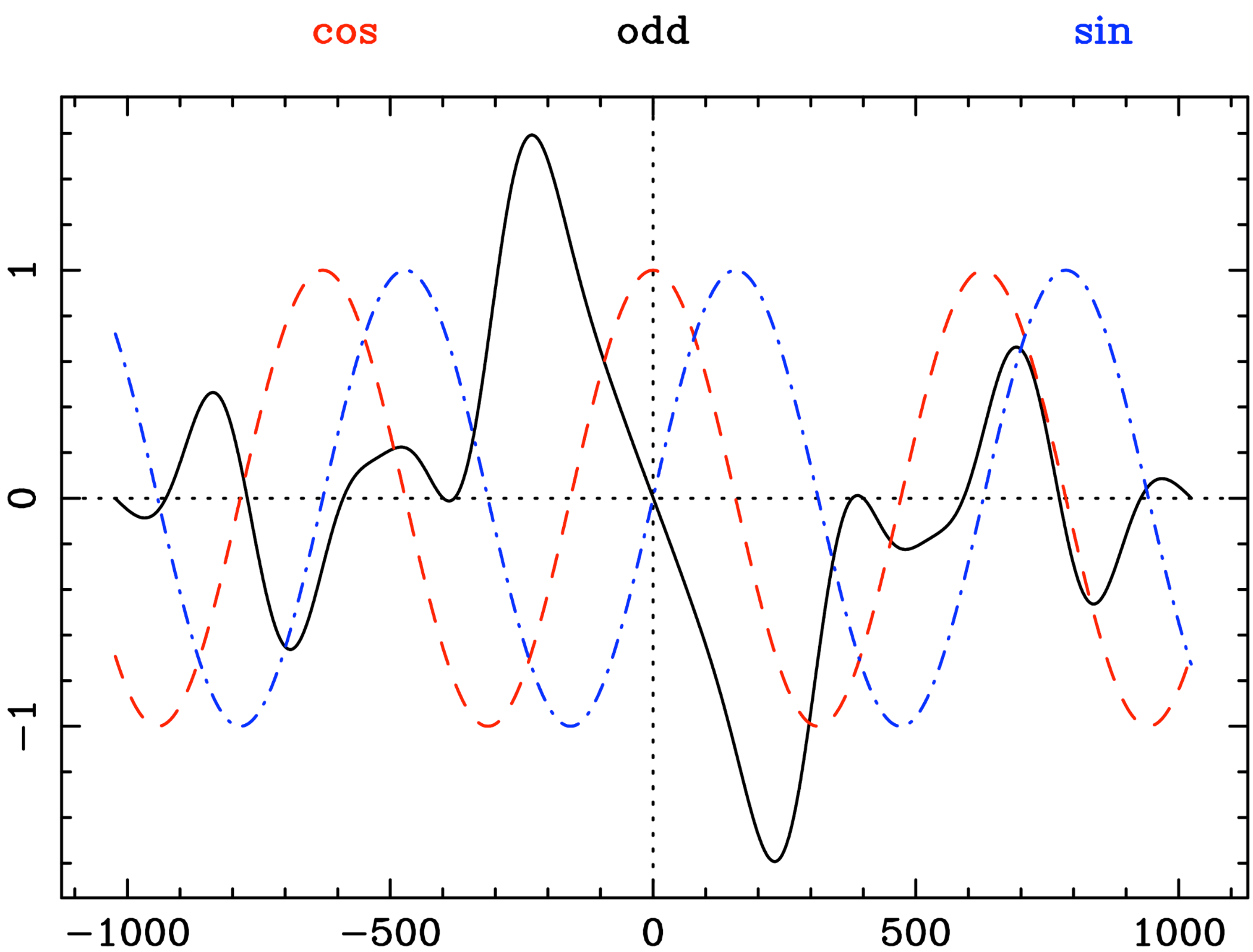
Even & Odd Decomposition



Even Function



Odd Function



Fourier Transform Properties: Symmetry

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x)$$

$$f_{\text{even}}(-x) = f_{\text{even}}(x) \quad f_{\text{odd}}(-x) = -f_{\text{odd}}(x)$$

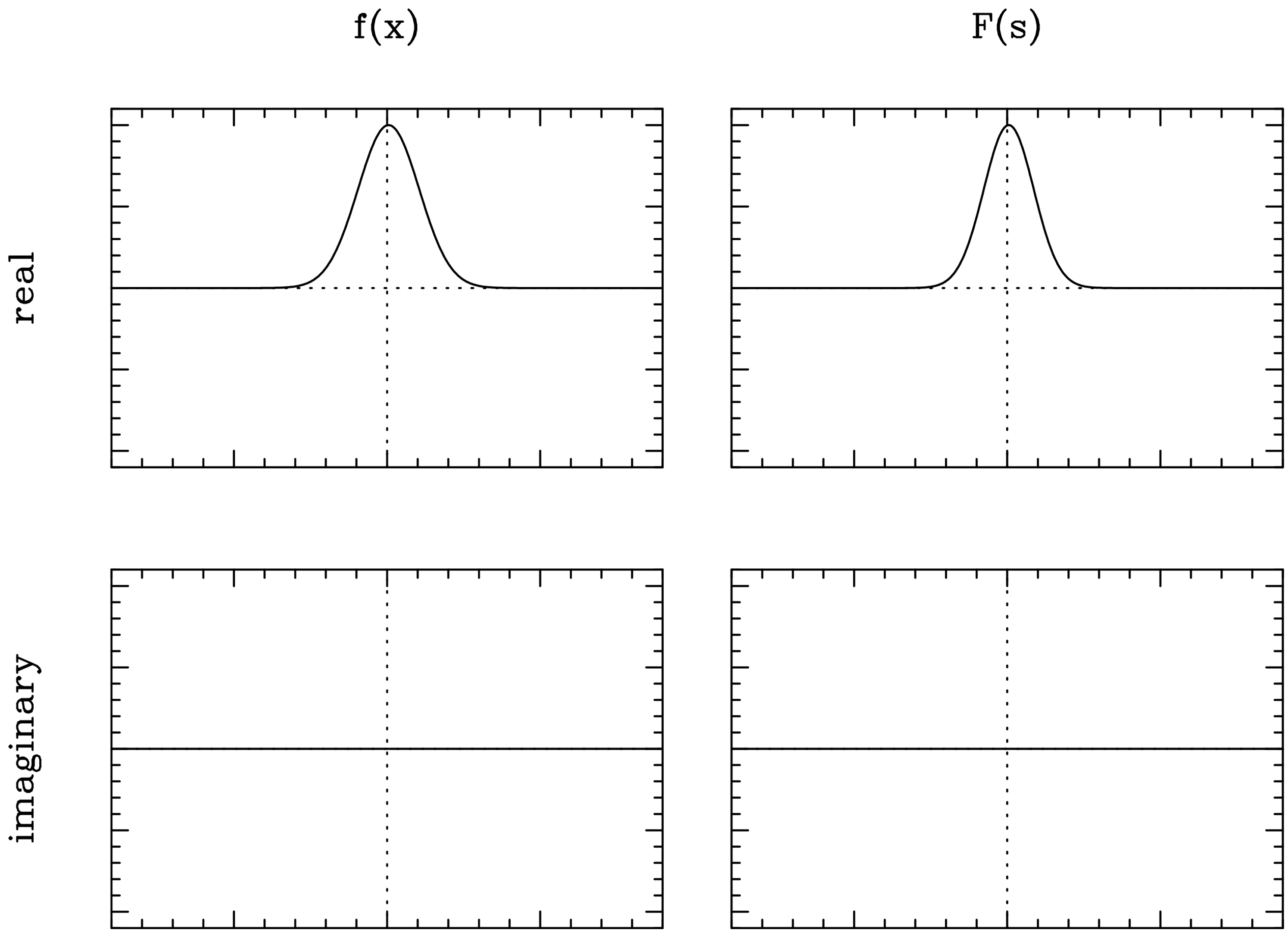
$$e^{-i2\pi xs} = \cos(2\pi xs) - i\sin(2\pi xs)$$

$$\Rightarrow F(s) = 2\int_0^{+\infty} f_{\text{even}}(x)\cos(2\pi xs)dx$$

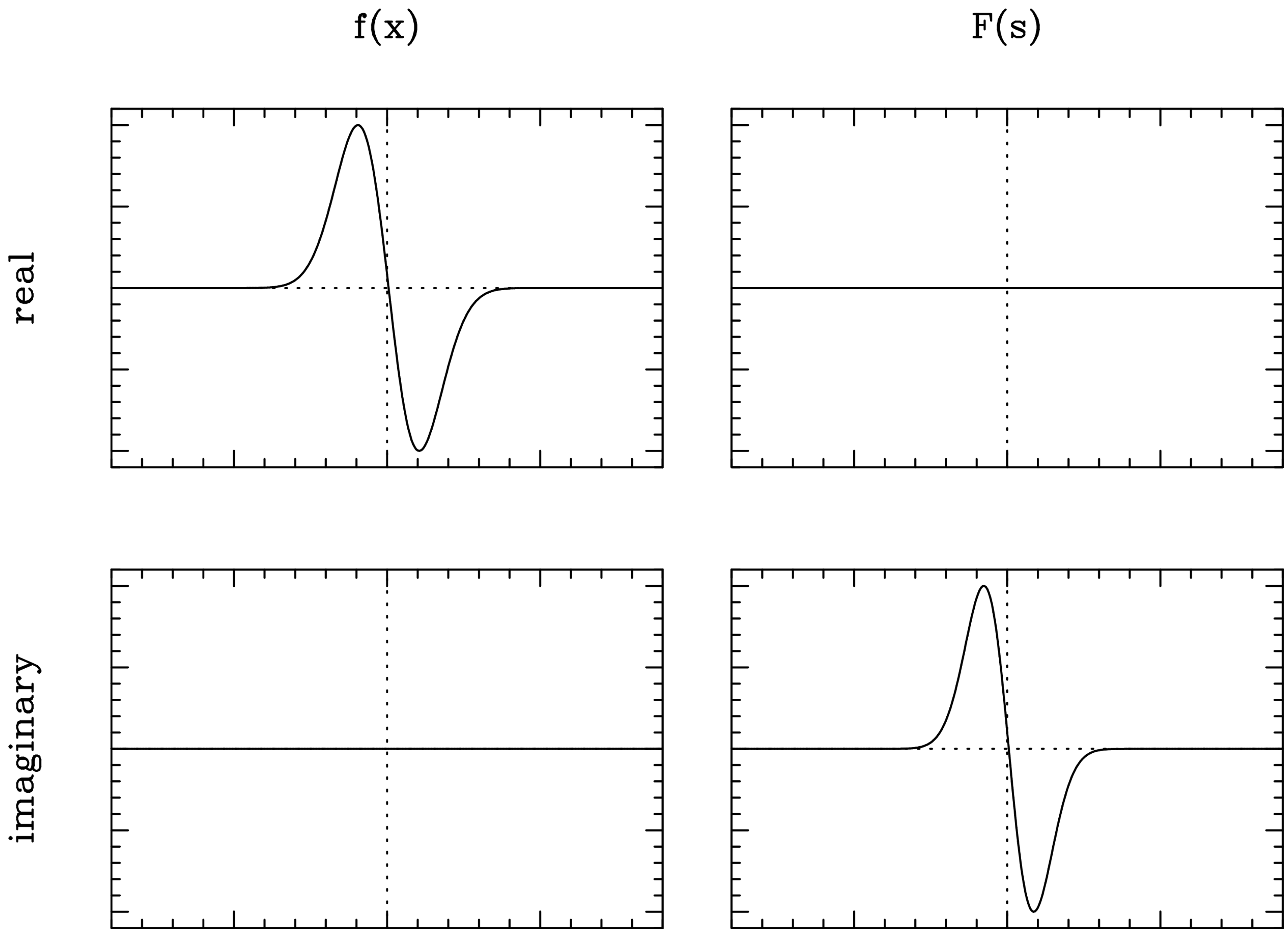
$$-i\ 2\int_0^{+\infty} f_{\text{odd}}(x)\sin(2\pi xs)dx$$

f(x) real: f_{even}(x) transforms to (even) real part of F(s),
f_{odd}(x) transforms to (odd) imaginary part of F(s).

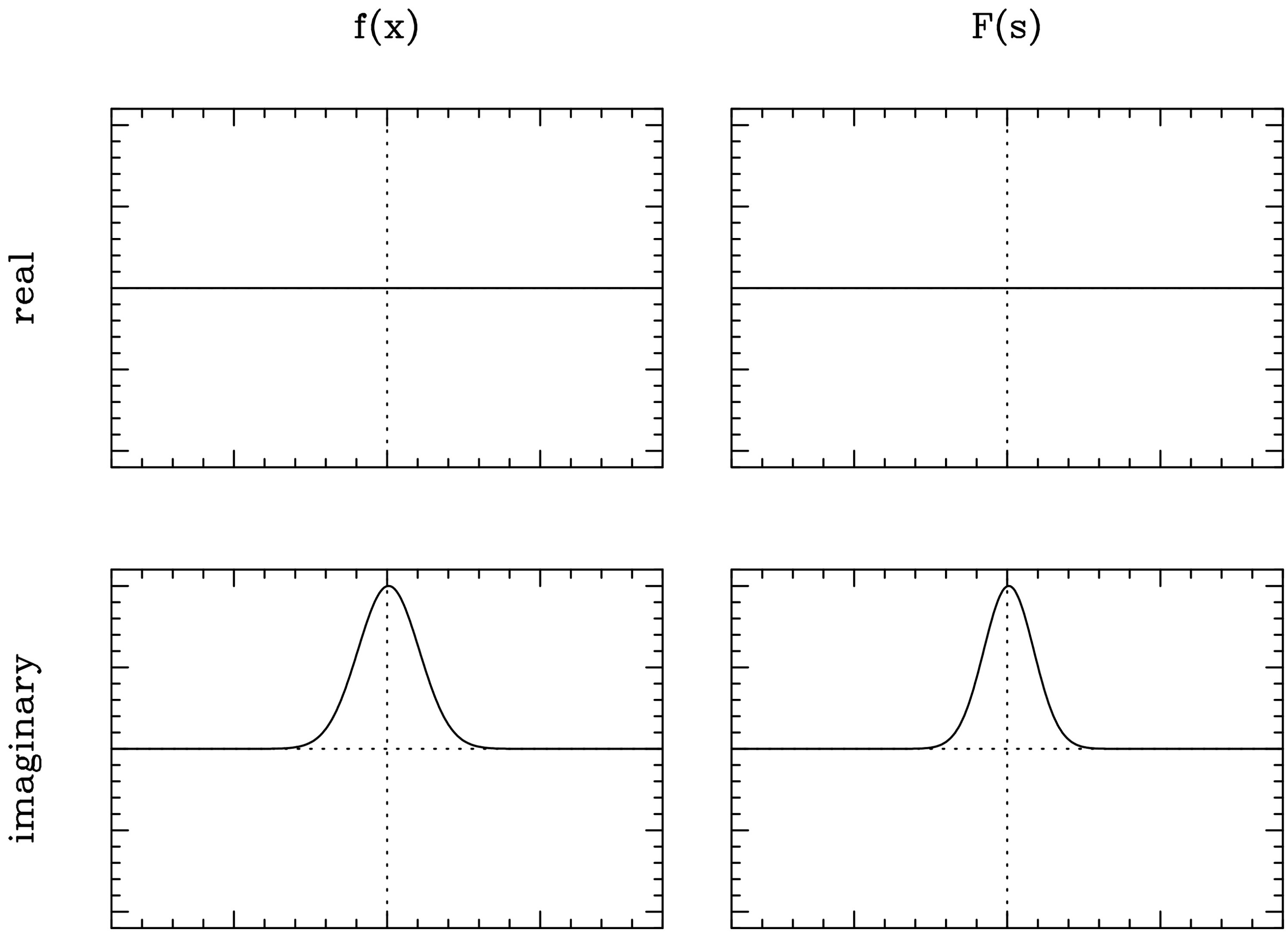
Real, Even



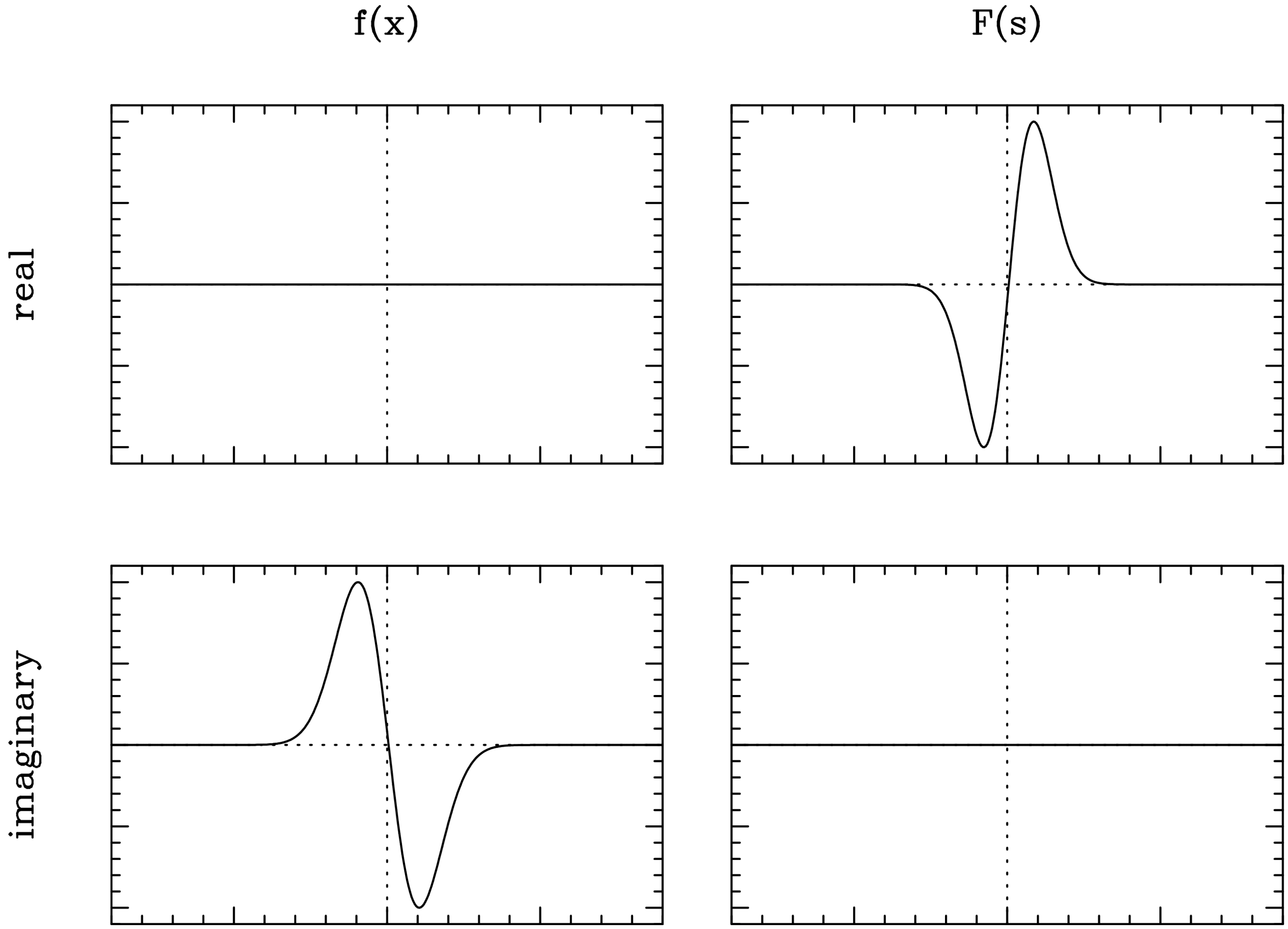
Real, Odd



Imaginary, Even

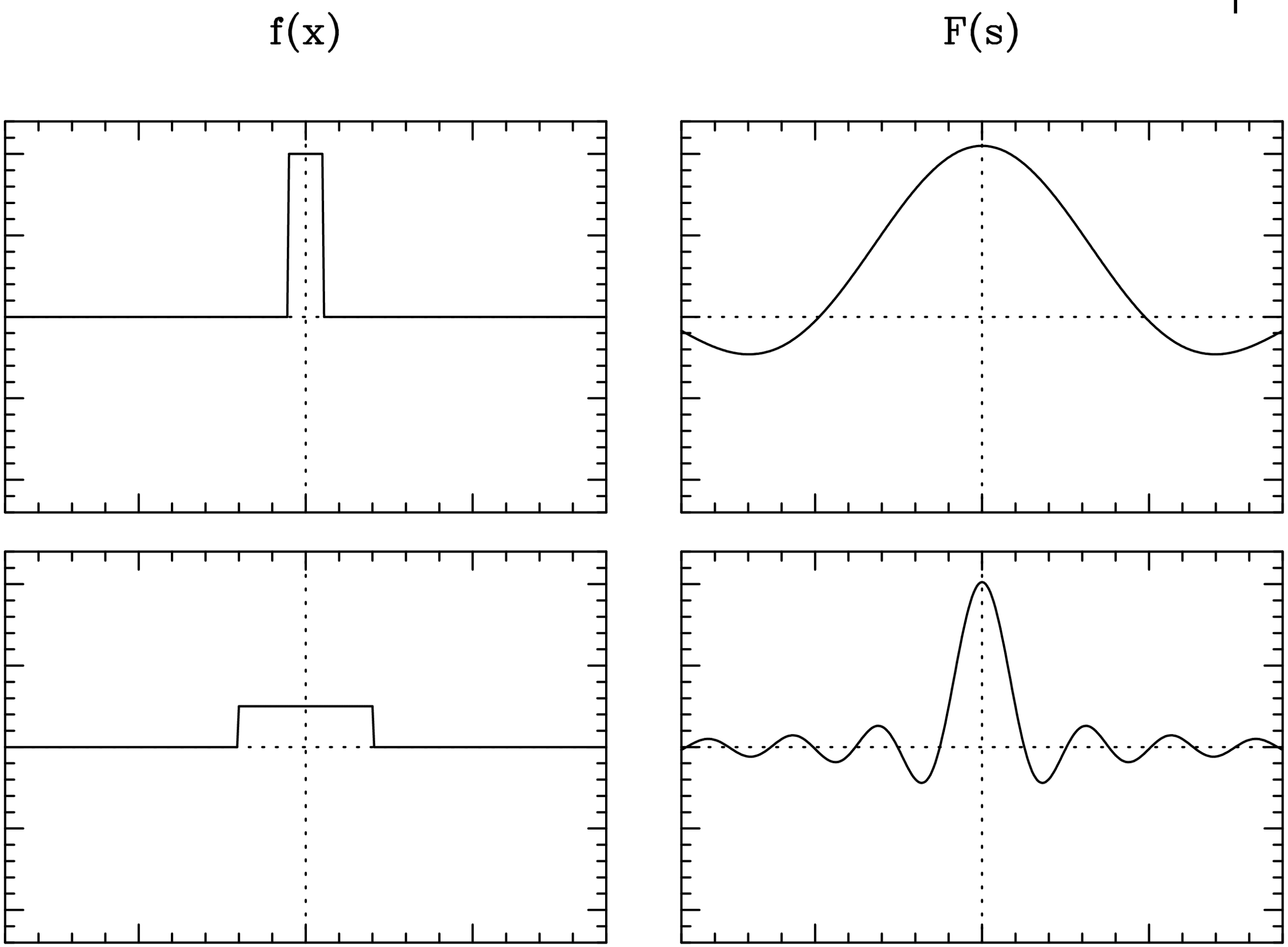


Imaginary, Odd



Fourier Transform Similarity

Expansion of *f(x)* contracts F*(s)*: $$f(x) \rightarrow f(ax) \Leftrightarrow \frac{1}{|a|} F\left(\frac{s}{a}\right)$$



Other Fourier Transform Properties

LINEARITY: $F(as) = a \cdot F(s)$

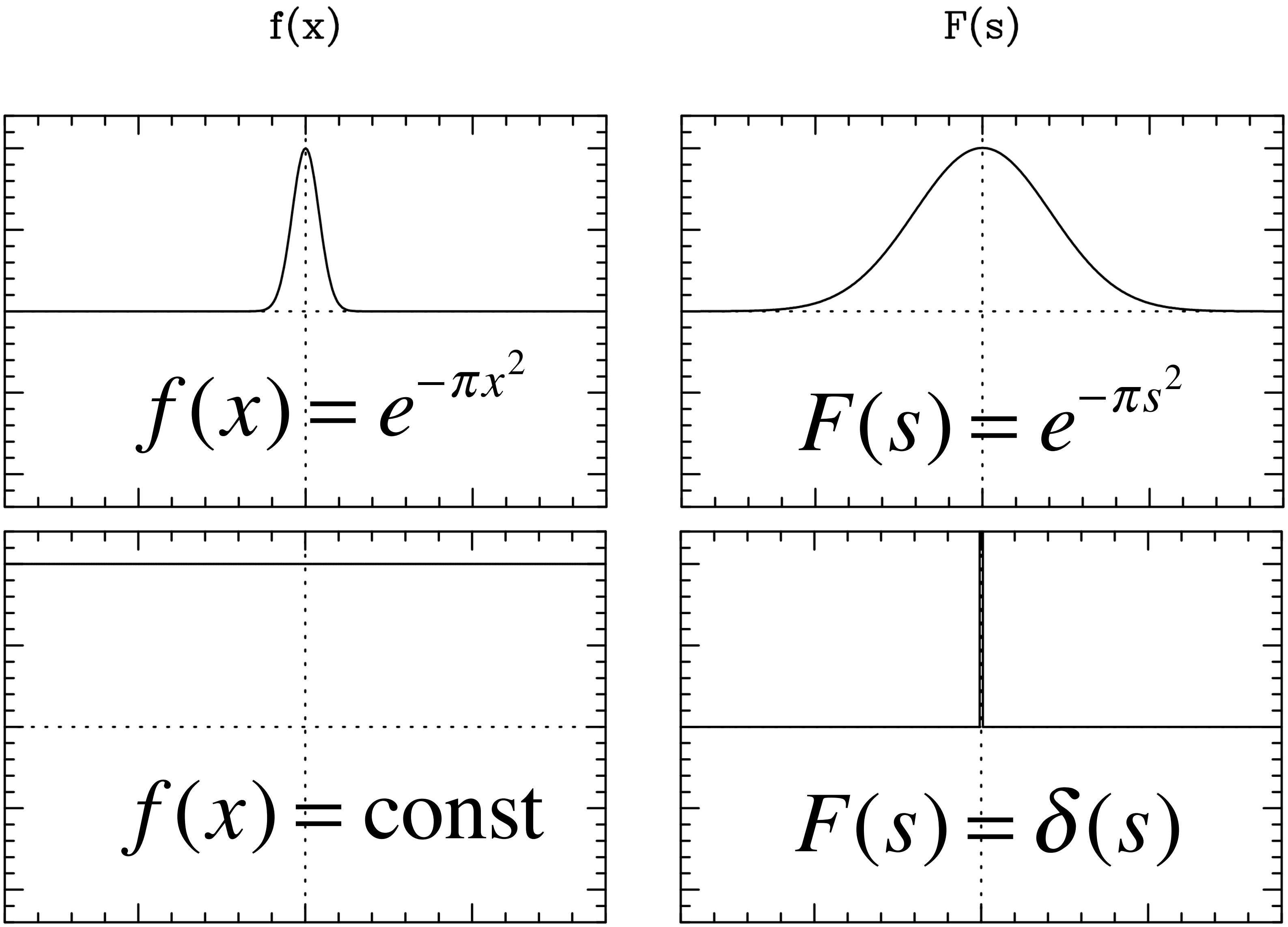
TRANSLATION: $f(x-a) \Leftrightarrow e^{-i2\pi as} F(s)$

DERIVATIVE: $\frac{\partial^n f(x)}{\partial x^n} \Leftrightarrow (i2\pi s)^n F(s)$

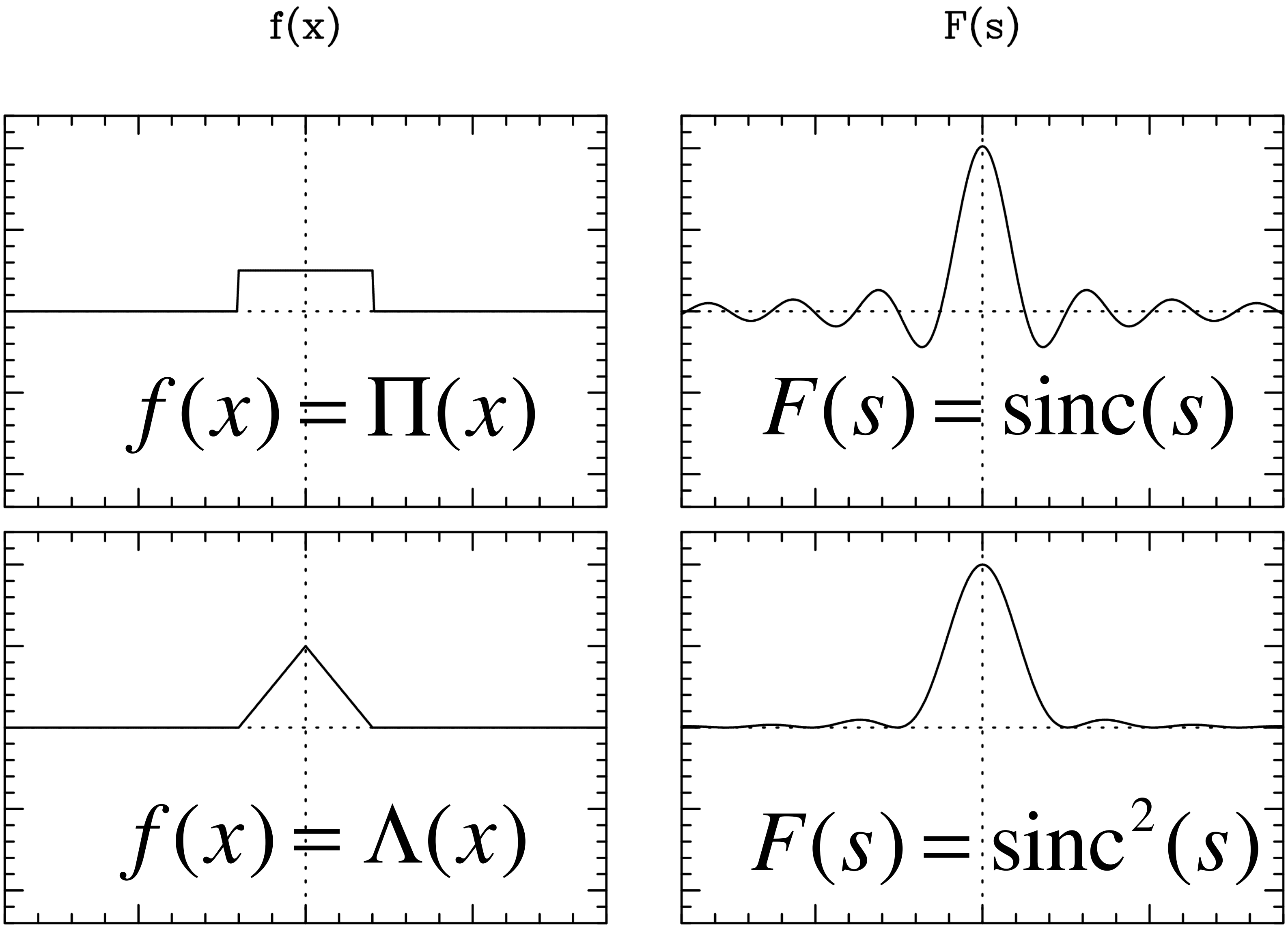
INTEGRAL: $\int f(x) \partial x \Leftrightarrow (i2\pi s)^{-1} F(s) + c\delta(s)$

ADDITION: $f(x) + g(x) \Leftrightarrow F(s) + G(s)$

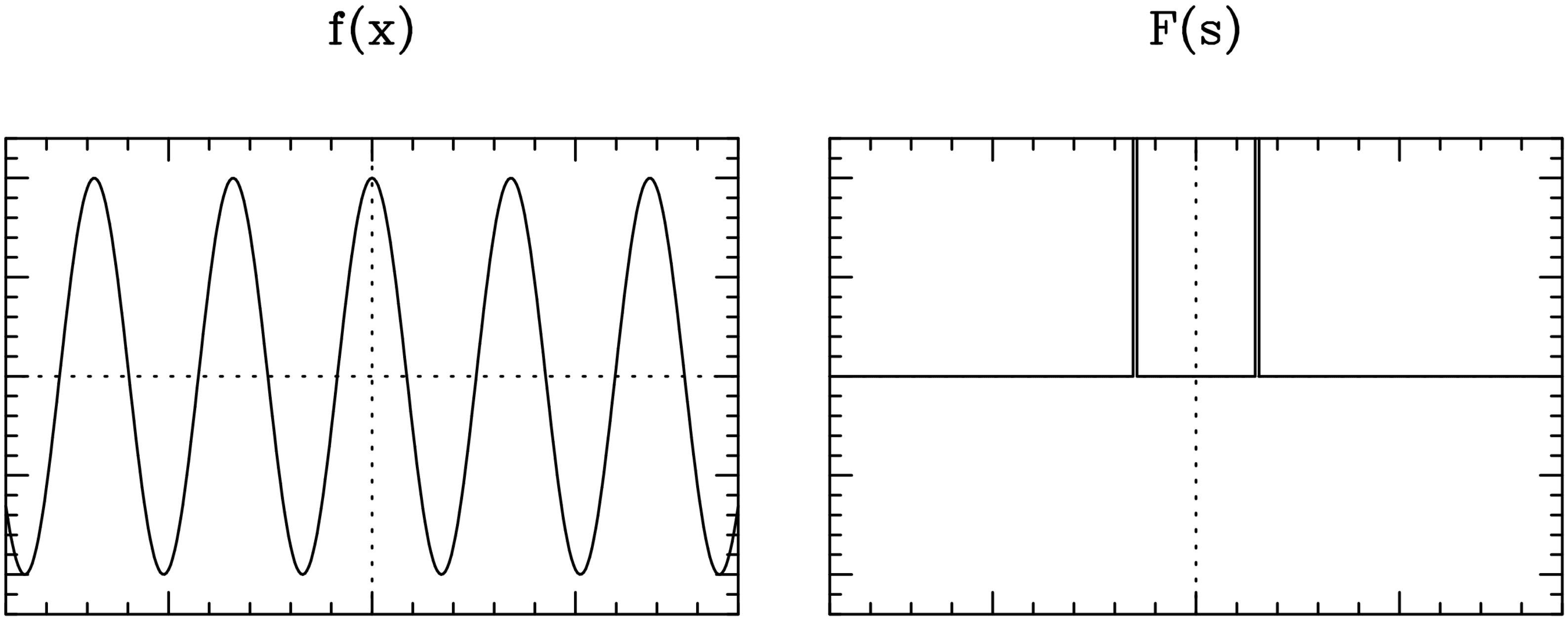
Important 1-D Fourier Pairs 1



Important 1-D Fourier Pairs 2



Important 1-D Fourier Pairs 3



$$f(x) = \cos(\pi x) \qquad F(s) = \delta(s \pm \frac{1}{2})$$

Numerical Fourier Transforms

- Problems with Fourier Transform
  - cannot integrate over $\pm\infty$
  - only know signal at discrete points (samples)
- Assumptions
  - signal is periodic beyond known interval
  - signal is sampled at discrete, evenly spaced points
  - signal is sampled at least twice as often as the highest frequency it contains (Nyquist or critical sampling)

Fourier Series of Periodic Functions

Decomposition using sines and cosines as orthonormal basis set

Periodic function: $f(x) = f(x+P)$

Fourier series: $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nx}{P}\right) + b_n \sin\left(\frac{2\pi nx}{P}\right)\right]$$

Fourier coefficients:
$$a_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\cos\left(\frac{2\pi nx}{P}\right)dx$$
$$b_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\sin\left(\frac{2\pi nx}{P}\right)dx$$

Period: P

Frequency: ν = 1/P

Angular frequency: ω = 2π/P

Orthonormal Basis Set

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx = \begin{cases} 1 \text{ for } n = m \\ 0 \text{ for } n \neq m \end{cases}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sin(mx)\cos(nx)\,dx = 0$$

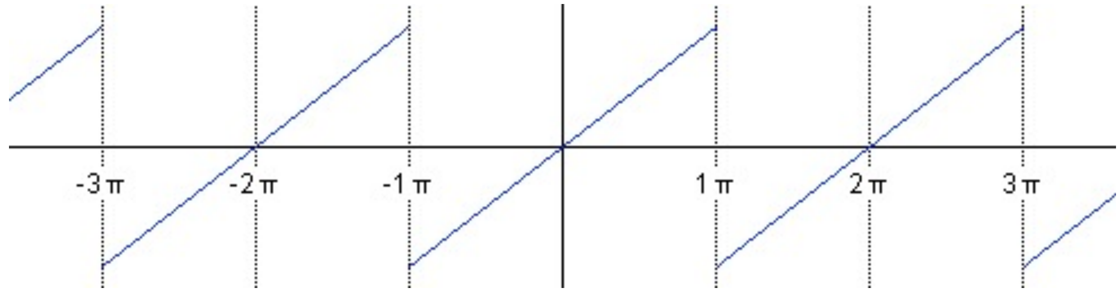
$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sin(mx)\sin(nx)\,dx = \begin{cases} 1 \text{ for } n = m \\ 0 \text{ for } n \neq m \end{cases}$$

Example: Sawtooth Function

Sawtooth function:

$$f(x) = x \qquad \text{for} \quad -\pi < x < \pi$$

$$f(x + 2\pi) = f(x)$$



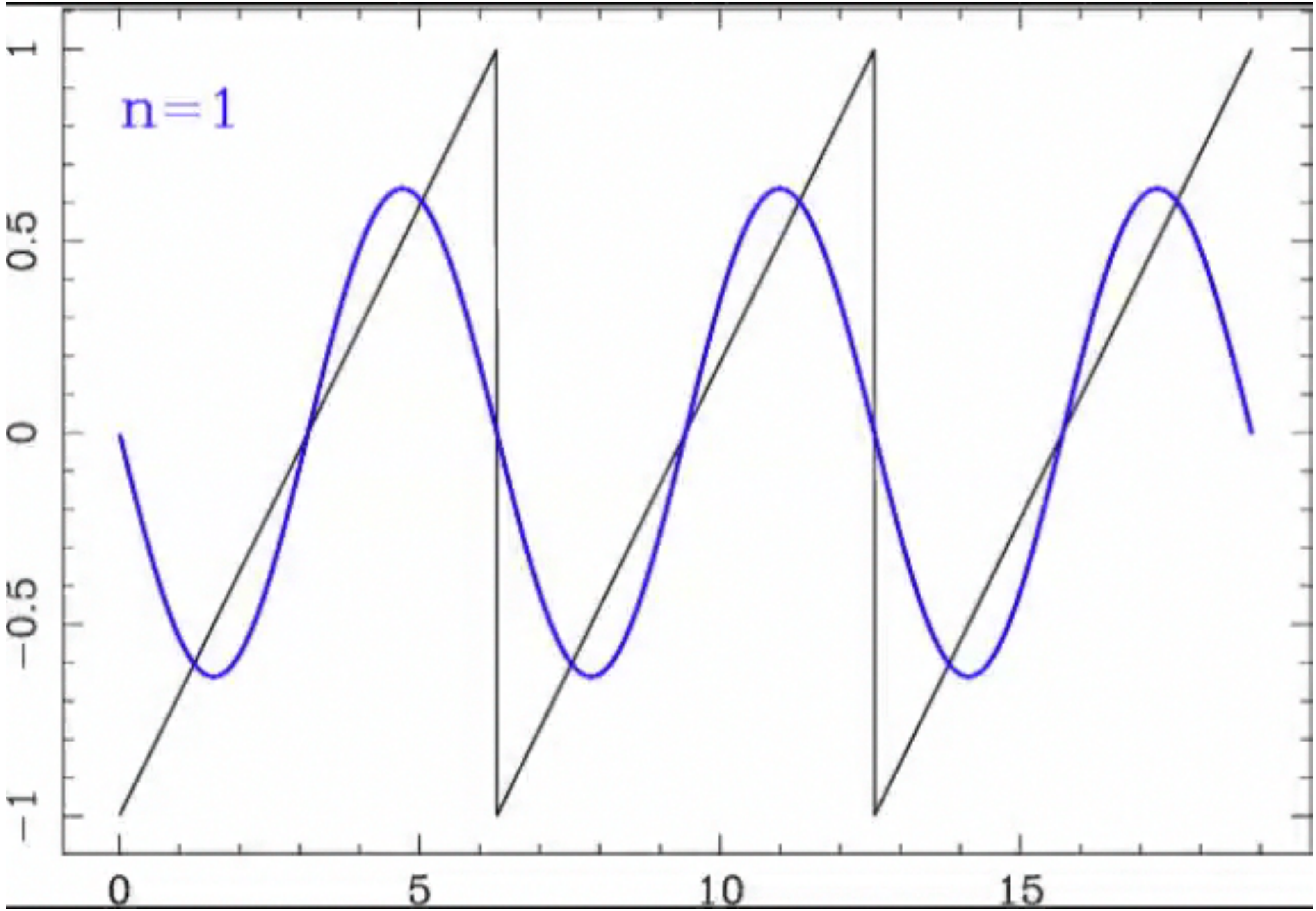
Fourier coefficients are:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx \overset{!}{=} 0 \qquad \text{(cos() is symmetric around 0)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = 2\frac{(-1)^{n+1}}{n}$$

and hence: $f(x) = \cancel{\frac{a_0}{2}} + \sum_{n=1}^{\infty}\left[\cancel{a_n\cos(nx)} + b_n\sin(nx)\right] = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$

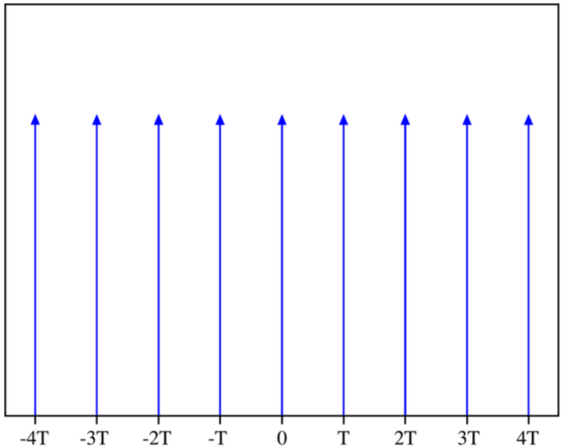
Sawtooth Approximation $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$



Dirac Comb

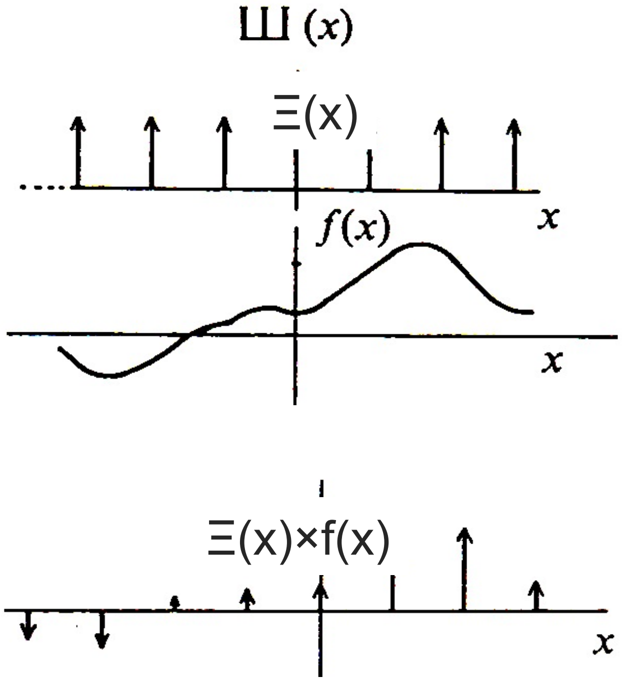
Dirac's delta "function":

$$f(x) = \delta(x) = \int_{-\infty}^{+\infty} e^{i2\pi sx}\, ds \;\rightarrow\; FT\{\delta(x)\} = 1$$



Dirac comb: infinite series of delta functions spaced at intervals of T:

$$\Xi_T(x) = \sum_{k=-\infty}^{\infty} \delta(x - kT) \underset{series}{\overset{Fourier}{=}} \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i2\pi nx/T}$$



- Fourier transform of Dirac comb is also a Dirac comb
- Dirac comb is also called impulse train or sampling function

Nyquist-Shannon Theorem

Sampling: signal at discrete values of x: $f(x) \rightarrow f(x) \cdot Ш\left(\frac{x}{\Delta x}\right)$

Interval between two successive readings is sampling rate

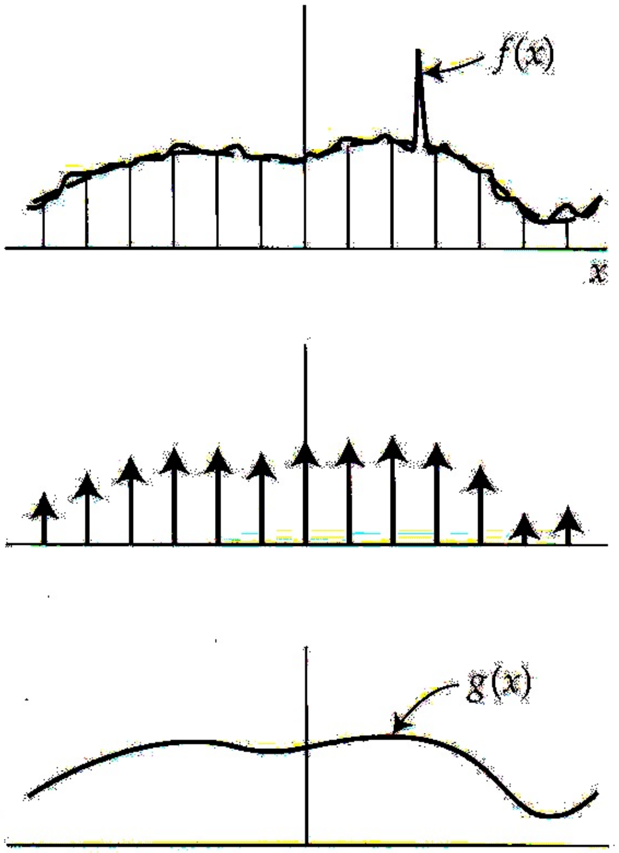
Critical sampling given by Nyquist-Shannon theorem

Given f(x), its Fourier Transform F(s) with bounded support $[-s_{max}, s_{max}]$.

Sampled distribution of the form

$$g(x) = f(x) \cdot Ш\left(\frac{x}{\Delta x}\right)$$

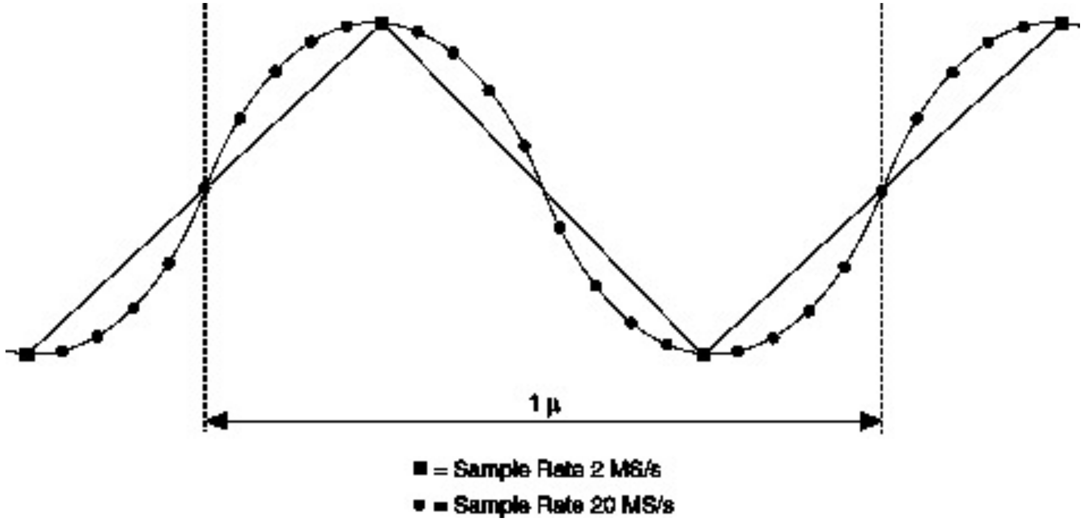
with a sampling rate of $\Delta x = 1/(2s_{max})$ is enough to reconstruct *f(x)* for all *x*.



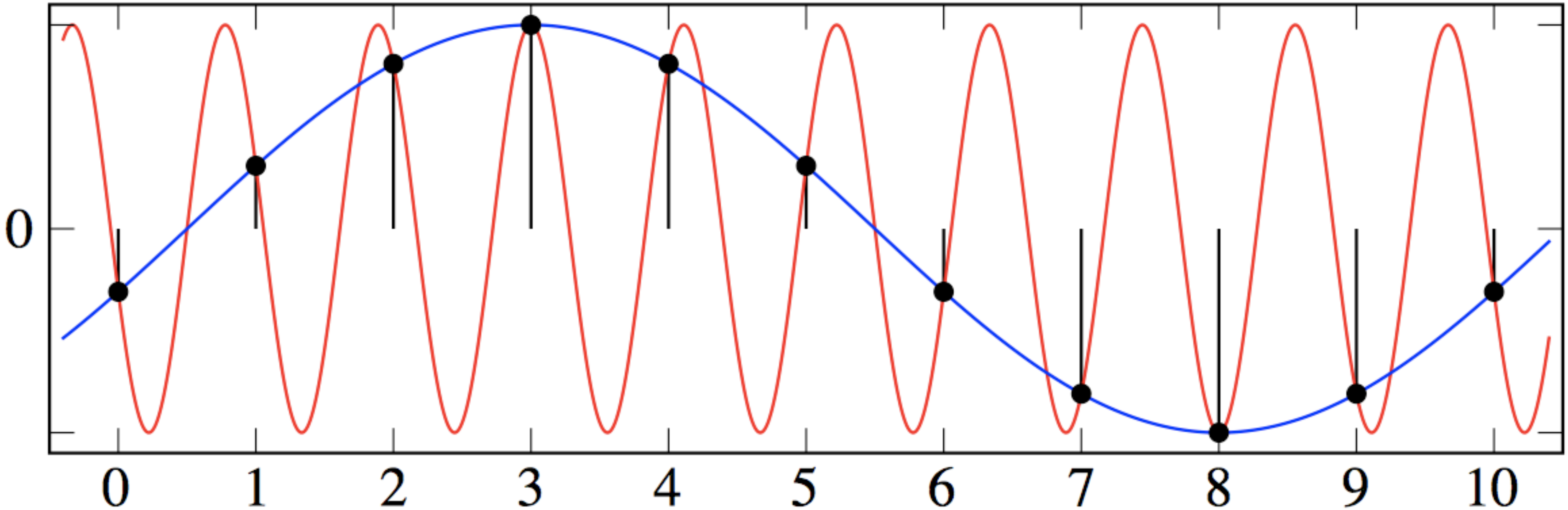
Sampling

Oversampling Sampling rate above critical sampling rate:
- redundant measurements
- often lowering the S/N

Undersampling Sampling rate below critical sampling rate:
- signal contains frequencies higher than $1/(2s_{max})$
- source signal cannot be determined after sampling
- loss of fine details
- must apply low-pass filter before sampling



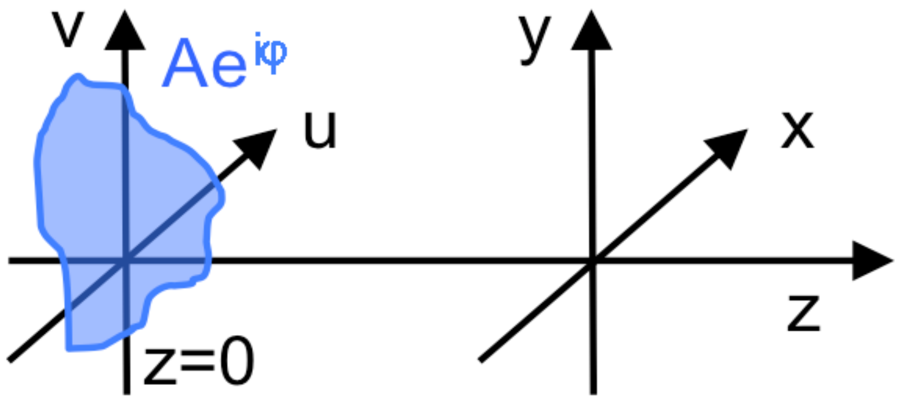
Aliasing



- unresolved, high frequencies look like resolved low frequencies
- create spurious components below Nyquist frequency
- may create major problems and uncertainties in determination of original signal

Point Spread Function

- Fraunhofer Diffraction: electric field in image plane is Fourier transf. of electric field in aperture



$$E(x,y,z)=\iint A(u,v)\,e^{i\varphi(u,v)}\,e^{-i\frac{2\pi}{\lambda z}(xu+yv)}\,du\,dv$$

- Point Spread Function (PSF)
  - image of a point source produced by optical system
  - PSF = $E(x,y,z)^2$

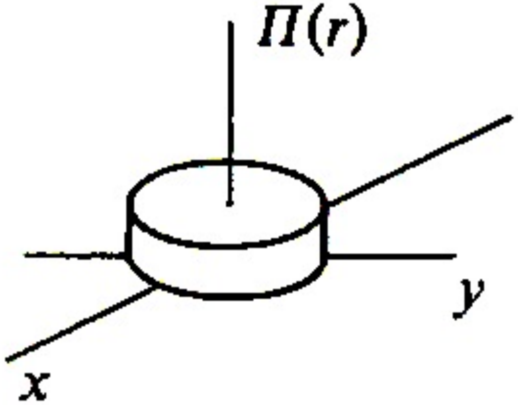
Fourier Pair in 2-D: Box Function

2-D box function with $r^2 = x^2 + y^2$:

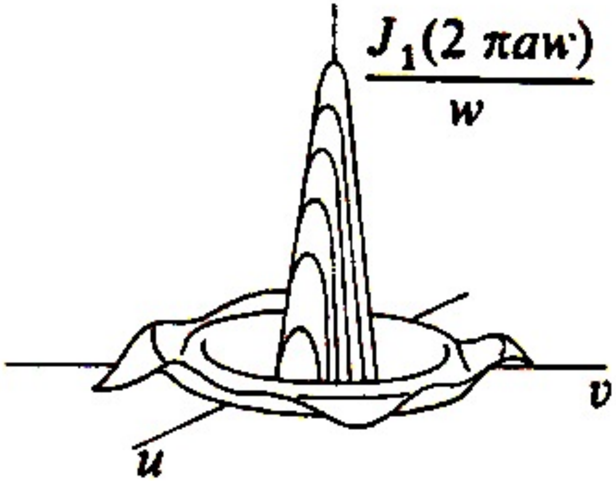
$$\Pi\left(\frac{r}{2}\right) = \begin{cases} 1 & \text{for } r < 1 \\ 0 & \text{for } r \geq 1 \end{cases}$$

Fourier Transform: $\Pi\left(\frac{r}{2}\right) \Leftrightarrow \frac{J_1(2\pi\omega)}{\omega}$ (1st order Bessel function J_1)

Electric Field in Telescope Aperture:



Electric Field in Focal plane:



Larger telescopes produce smaller Point Spread Functions (PSFs)!

Bessel Functions

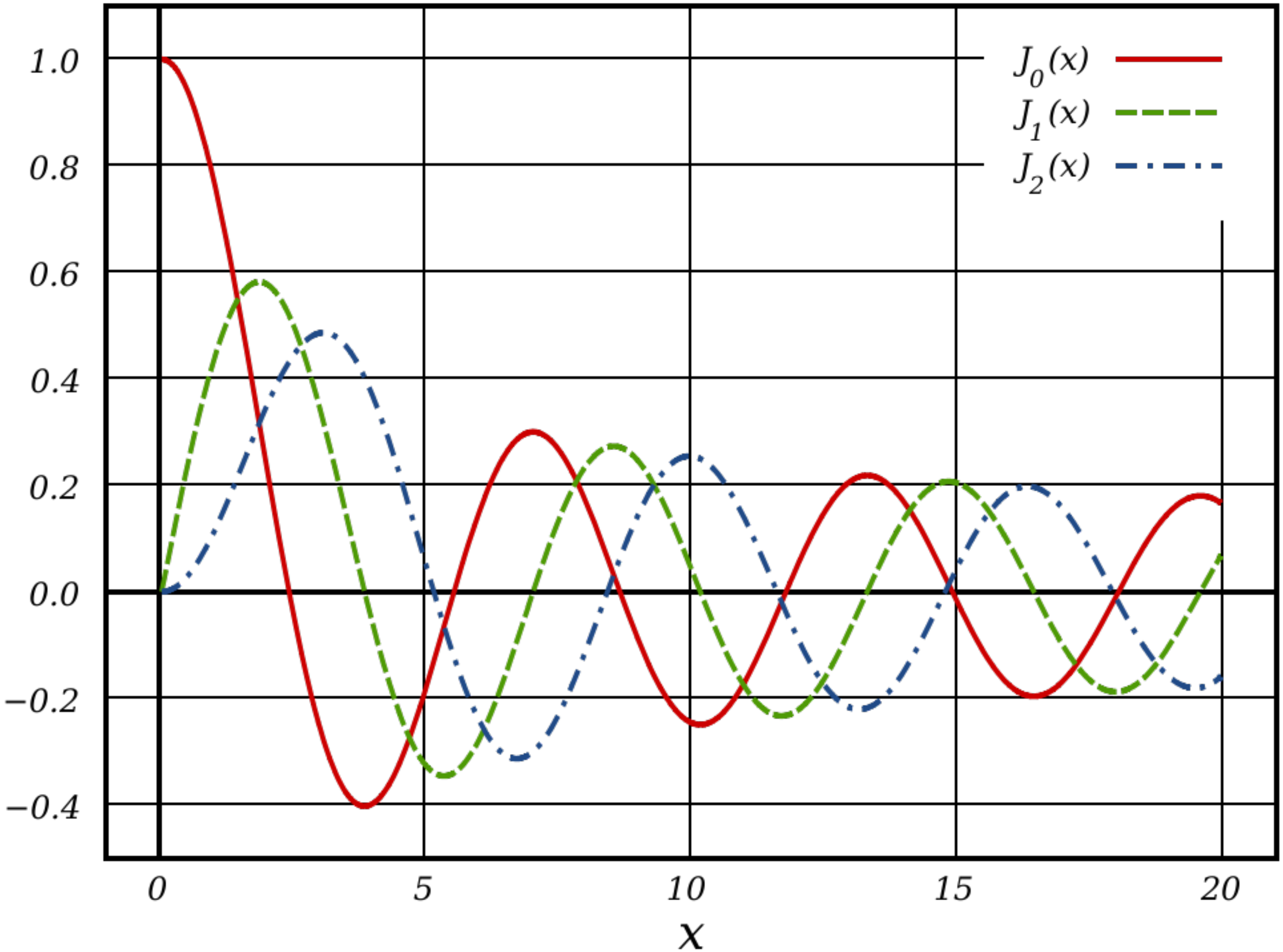
Bessel functions are canonical solutions *y(x)* of Bessel's differential equation:

$$x^2 \frac{\partial^2 y}{\partial x^2} + x \frac{\partial y}{\partial x} + \left(x^2 - n^2\right) y = 0$$

for an arbitrary real or complex number *n*, the order of the Bessel function.

Solutions = Bessel Functions: $J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+n}}{k!(k+n)!}$

Bessel Functions J_0, J_1, J_2

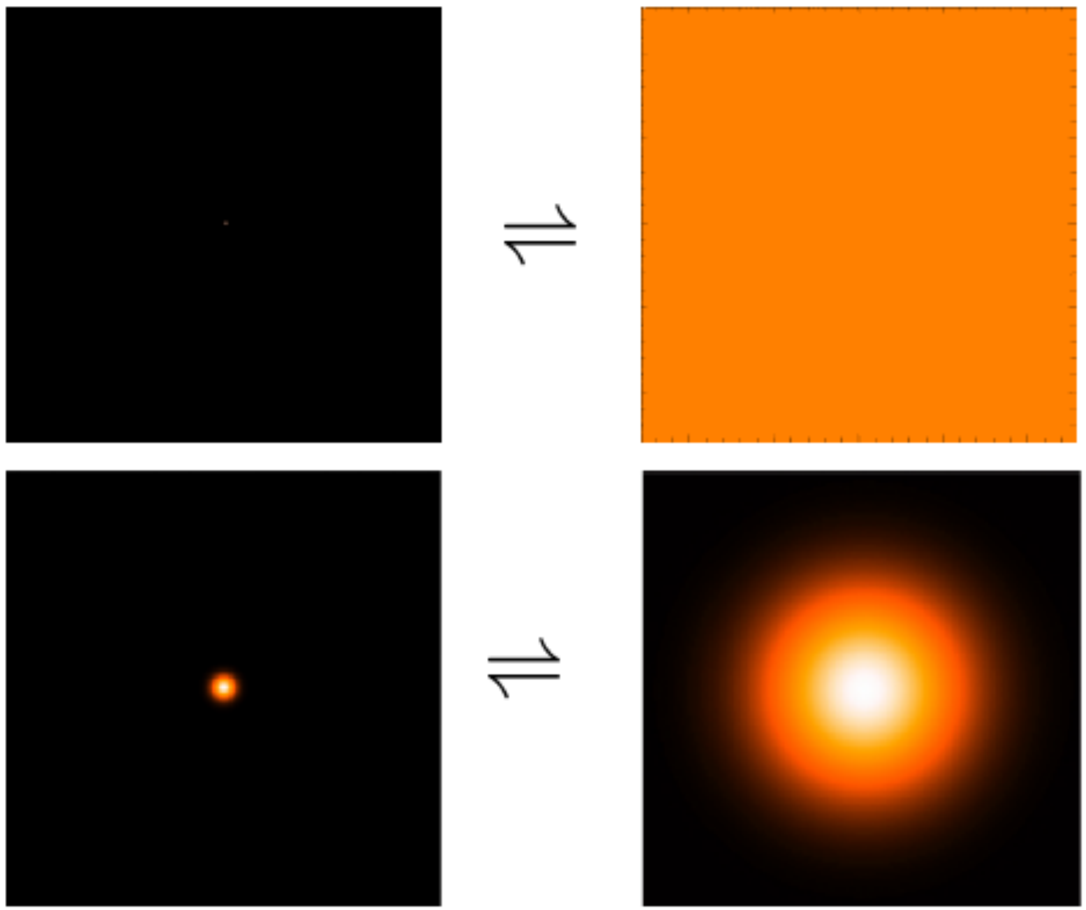


Telescope Aperture ⇔ Focal Plane 1

Amp{V(u,v)}

δ Function ⇌ Constant

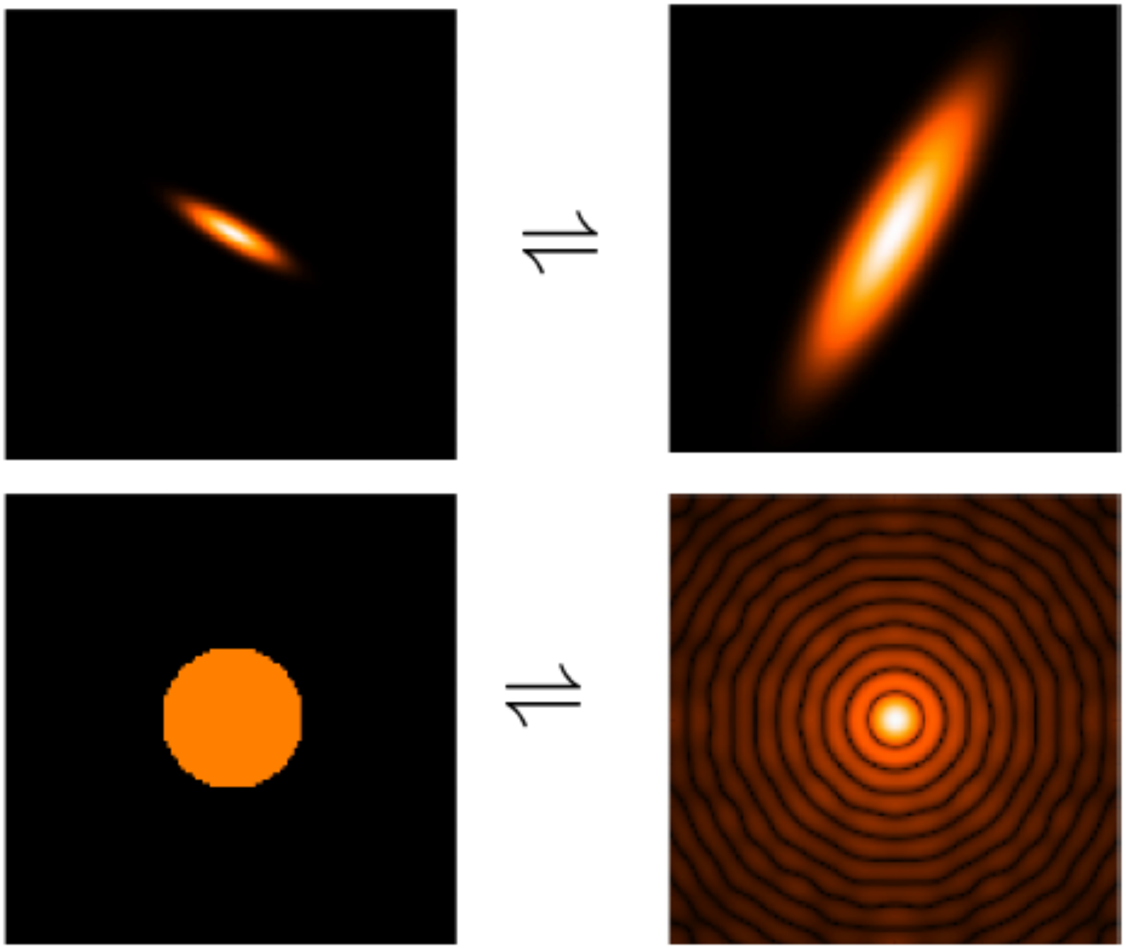
Gaussian ⇌ Gaussian



Telescope Aperture ⇔ Focal Plane 2

Amp{V(u,v)}

elliptical Gaussian

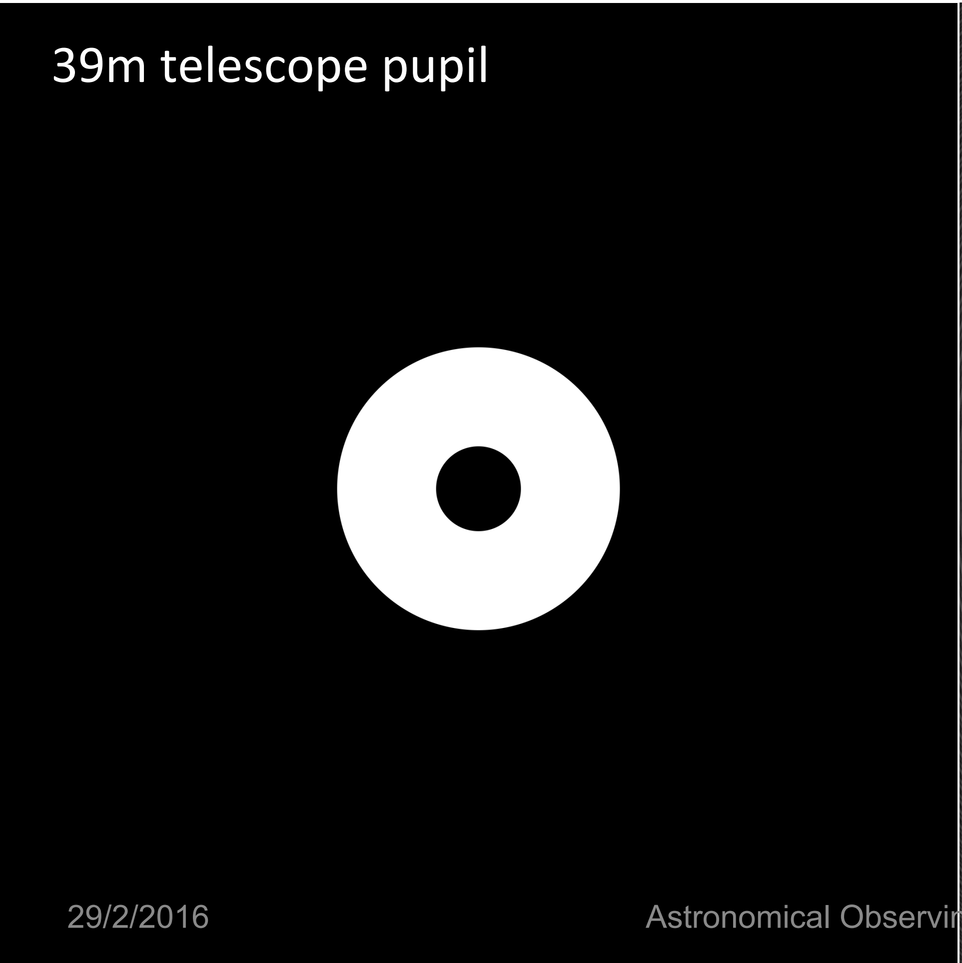


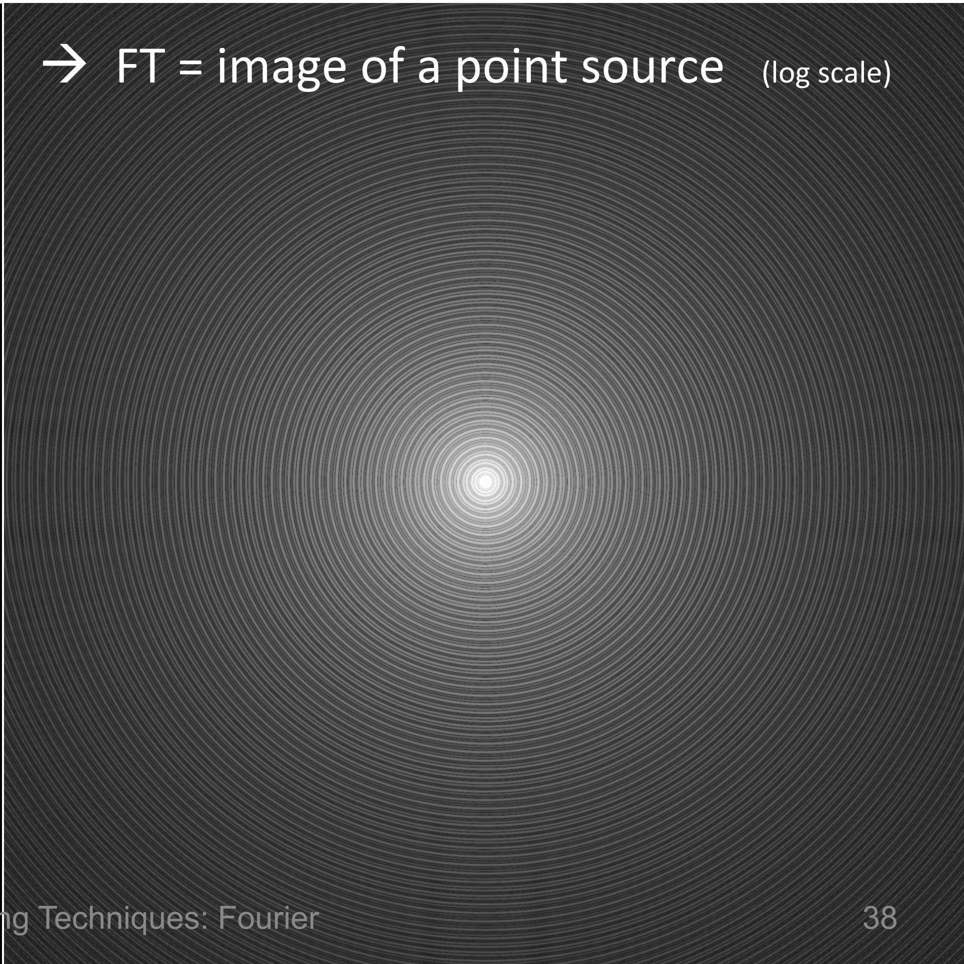
elliptical Gaussian

Disk

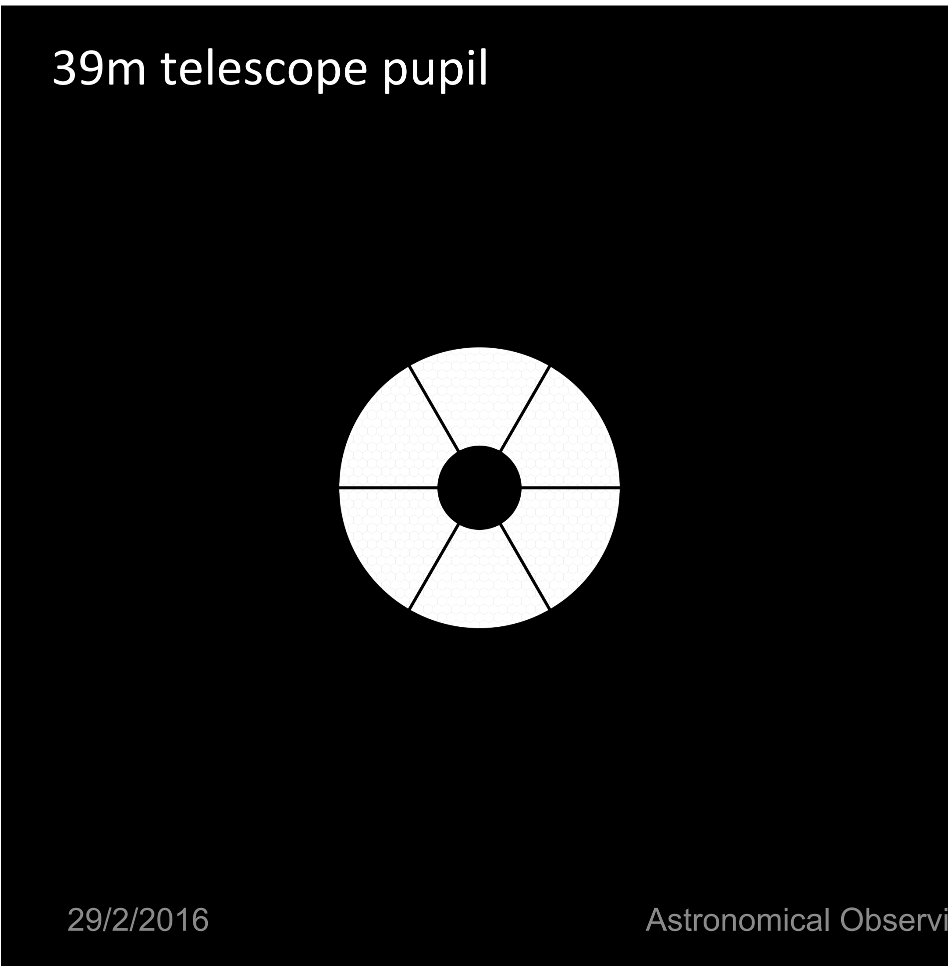
Bessel

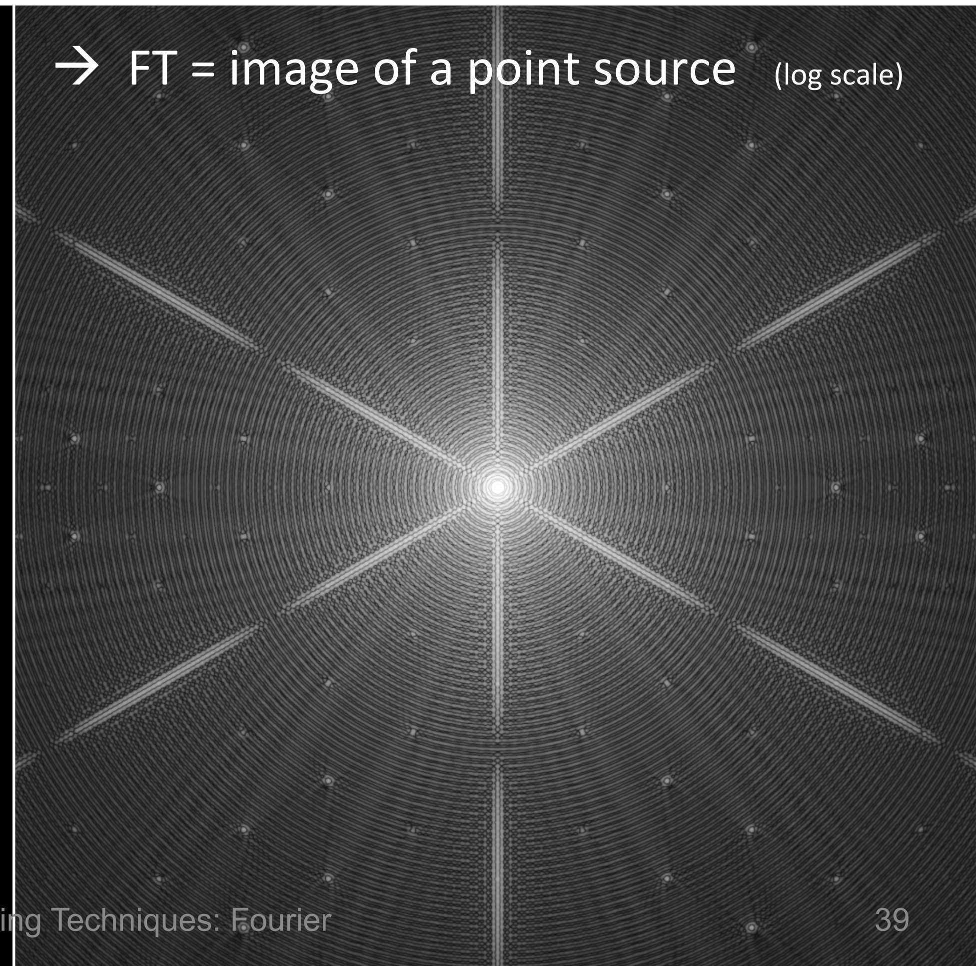
PSF Example
central obscuration, monolithic mirror (pupil) no support-spiders



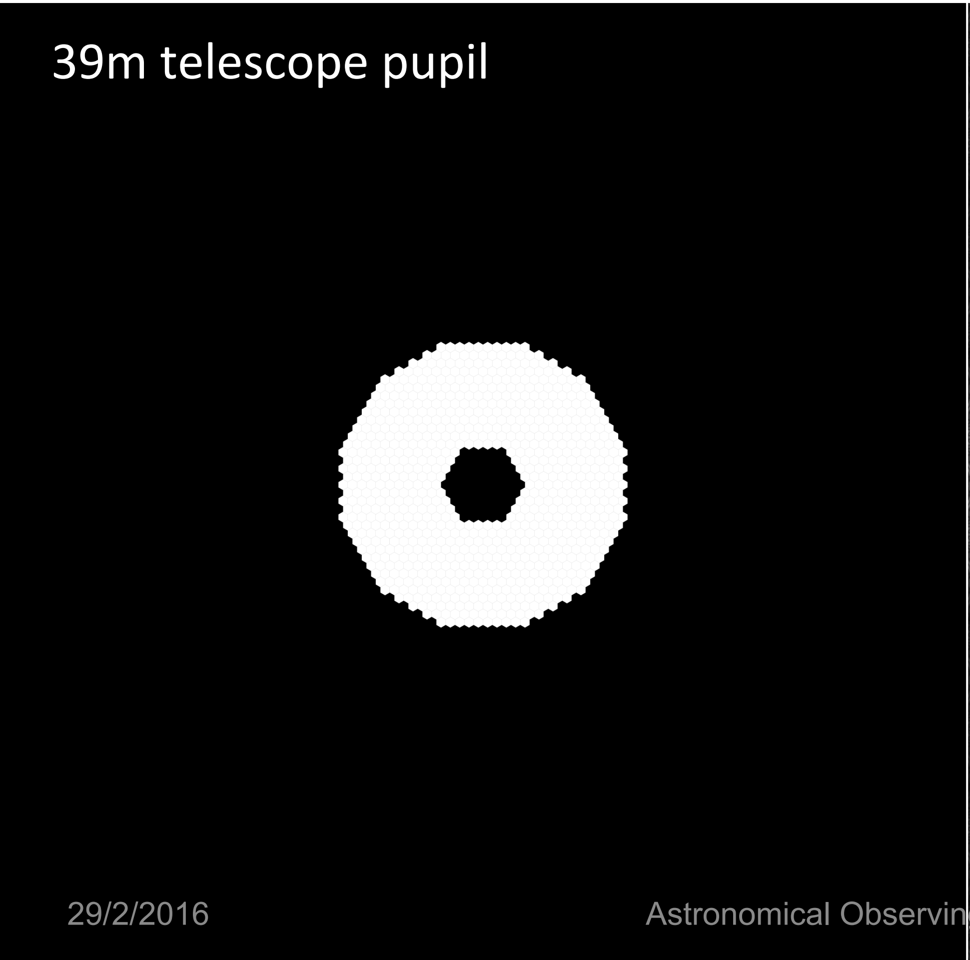


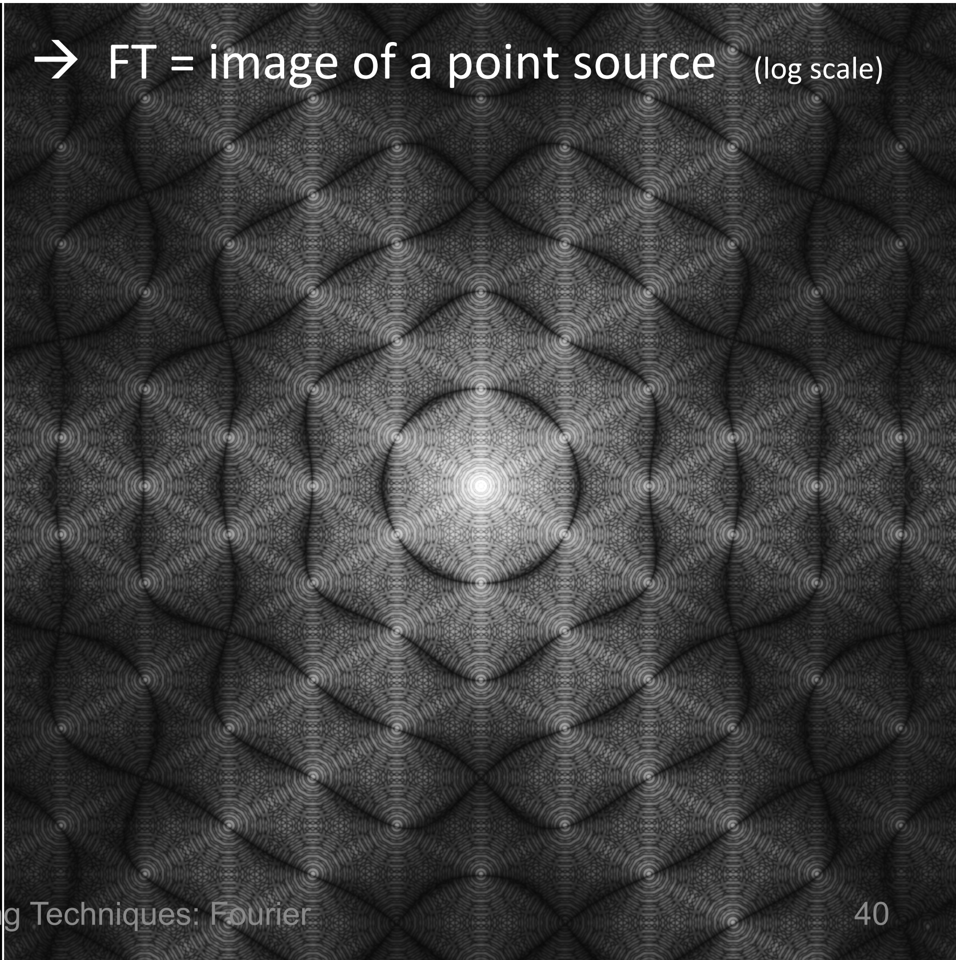
central obscuration, monolithic mirror (pupil) with 6 support-spiders





central obscuration,
segmented mirror (pupil)
no support-spiders

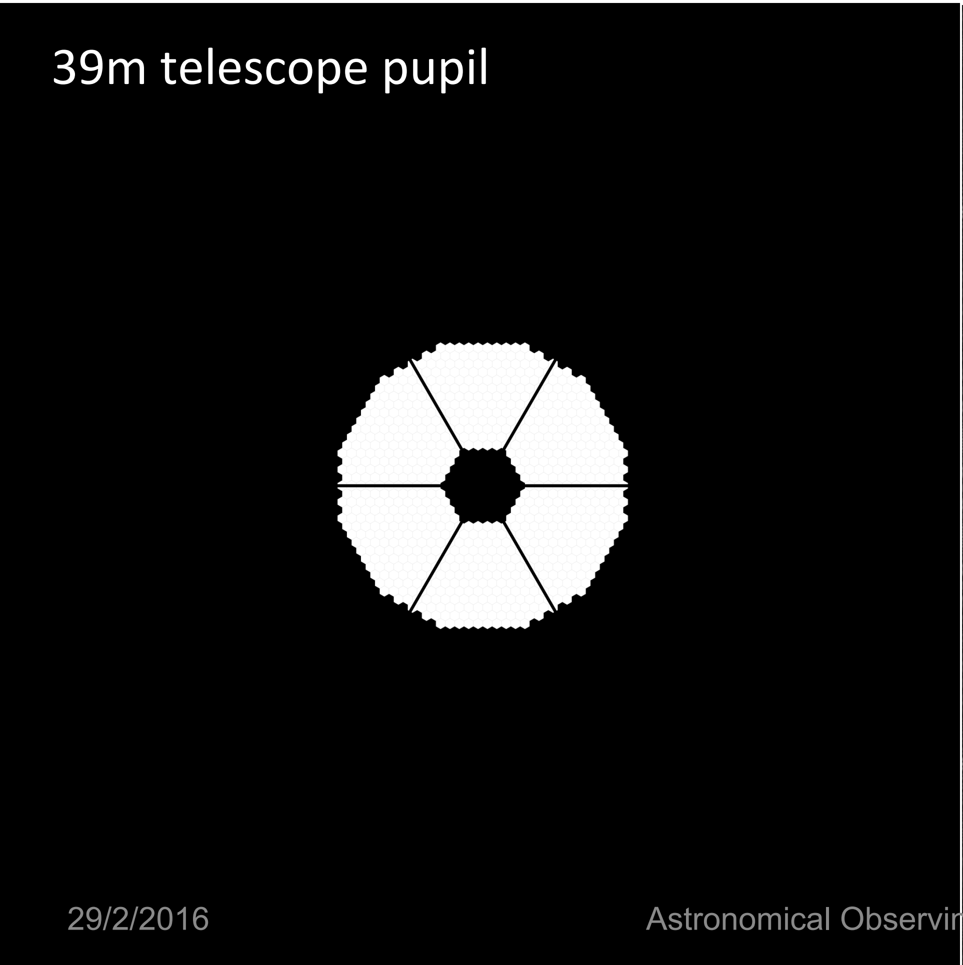


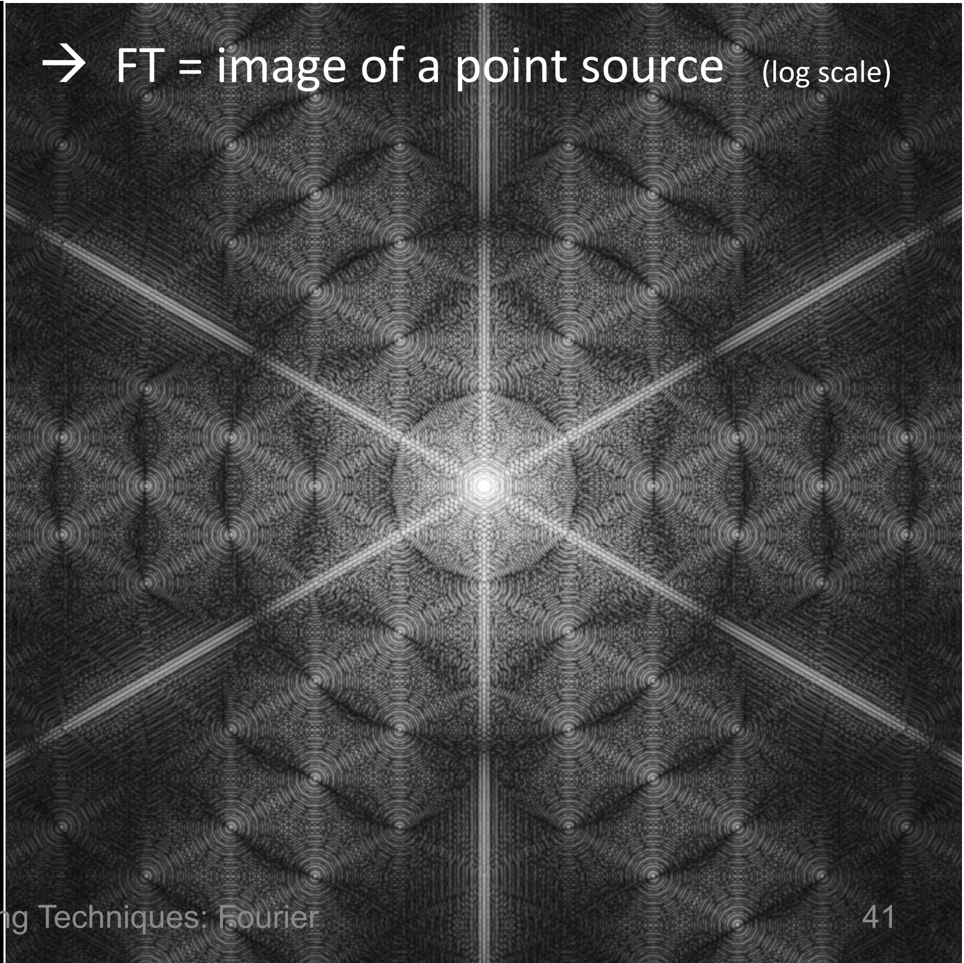


central obscuration,

segmented mirror (pupil)

with 6 support-spiders





Convolution

Convolution of two functions, f*g, is integral of product of functions after one is reversed and shifted:

$$h(x) = f(x) * g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x-u)\, du$$

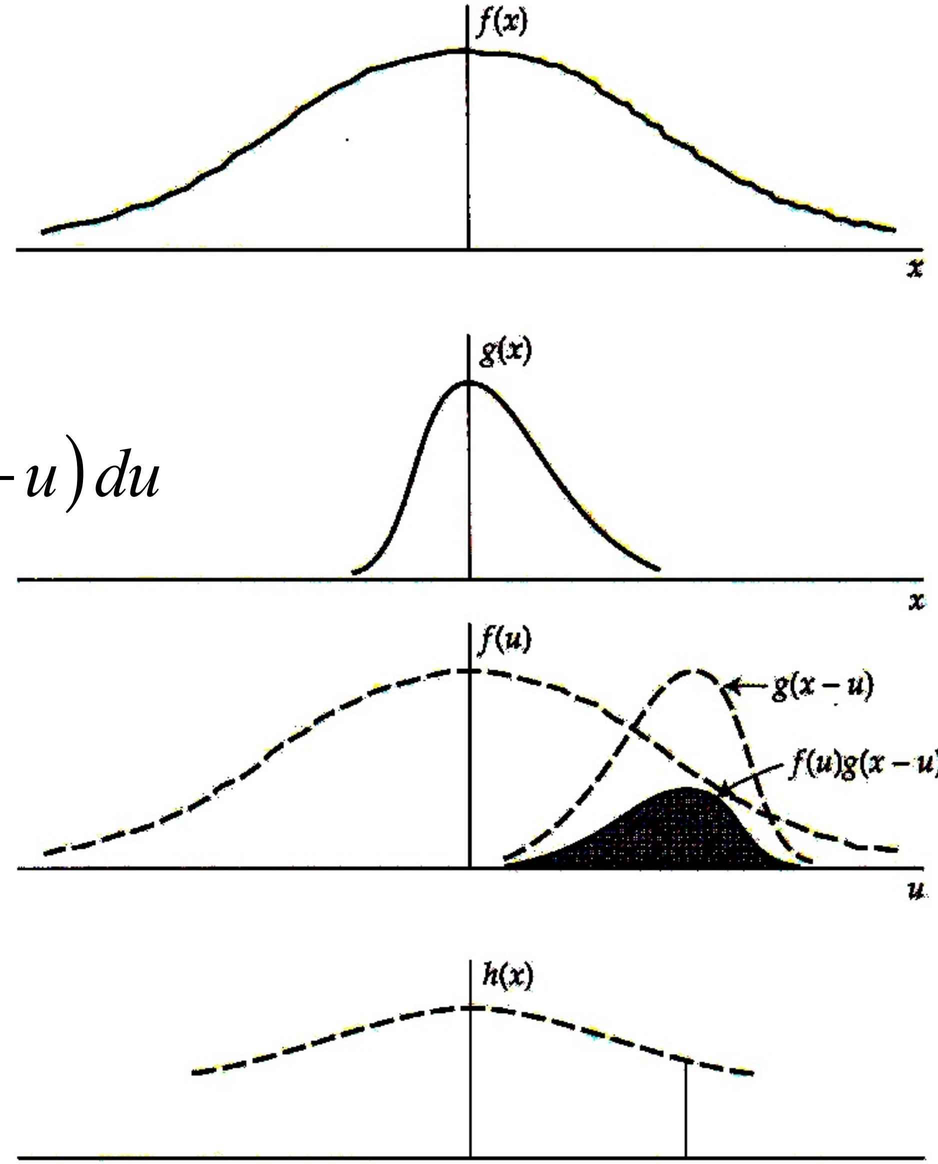
$$f(x) \Leftrightarrow F(s)$$

$$g(x) \Leftrightarrow G(s)$$

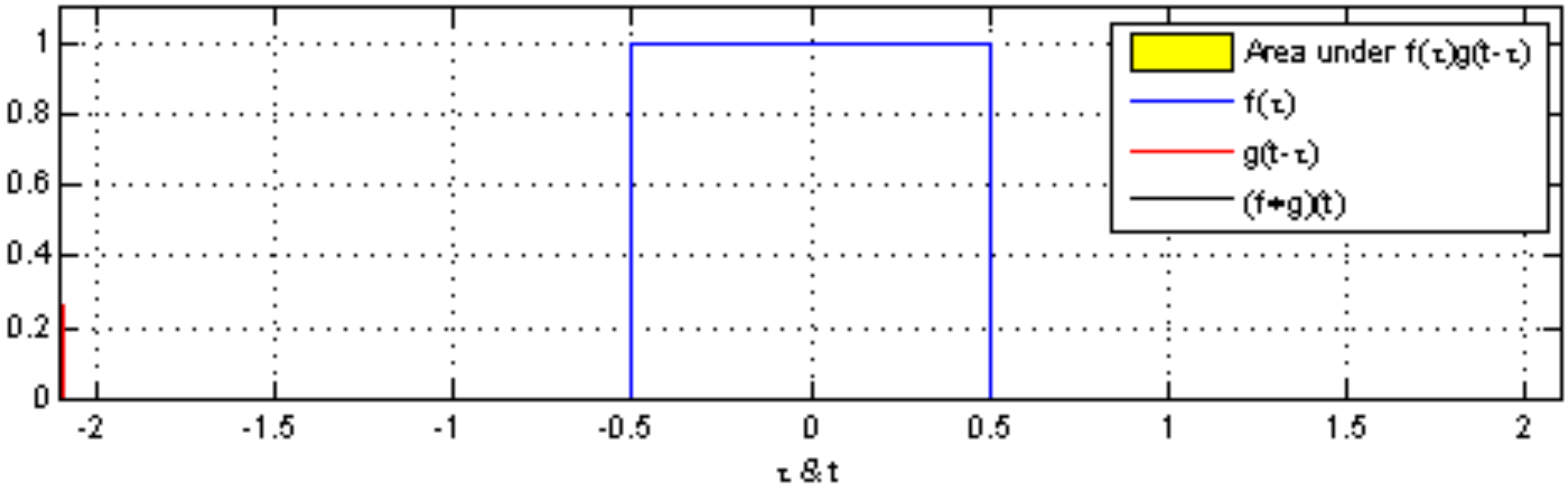
$$h(x) = f(x) * g(x)$$

$$\Leftrightarrow$$

$$F(s) \cdot G(s) = H(s)$$



Convolution: Example



Convolution: Applications

Example:

f(x) : object in sky

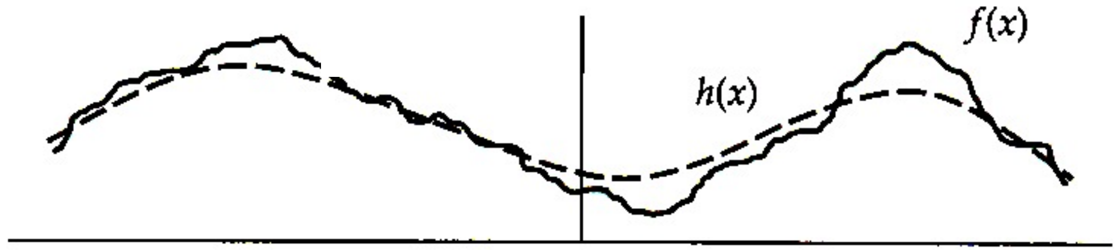
g(x): point spread function of telescope

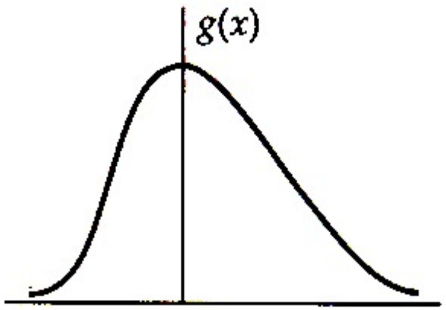
h(x): observed image

$$f(x) * g(x) = h(x)$$

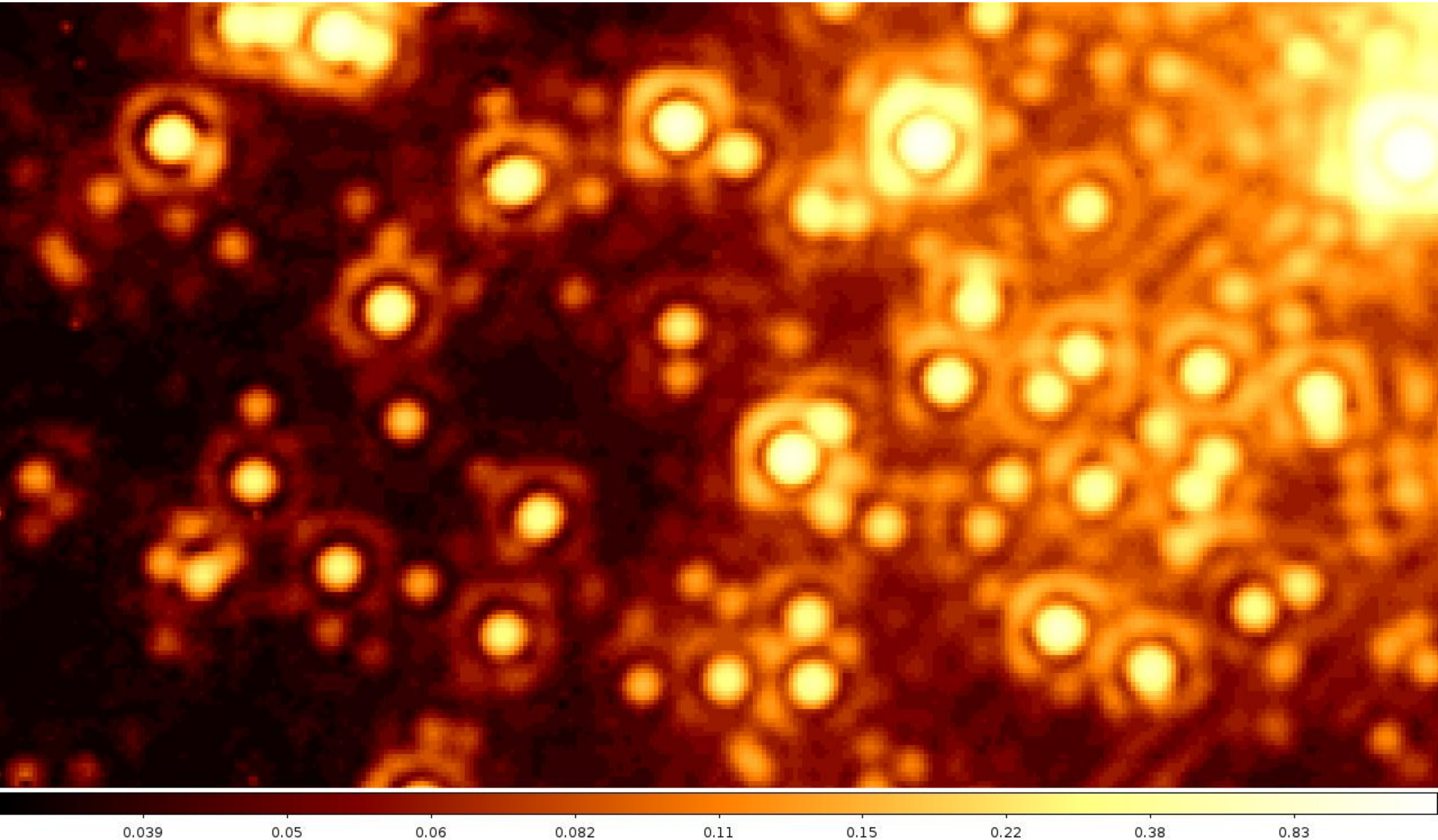
Example:

Convolution of *f(x)* with a smooth kernel g(x) can be used to smoothen *f(x)*





Star cluster observed with HST/NICMOS



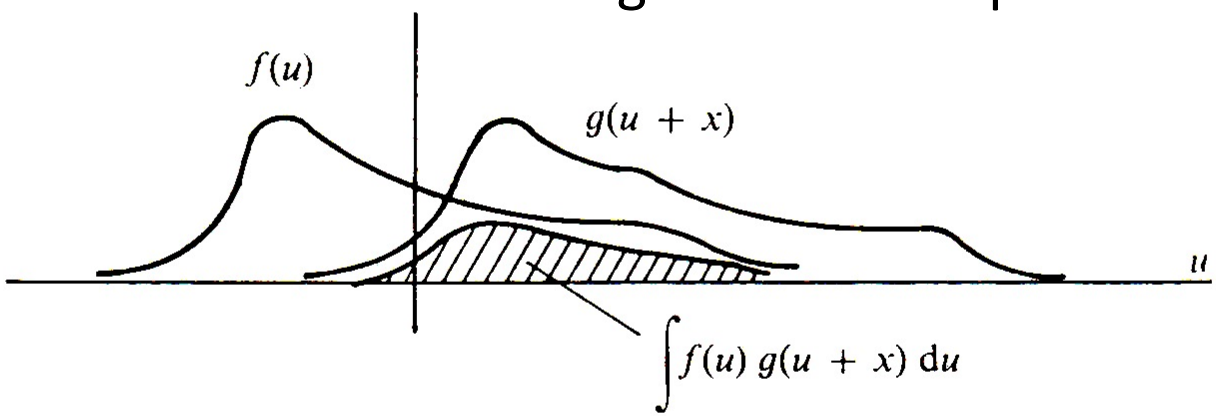
Cross-Correlation

Cross-correlation (or covariance) is measure of similarity of two waveforms as function of time-lag between them.

$$k(x) = f(x) \otimes g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x+u)\, du$$

Difference between cross-correlation and convolution:

- Convolution reverses the signal ('-' sign)
- Cross-correlation shifts the signal and multiplies it with another



Interpretation: By how much (x) must *g(u)* be shifted to match *f(u)*? Answer given by maximum of *k(x)*

Cross-Correlation in Fourier Space

$$f(x) \Leftrightarrow F(s)$$
$$g(x) \Leftrightarrow G(s)$$
$$h(x) = f(x) \otimes g(x) \Leftrightarrow F(s) \cdot G^*(s) = H(s)$$

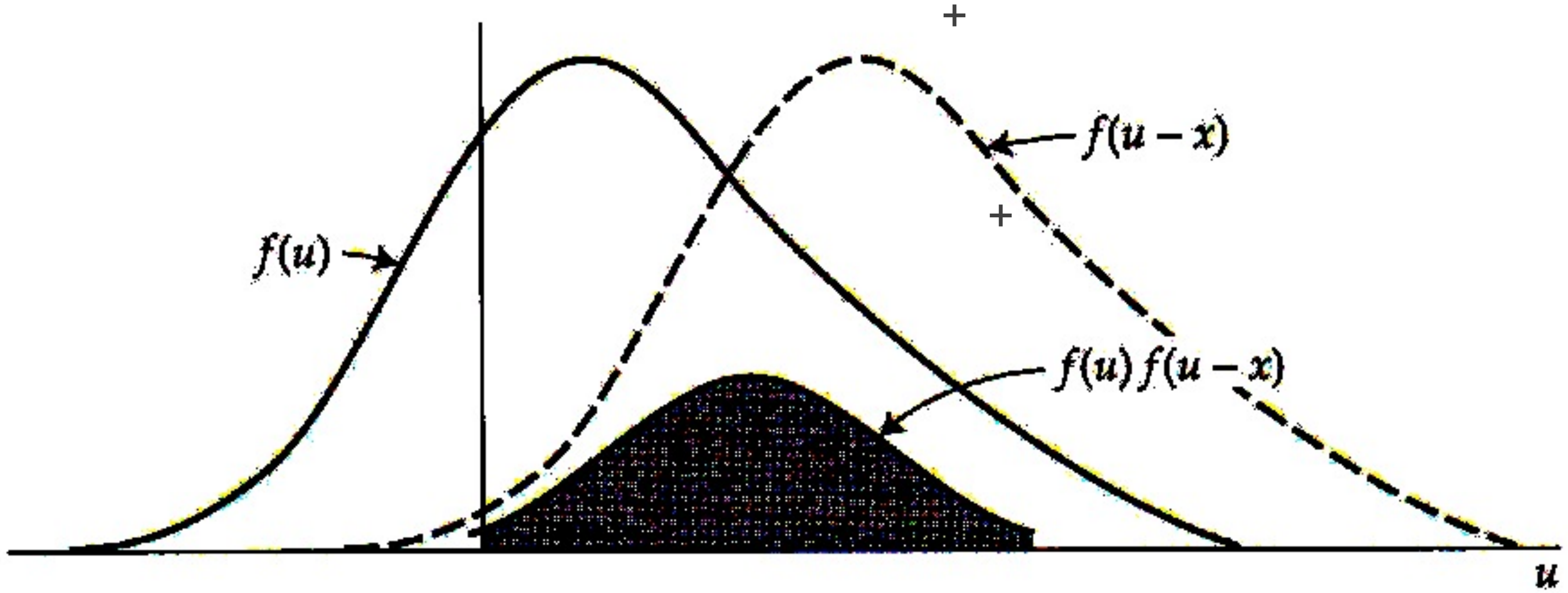
In contrast to convolution, in general

$$f \otimes g \neq g \otimes f$$

Auto-Correlation Theorem

Auto-correlation is cross-correlation of function with itself:

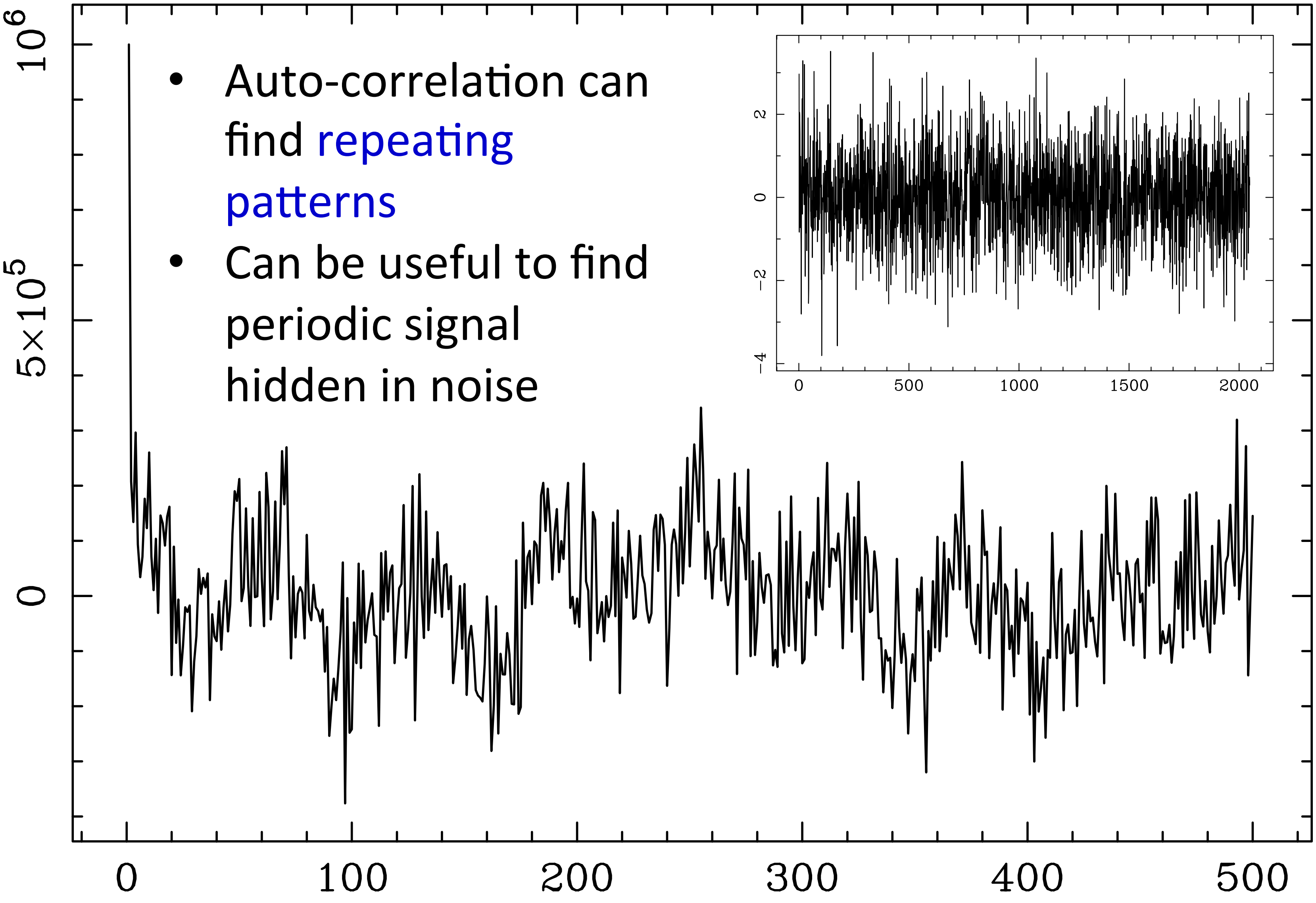
$$k(x) = f(x) \otimes f(x) = \int_{-\infty}^{+\infty} f(u) \cdot f(x+u)\, du$$



$$f(x) \otimes f(x) \Leftrightarrow F(s) F^{*}(s) = |F(s)|^{2}$$

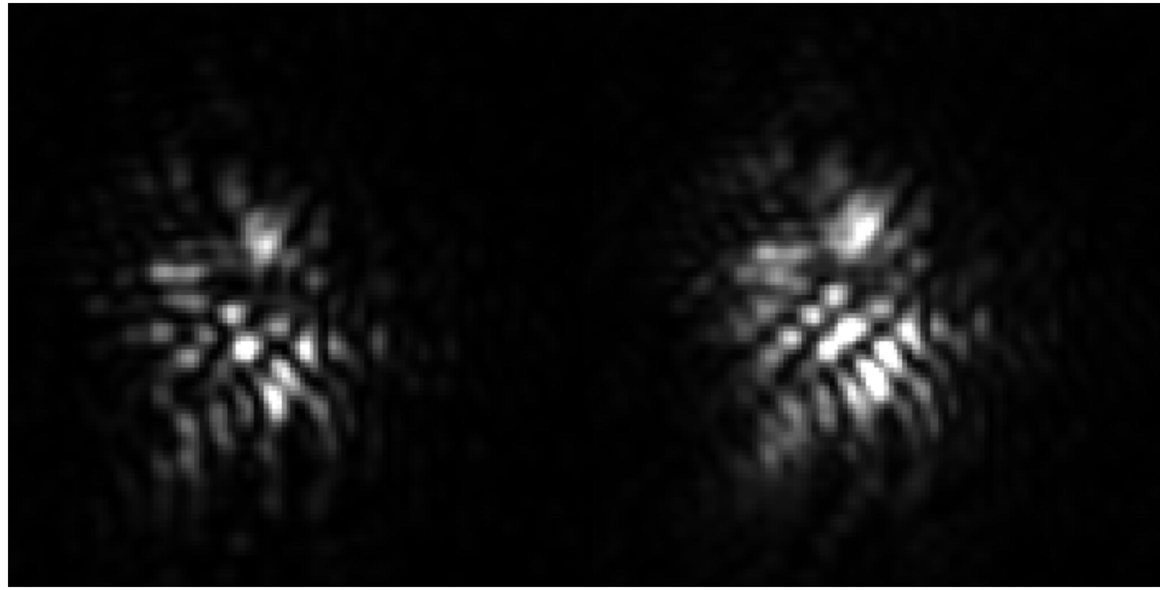
Auto-Correlation: Application

- Auto-correlation can find repeating patterns
- Can be useful to find periodic signal hidden in noise



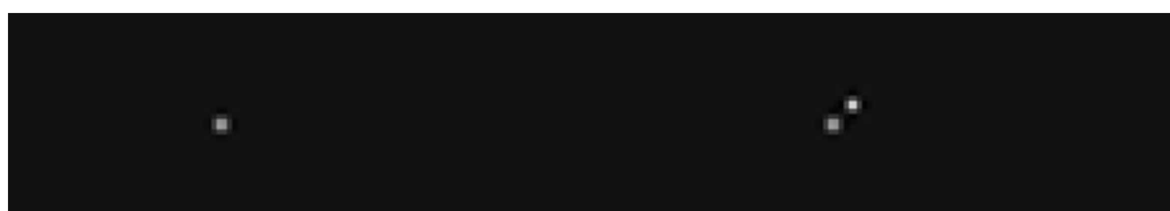
Speckle Interferometry

- average auto-correlation of short-exposure images
- preserves high-resolution information



average cross-correlation

perfect image



Power Spectrum

Power Spectrum S_f of $f(x)$ (or the Power Spectral Density, PSD) describes how the power of a signal is distributed with frequency.

Power is often defined as squared value of signal:

$$S_f(s) = |F(s)|^2$$

Power spectrum is Fourier transform of autocorrelation and indicates what frequencies carry most of the energy.

Total energy of a signal is: $$\int_{-\infty}^{+\infty} S_f(s)\,ds$$

Applications: spectrum analyzers, calorimeters of light sources, …

Parseval's Theorem

Parseval's theorem (or Rayleigh's Energy Theorem):
Sum of square of a function is the same as sum of square of the Fourier transform:

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |F(s)|^2 ds$$

Interpretation: Total energy contained in signal *f(x)*, summed over all x is equal to total energy of signal's Fourier transform *F(s)* summed over all frequencies *s*.

Wiener-Khinchin Theorem

Wiener–Khinchin theorem states that the power spectral density S_f of a function *f(x)* is the Fourier transform of its auto-correlation function:

$$\left|F(s)\right|^2 = FT\{f(x) \otimes f(x)\}$$

$$\updownarrow$$

$$F(s) \cdot F^*(s)$$

<u>Applications:</u> E.g. in the analysis of linear time-invariant systems, when the inputs and outputs are not square integrable, i.e. their Fourier transforms do not exist.

Fourier Relation Summary

Convolution	$h(x)=f(x)*g(x)=\int_{-\infty}^{+\infty} f(u)\cdot g(x-u)\,du$
Cross-correlation	$k(x)=f(x)\otimes g(x)=\int_{-\infty}^{+\infty} f(u)\cdot g(x+u)\,du$
Auto-correlation	$k(x)=f(x)\otimes f(x)=\int_{-\infty}^{+\infty} f(u)\cdot f(x+u)\,du$
Power spectrum	$S_f(s)=\lvert F(s)\rvert^2$
Parseval's theorem	$\int_{-\infty}^{+\infty} \lvert f(x)\rvert^2 dx = \int_{-\infty}^{+\infty} \lvert F(s)\rvert^2 ds$
Wiener-Khinchin theorem	$\lvert F(s)\rvert^2 = FT\{f(x)\otimes f(x)\} = F(s)\cdot F^*(s)$