Astronomical Observing Techniques

Lecture 5: Fourier

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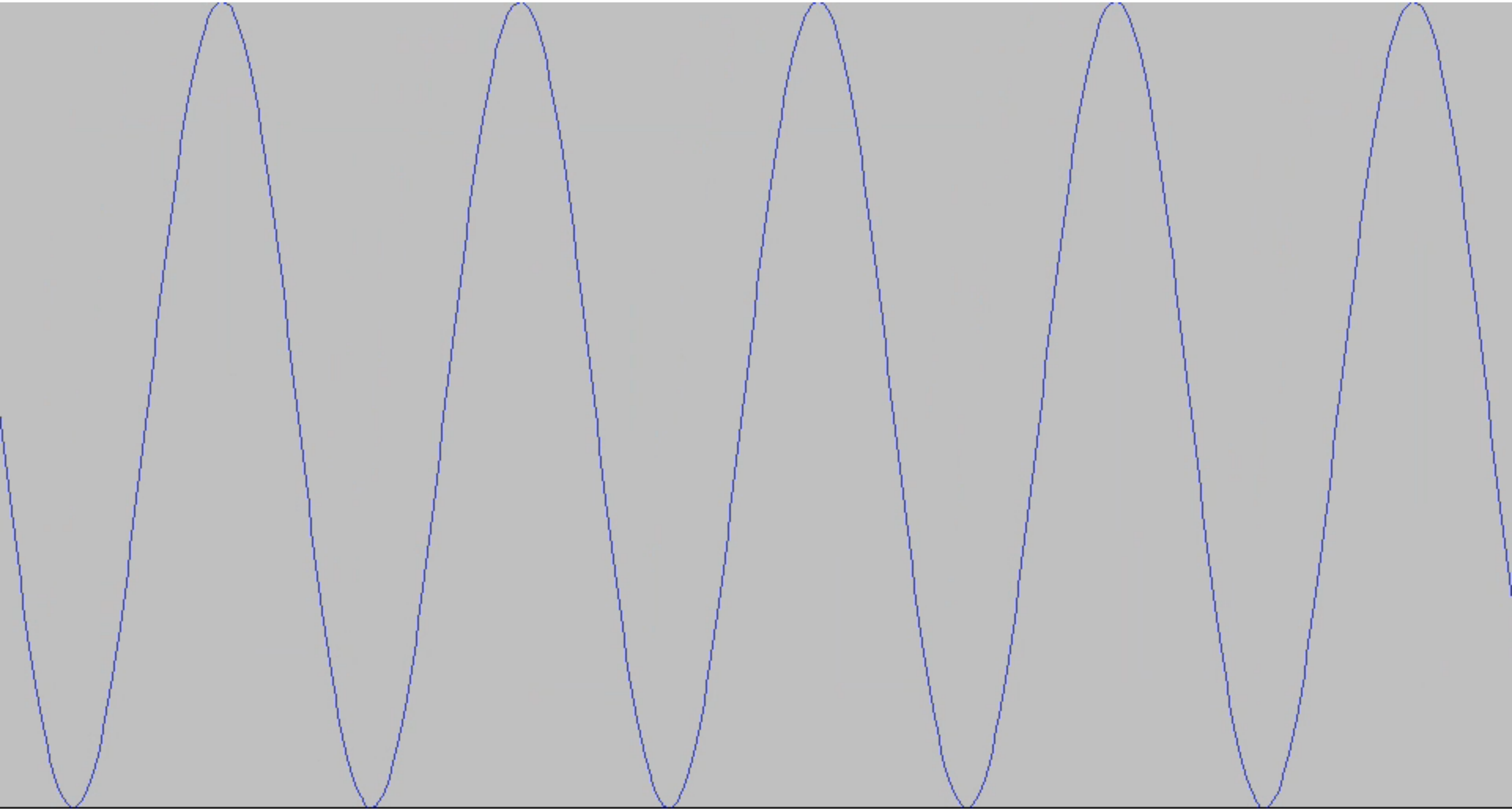
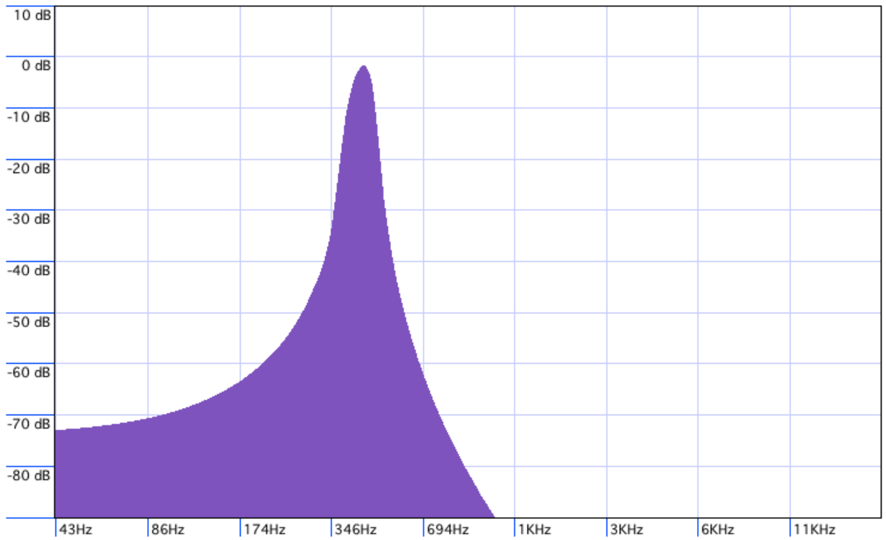
Outline

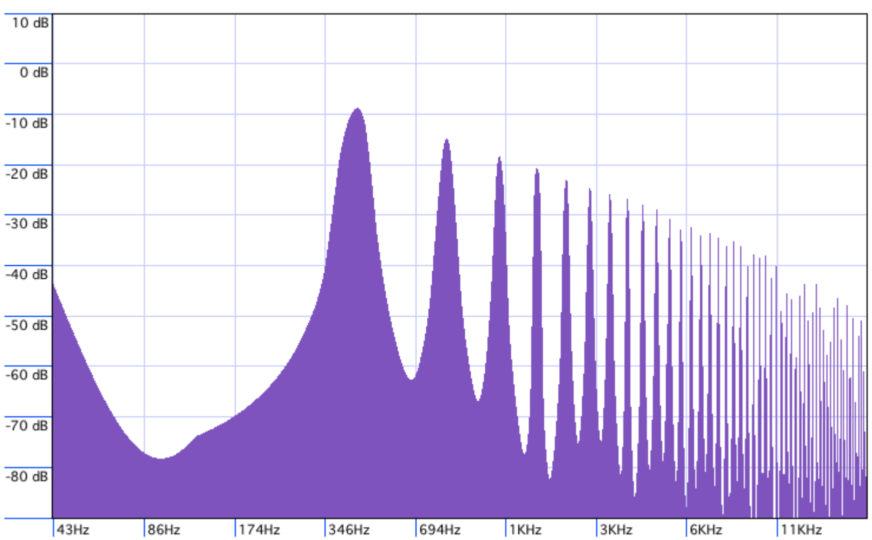
1. Fourier Series
2. Fourier Transform
3. FT Examples in 1D, 2D
4. Telescope ⬄ PSF
5. Important Theorems

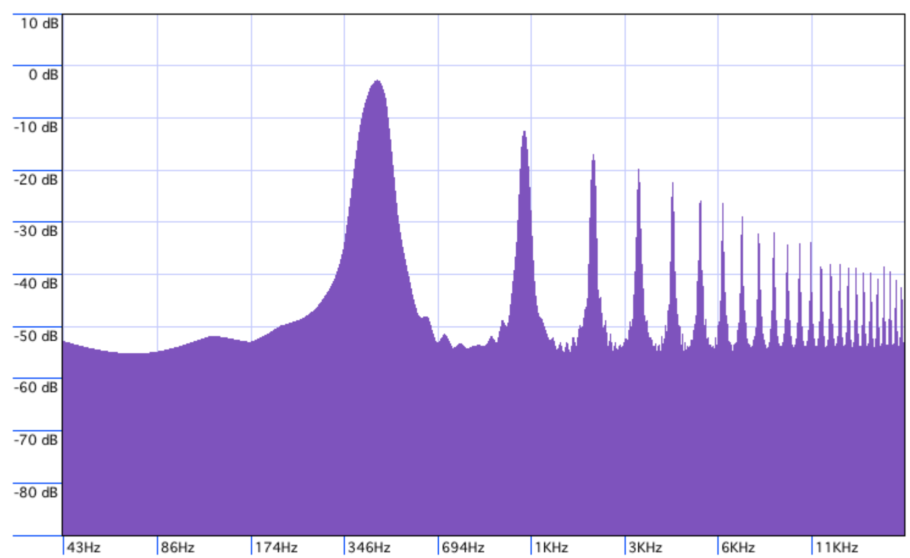
Hear the Difference



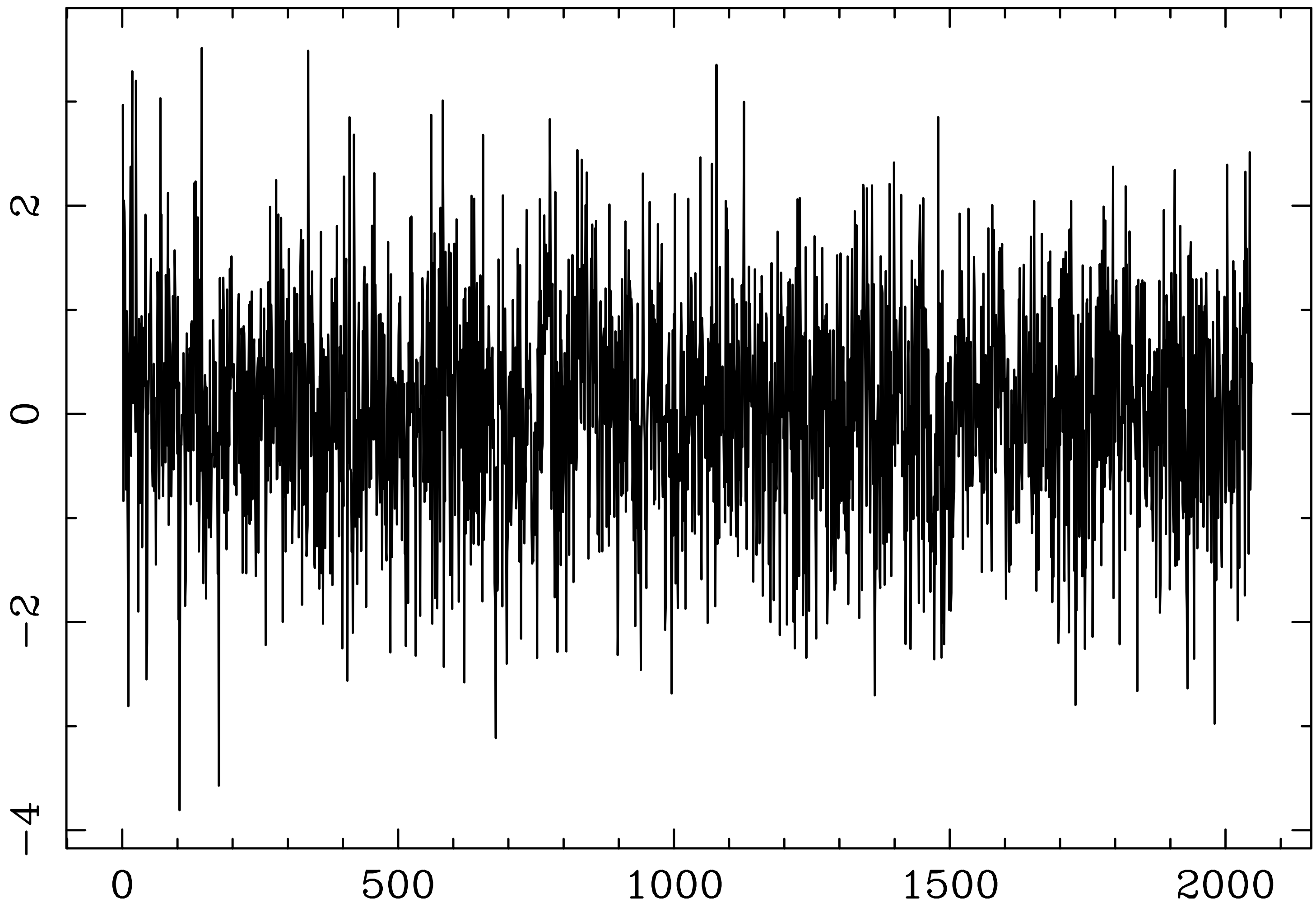
See the Difference



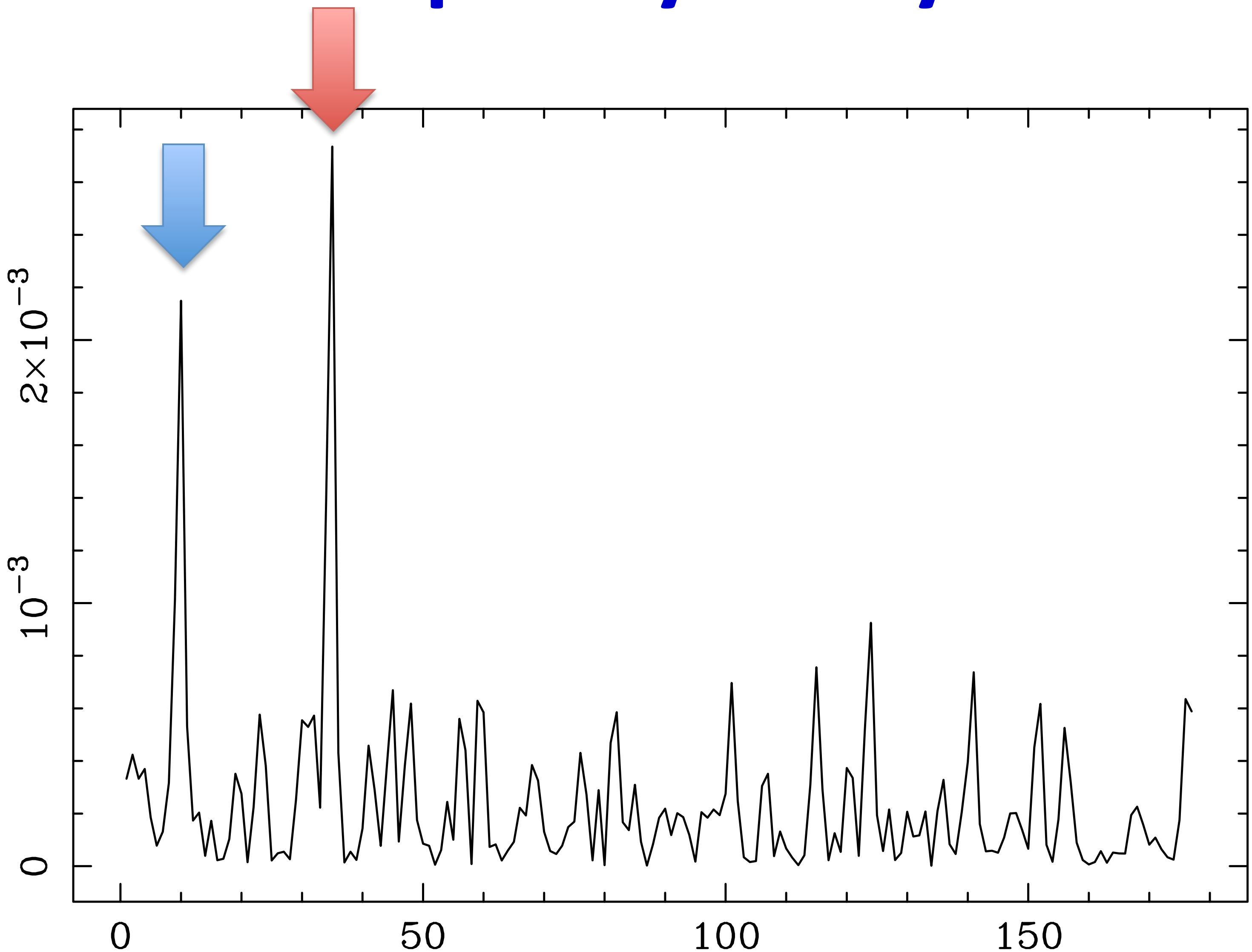




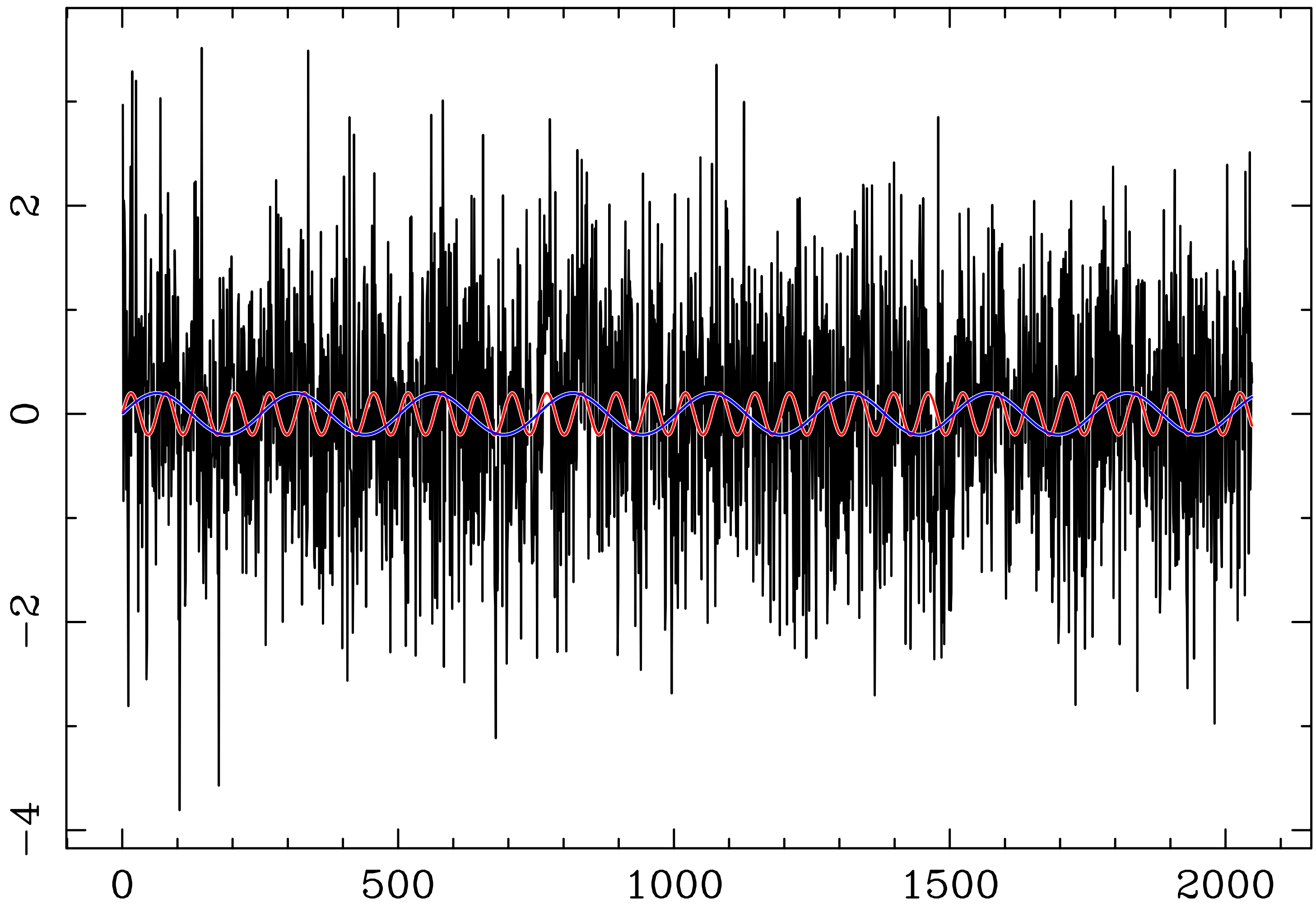
Find the Signal



Frequency Analysis



See the Periodic Signal



Fourier Series

Decomposition using sines and cosines as orthonormal basis set

Periodic function: $f(x) = f(x+P)$

Fourier series: $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nx}{P}\right) + b_n \sin\left(\frac{2\pi nx}{P}\right)\right]$$

Fourier coefficients:

$$a_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\cos\left(\frac{2\pi nx}{P}\right)dx$$

$$b_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\sin\left(\frac{2\pi nx}{P}\right)dx$$

Period: P

Frequency: $\nu = 1/P$

Angular frequency: $\omega = 2\pi/P$

Orthonormal Basis Set

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx = \begin{cases} 1 \text{ for } n = m \\ 0 \text{ for } n \neq m \end{cases}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sin(mx)\cos(nx)\,dx = 0$$

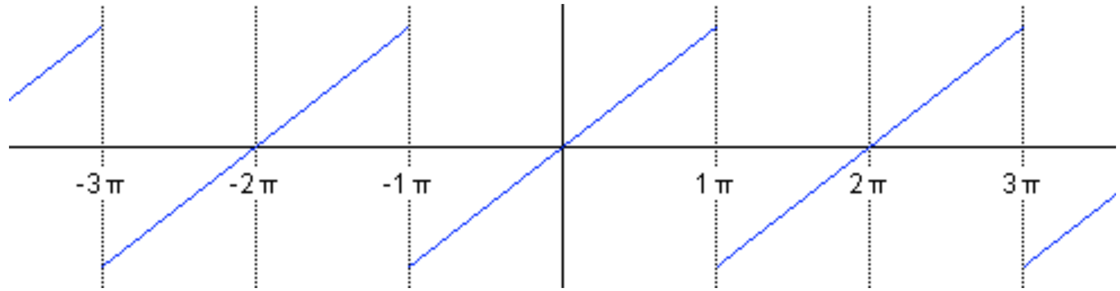
$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sin(mx)\sin(nx)\,dx = \begin{cases} 1 \text{ for } n = m \\ 0 \text{ for } n \neq m \end{cases}$$

Example: Sawtooth Function

Sawtooth function:

$$f(x) = x \quad \text{for} \quad -\pi < x < \pi$$

$$f(x + 2\pi) = f(x)$$



Fourier coefficients are:

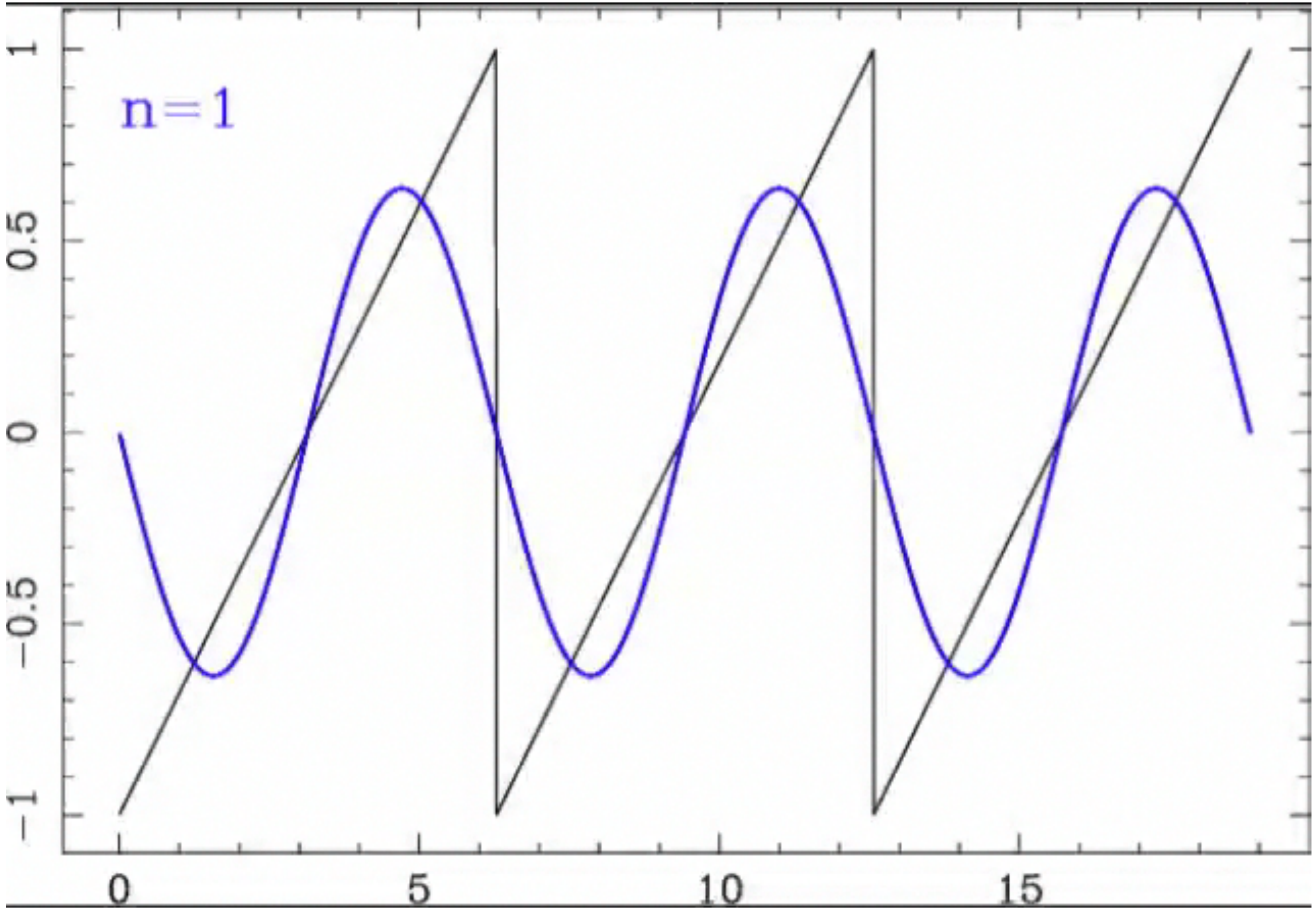
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx)\, dx \overset{!}{=} 0 \qquad \text{(cos() is symmetric around 0)}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\, dx = 2\frac{(-1)^{n+1}}{n}$$

and hence: $f(x) = \cancel{\frac{a_0}{2}} + \sum_{n=1}^{\infty} \left[\cancel{a_n \cos(nx)} + b_n \sin(nx) \right] = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$

Sawtooth Approximation

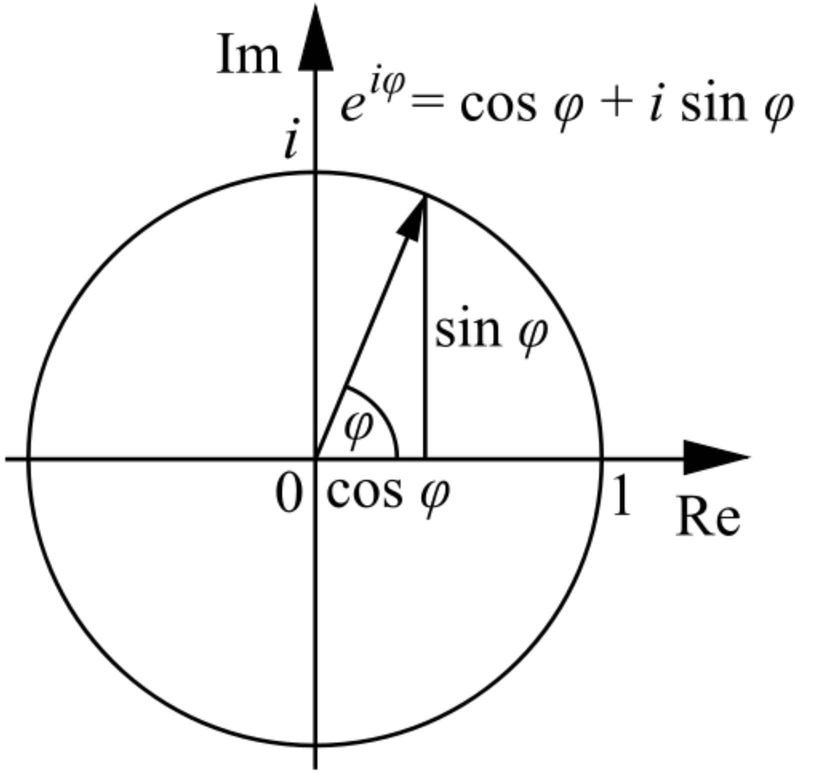
$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$$



Euler's Formula

Euler's formula: relation between trigonometric functions and complex exponential function:

$$e^{i2\pi\theta} = \cos(2\pi\theta) + i\sin(2\pi\theta)$$



Rewrite Fourier series in terms of waves with amplitudes and phases:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi nx}{P}}$$

$$c_n = \frac{1}{P}\int_{-P/2}^{P/2} f(x) e^{-i\frac{2\pi nx}{P}} dx$$

Fourier Transformation

Functions *f(x)* and *F(s)* are Fourier pairs

$$F(s) = \int_{-\infty}^{+\infty} f(x) \cdot e^{-i2\pi xs} dx$$

$$f(x) = \int_{-\infty}^{+\infty} F(s) \cdot e^{i2\pi xs} ds$$

- *x*, *s* can be scalar or vector (*xs* becomes scalar product)
- Fourier transform is reciprocal (exponent sign changes)
- exponent sign and normalization are not well defined

Fourier Transform Properties: Symmetry

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x)$$

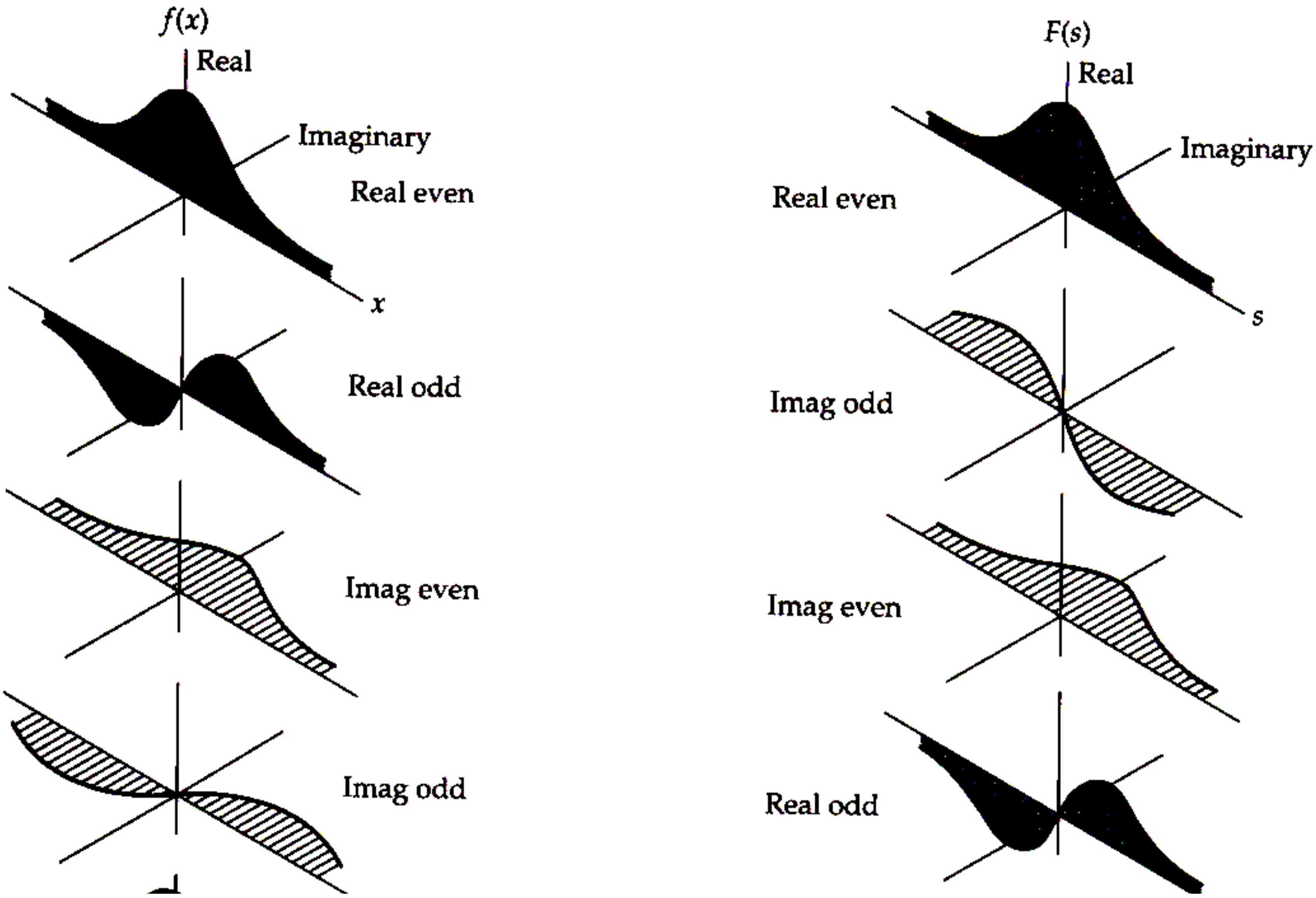
$$f_{\text{even}}(-x) = f_{\text{even}}(x) \quad f_{\text{odd}}(-x) = -f_{\text{odd}}(x)$$

$$\Rightarrow F(s) = 2\int_0^{+\infty} f_{\text{even}}(x)\cos(2\pi xs)dx$$

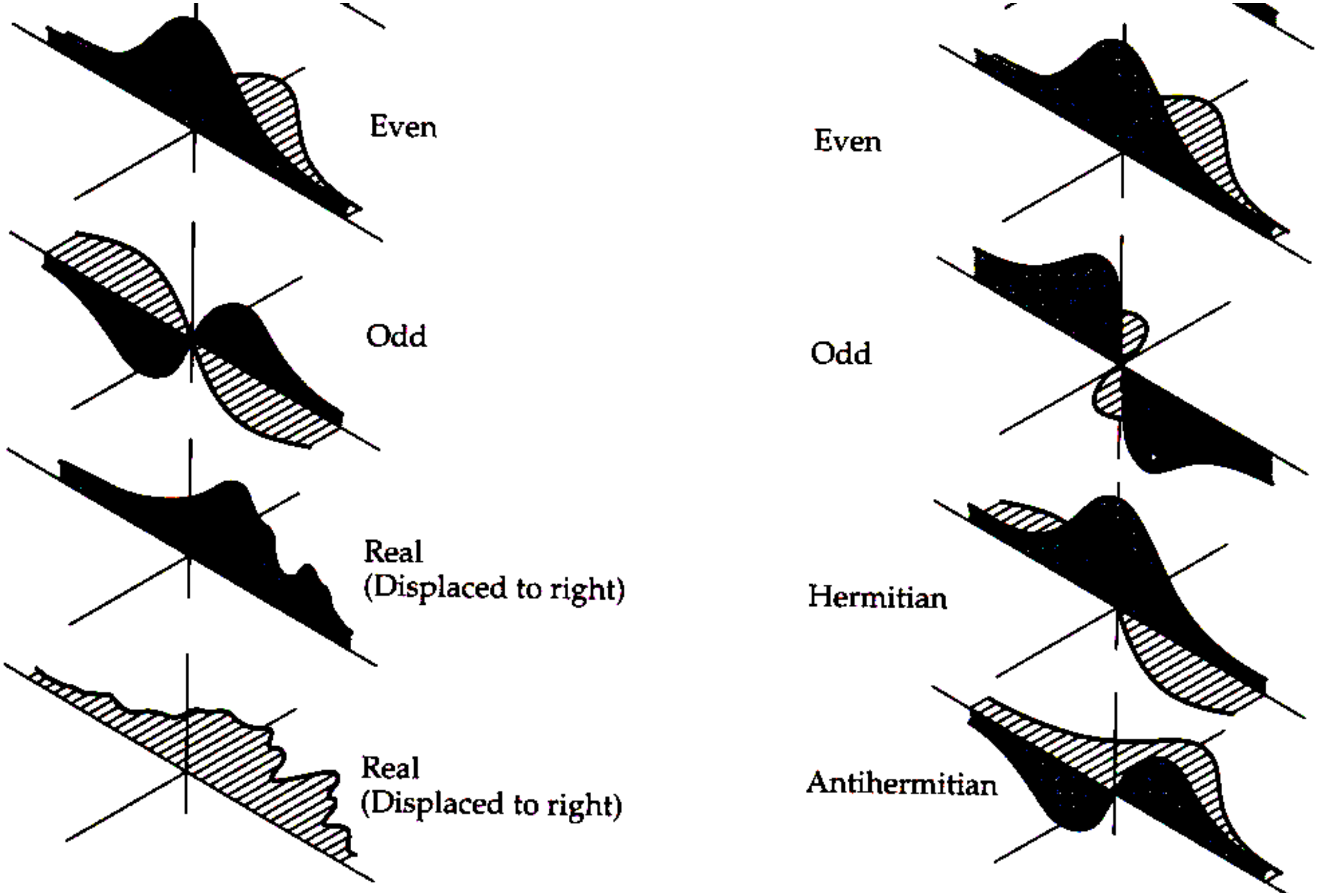
$$-i\ 2\int_0^{+\infty} f_{\text{odd}}(x)\sin(2\pi xs)dx$$

If f(x) is real, the even part of f(x) corresponds to the (even) real part of the Fourier transform F(s), and the odd part of f(x) corresponds to the (odd) imaginary part of F(s).

Fourier Transform Symmetries



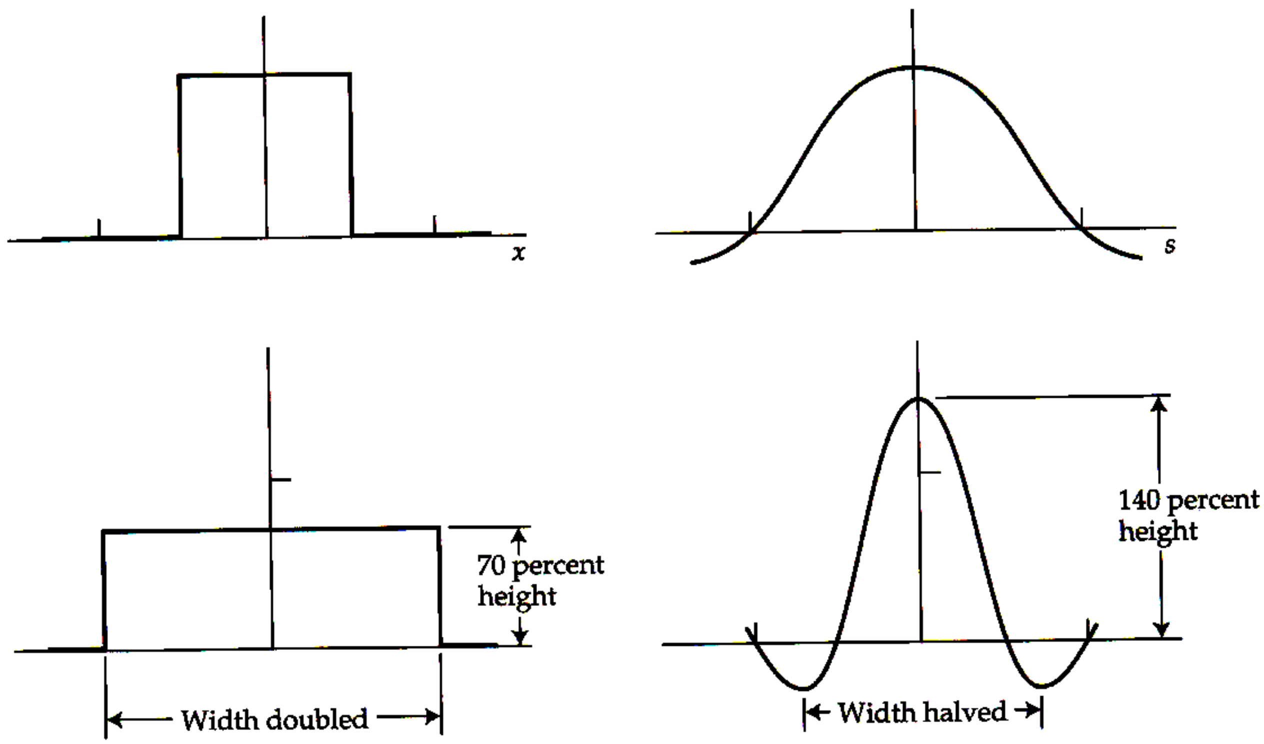
Fourier Transform Symmetries



Fourier Transform Properties: Similarity

Expansion of function *f(x)* causes contraction of its transform *F(s)*:

$$f(x) \rightarrow f(ax) \Leftrightarrow \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

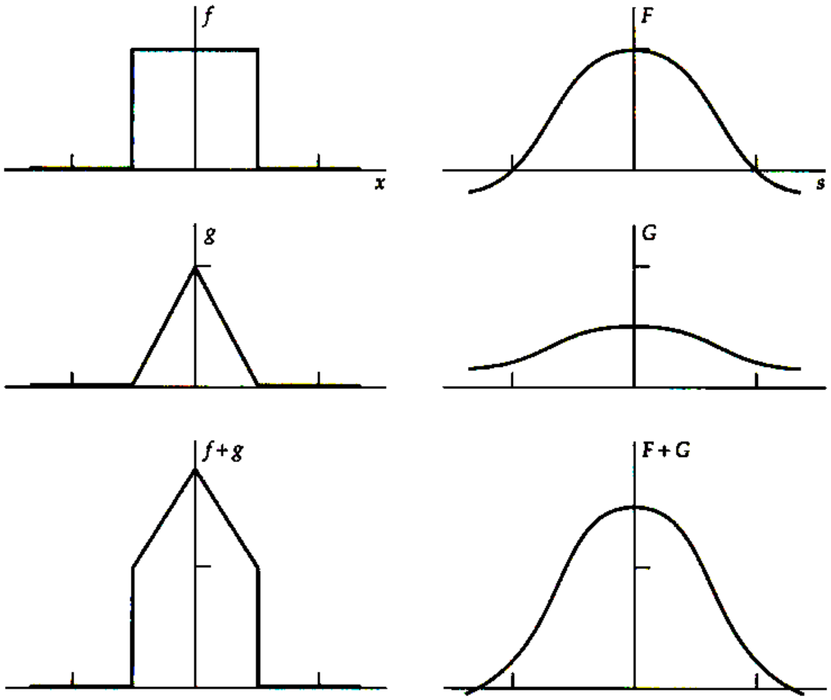


Other Fourier Transform Properties

LINEARITY: $F(as) = a \cdot F(s)$

TRANSLATION: $f(x-a) \Leftrightarrow e^{-i2\pi as} F(s)$

DERIVATIVE: $\frac{\partial^n f(x)}{\partial x^n} \Leftrightarrow (i2\pi s)^n F(s)$



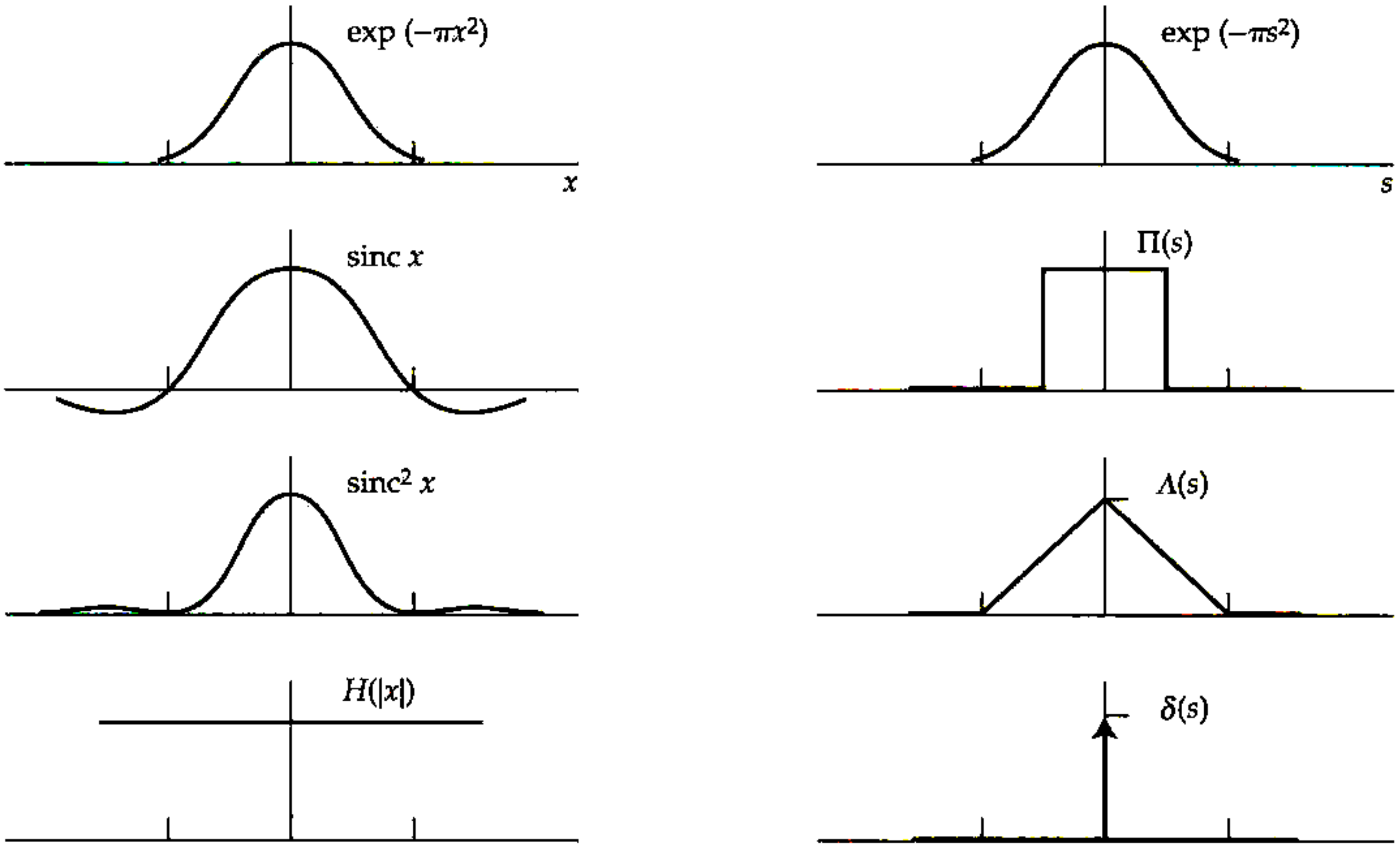
INTEGRAL:

$$\int f(x)\partial x \Leftrightarrow (i2\pi s)^{-1} F(s) + c\delta(s)$$

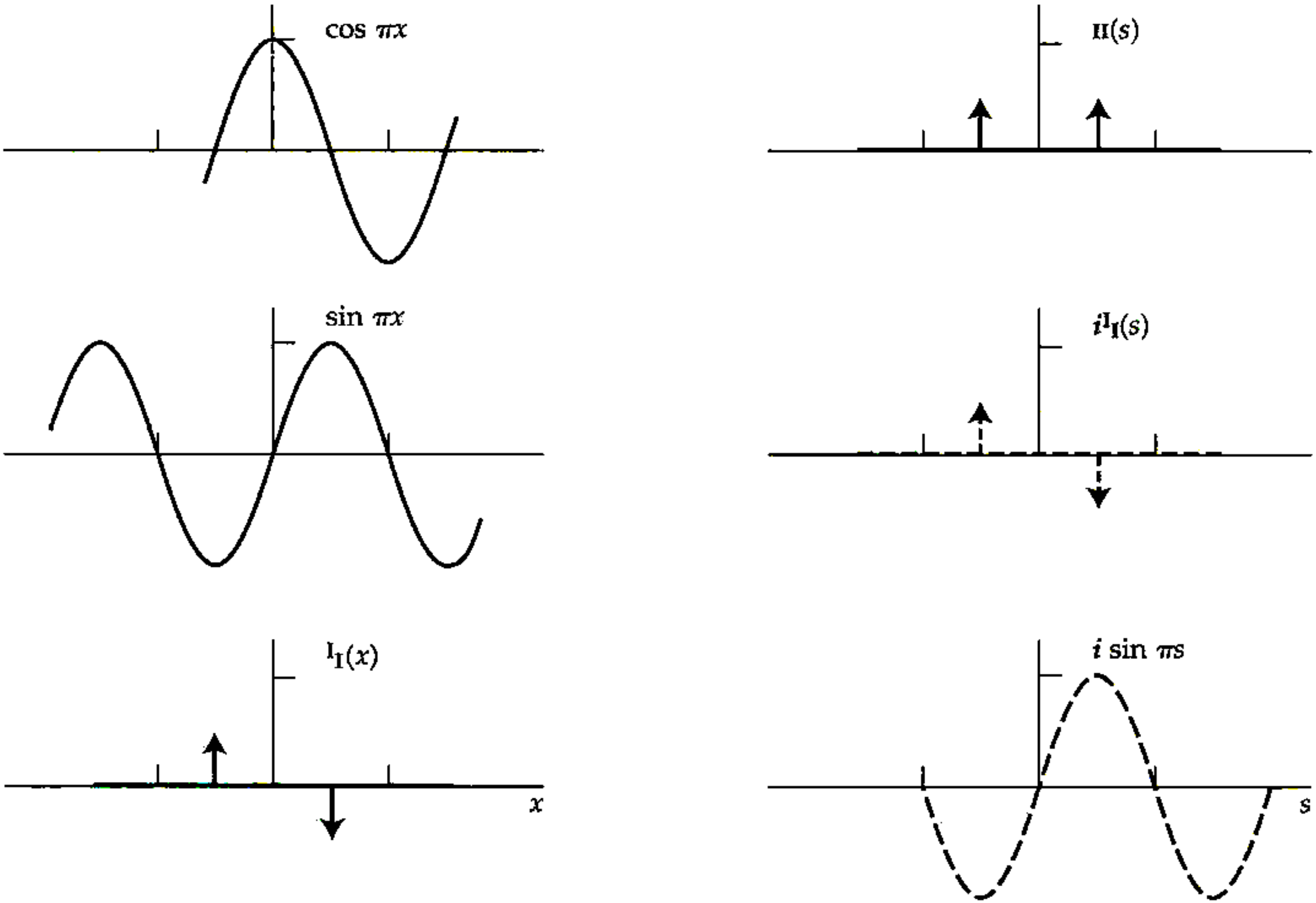
ADDITION:

$$f(x) + g(x) \Leftrightarrow F(s) + G(s)$$

Important 1-D Fourier Pairs



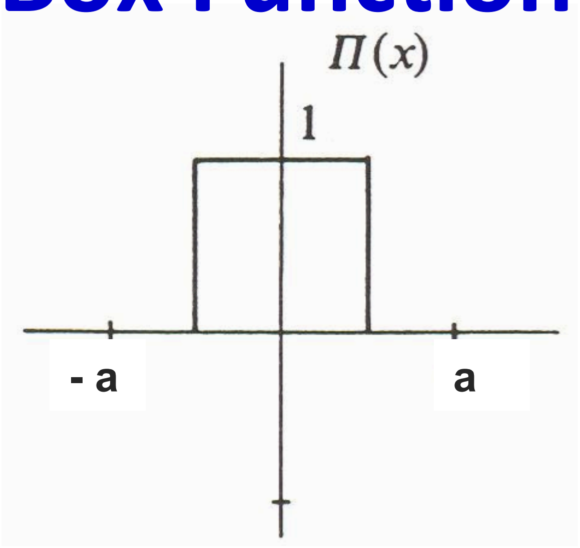
Important 1-D Fourier Pairs



Special 1-D Pairs (1): Box Function

Box function:

$$\Pi\left(\frac{x}{a}\right)=\begin{cases}1 & \text{for } -\frac{a}{2}<x<\frac{a}{2}\\ 0 & \text{elsewhere}\end{cases}$$

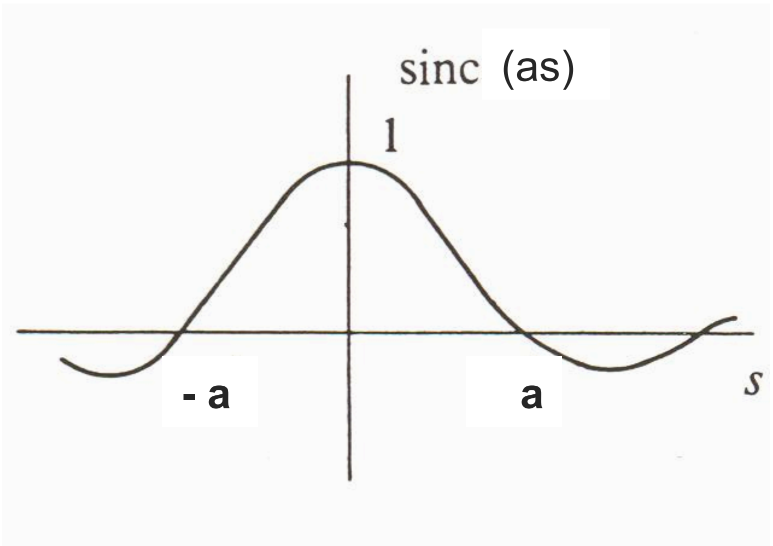


With the Fourier pairs

$$\Pi(x)\Leftrightarrow\frac{\sin(\pi s)}{\pi s}\equiv\operatorname{sinc}(s)$$

and using the similarity relation:

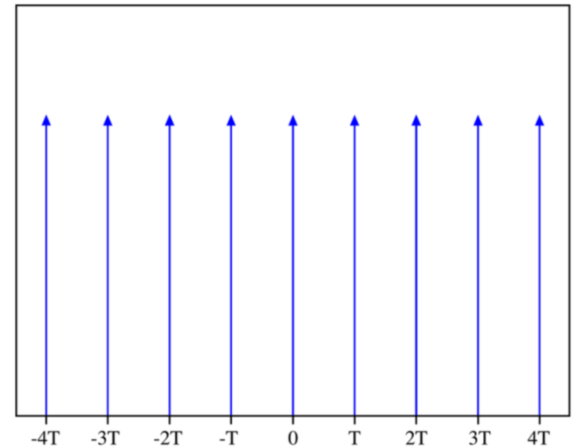
$$\Pi\left(\frac{x}{a}\right)\Leftrightarrow|a|\cdot\operatorname{sinc}(as)$$



Dirac Comb

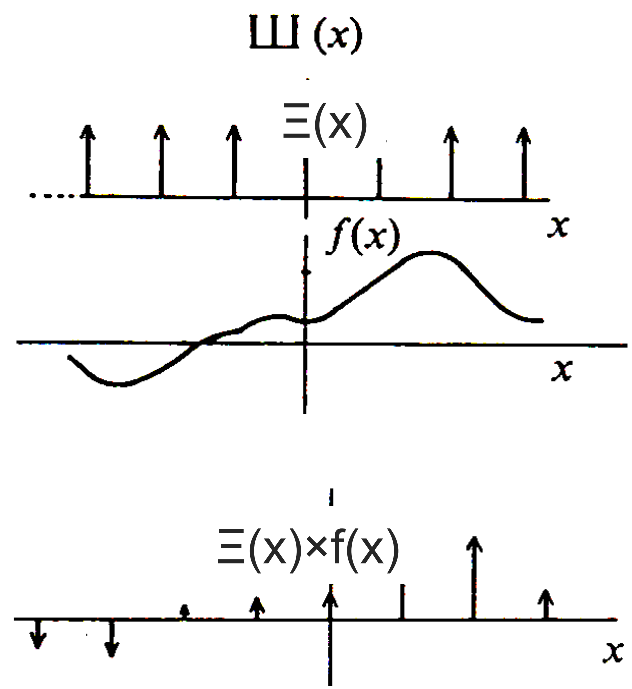
Dirac's delta "function":

$$f(x) = \delta(x) = \int_{-\infty}^{+\infty} e^{i2\pi sx}\, ds \;\rightarrow\; FT\{\delta(x)\} = 1$$



Dirac comb: infinite series of delta functions spaced at intervals of T:

$$\Xi_T(x) = \sum_{k=-\infty}^{\infty} \delta(x - kT) \underset{series}{\overset{Fourier}{=}} \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i2\pi nx/T}$$



- Fourier transform of Dirac comb is also a Dirac comb
- Dirac comb is also called impulse train or sampling function

Nyquist-Shannon Theorem

Sampling: signal at discrete values of x: $f(x) \rightarrow f(x) \cdot \Xi\left(\frac{x}{\Delta x}\right)$

Interval between two successive readings is sampling rate

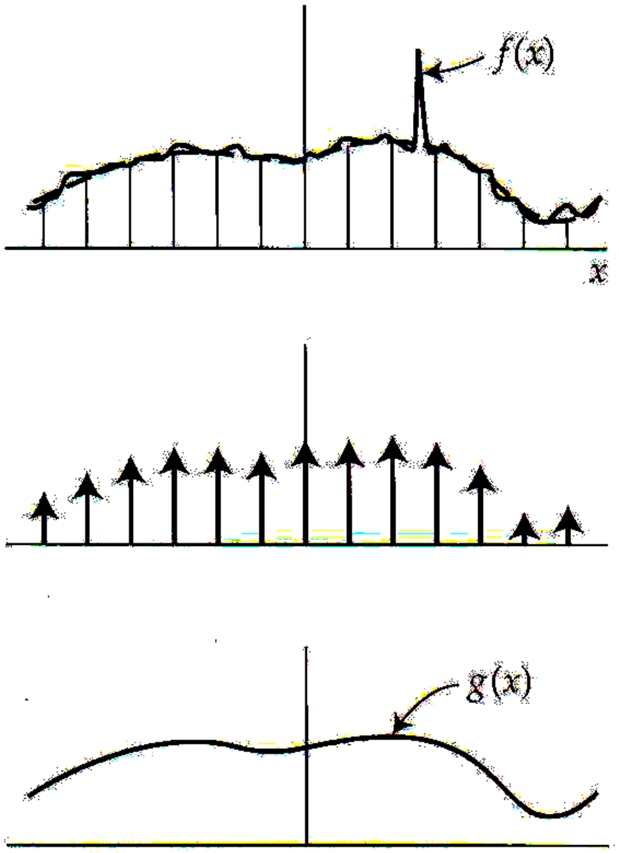
Critical sampling given by Nyquist-Shannon theorem

Given f(x), its Fourier Transform F(s) with bounded support [$-s_{max}$, s_{max}].

Sampled distribution of the form

$$g(x) = f(x) \cdot \Xi\left(\frac{x}{\Delta x}\right)$$

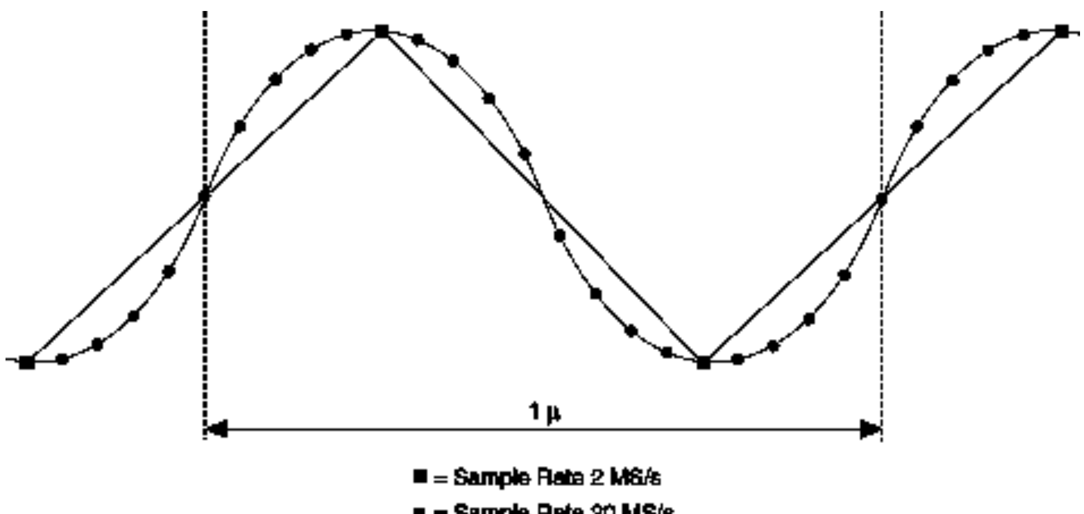
with a sampling rate of $\Delta x = 1/(2s_{max})$ is enough to reconstruct *f(x)* for all *x*.



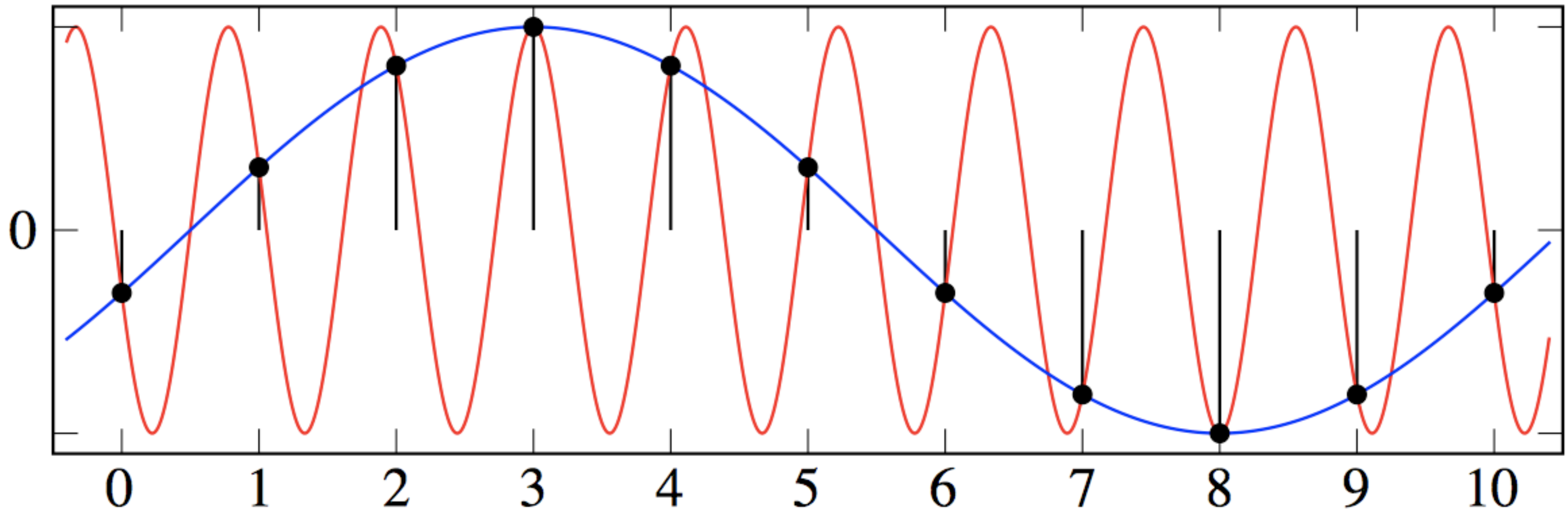
Sampling

Oversampling Sampling rate above critical sampling rate:
- redundant measurements
- often lowering the S/N

Undersampling Sampling rate below critical sampling rate:
- signal contains frequencies higher than $1/(2s_{max})$
- source signal cannot be determined after sampling
- loss of fine details
- must apply low-pass filter before sampling



Aliasing



- unresolved, high frequencies beat with measured frequencies
- produce spurious components in frequency domain below Nyquist frequency
- may give rise to major problems and uncertainties in the determination of original signal

Bessel Functions

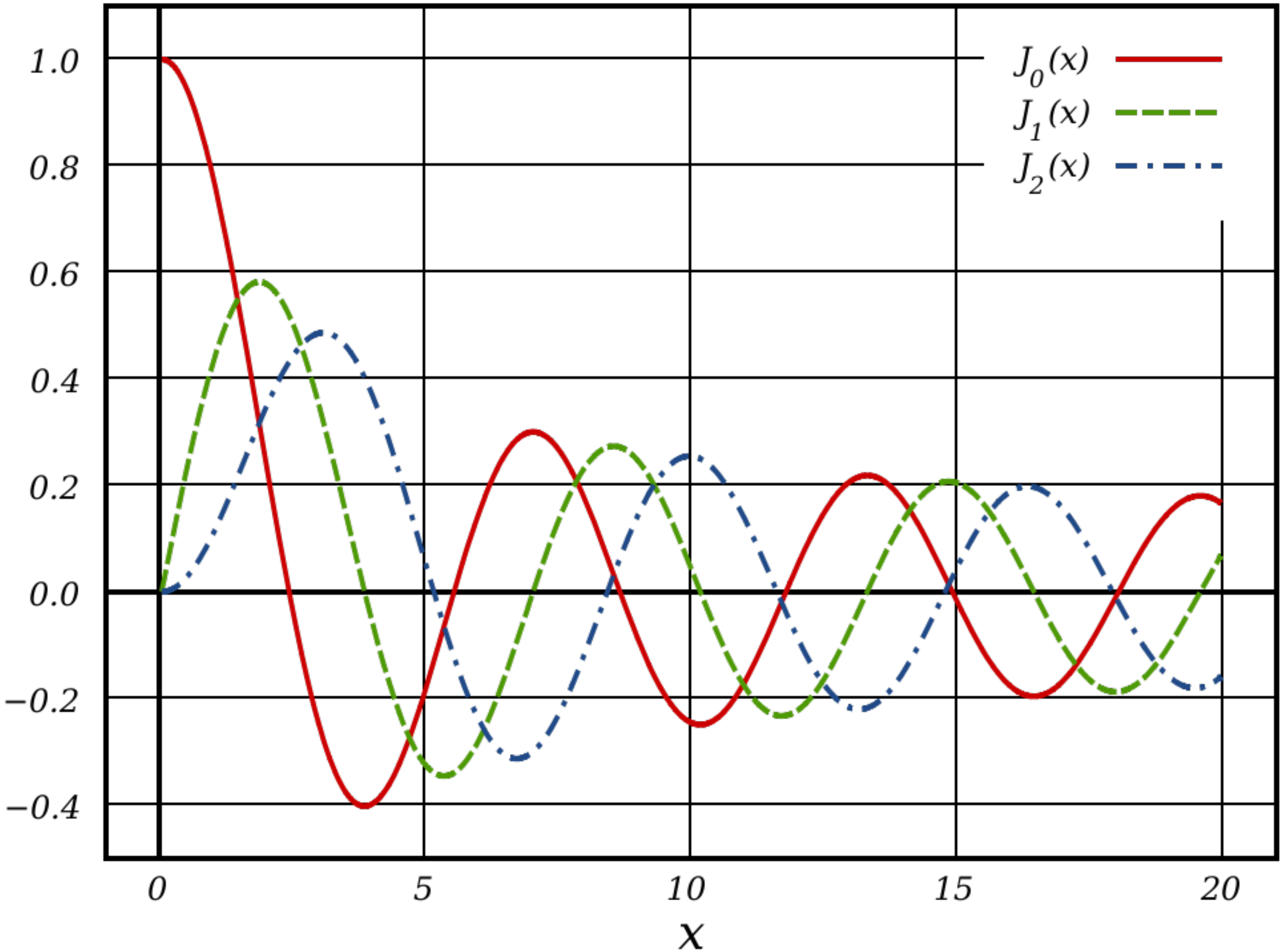
Bessel functions are canonical solutions *y(x)* of Bessel's differential equation:

$$x^2 \frac{\partial^2 y}{\partial x^2} + x \frac{\partial y}{\partial x} + \left(x^2 - n^2\right) y = 0$$

for an arbitrary real or complex number *n*, the order of the Bessel function.

Solutions = Bessel Functions:
$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+n}}{k!(k+n)!}$$

Bessel Functions J_0, J_1, J_2

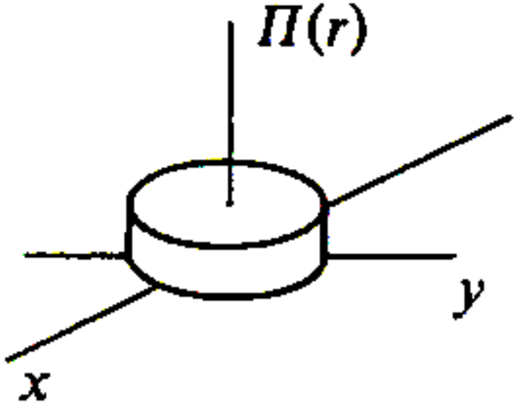


Special 2-D Pairs (1): Box Function

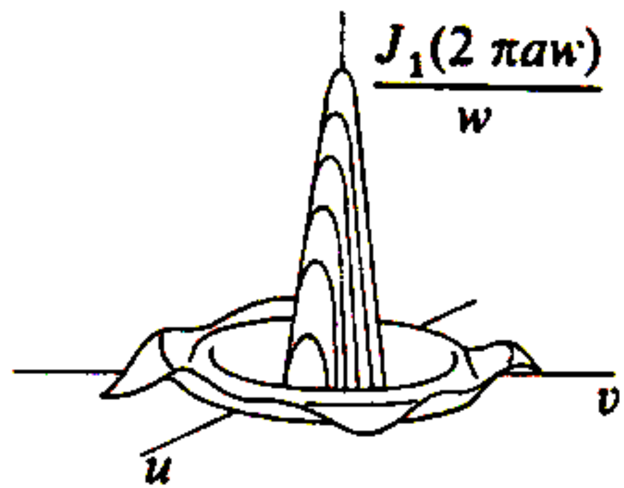
2-D box function with $r^2 = x^2 + y^2$: $\Pi\left(\frac{r}{2}\right) = \begin{cases} 1 & \text{for } r < 1 \\ 0 & \text{for } r \geq 1 \end{cases}$

Fourier Transform: $\Pi\left(\frac{r}{2}\right) \Leftrightarrow \frac{J_1(2\pi\omega)}{\omega}$ (1st order Bessel function J_1)

Telescope Aperture:



Focal plane:

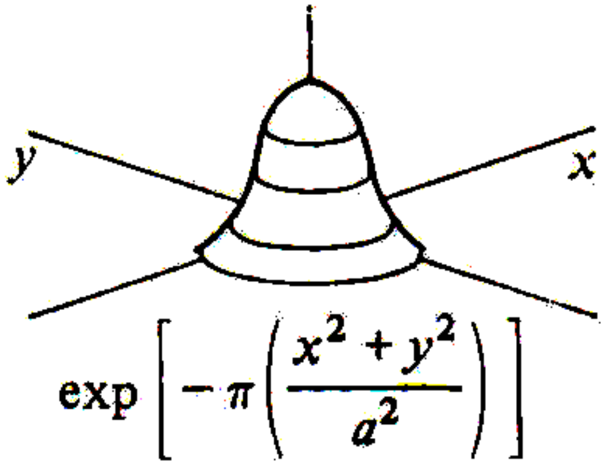


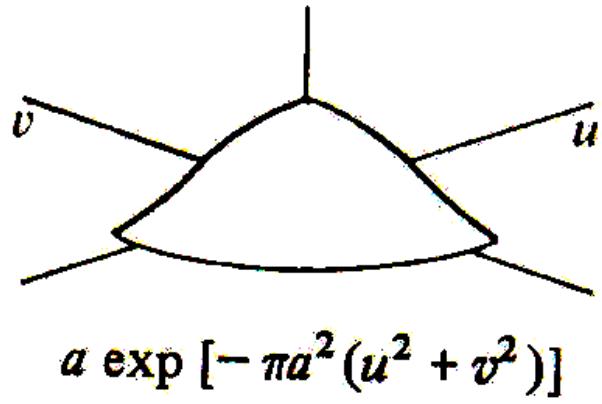
Larger telescopes produce smaller Point Spread Functions (PSFs)!

Special 2-D Pairs (2): Gauss Function

2-D Gauss function with $r^2 = x^2 + y^2$:

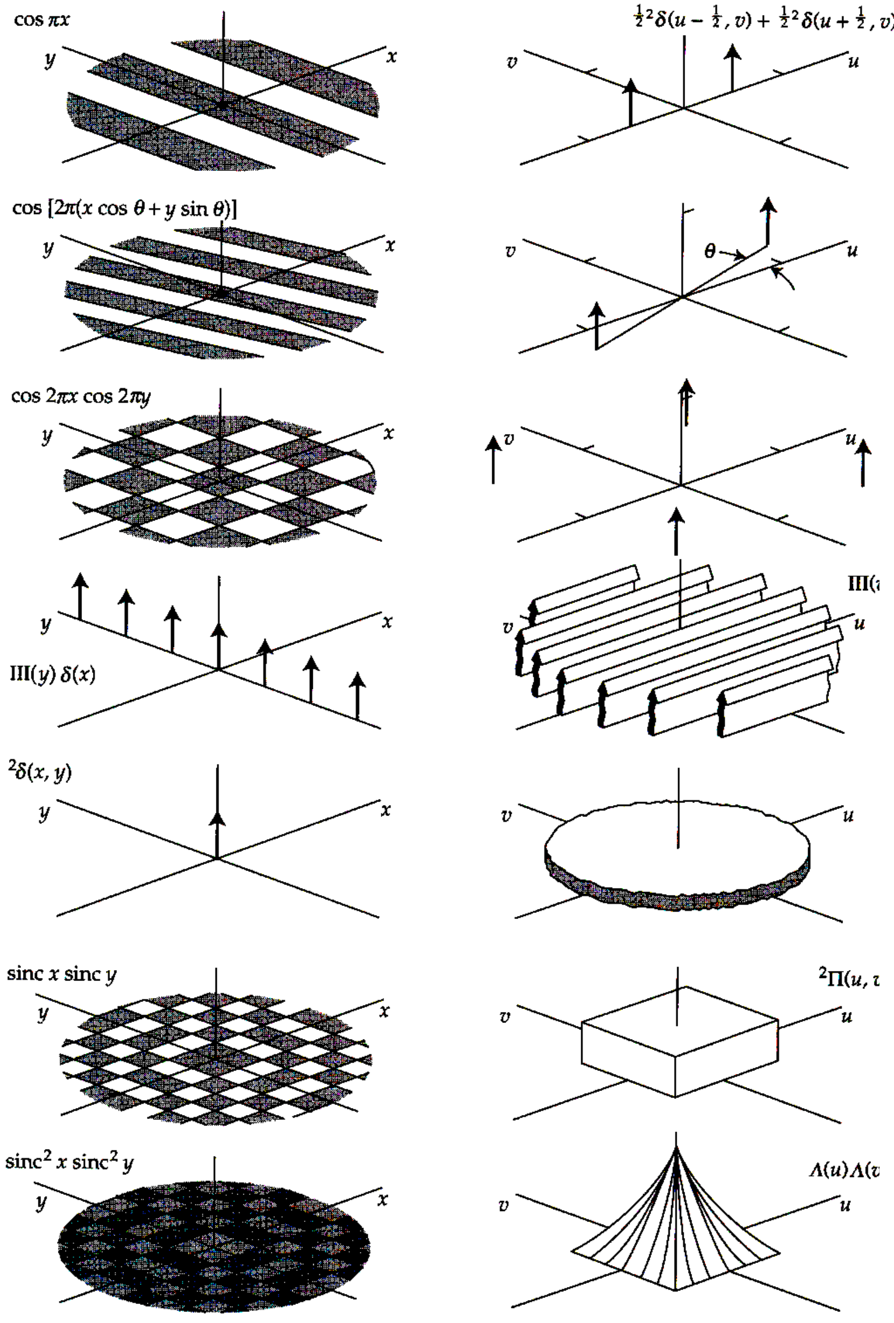
$$e^{-\pi r^2} \Leftrightarrow e^{-\pi\omega^2} \xrightarrow{\text{similarity}} e^{-\pi\left(\frac{r}{a}\right)^2} \Leftrightarrow |a| \cdot e^{-\pi(a\omega)^2}$$

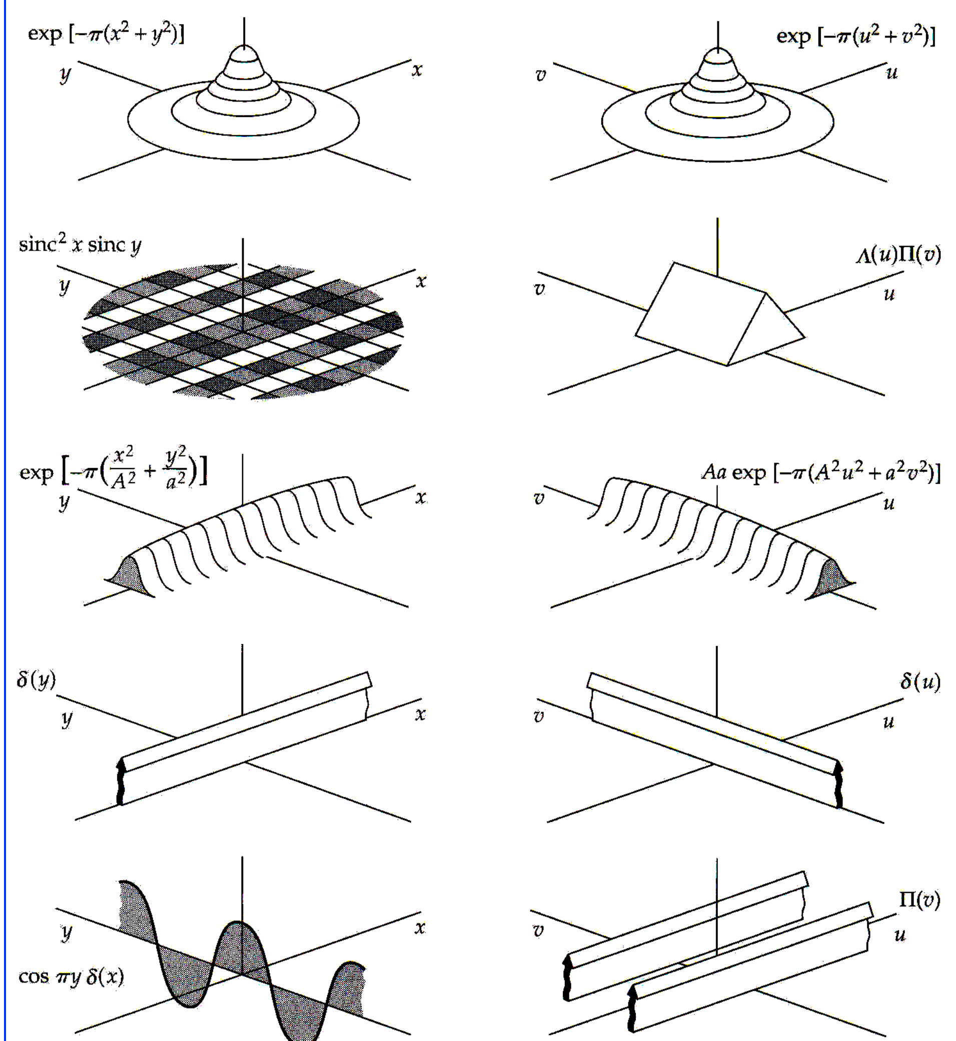




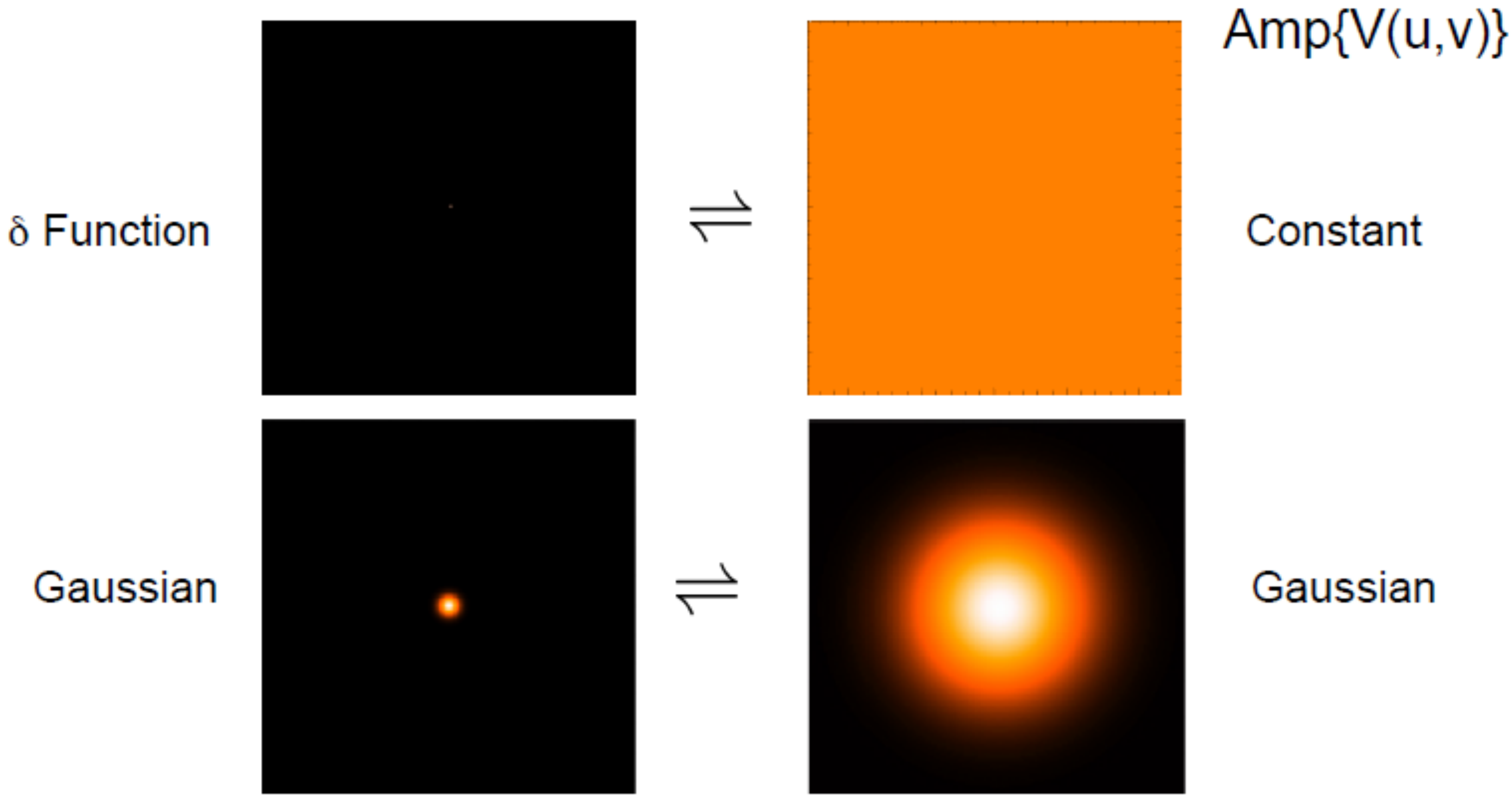
Gauss function Fourier transforms into Gauss function

Important 2-D Fourier Pairs



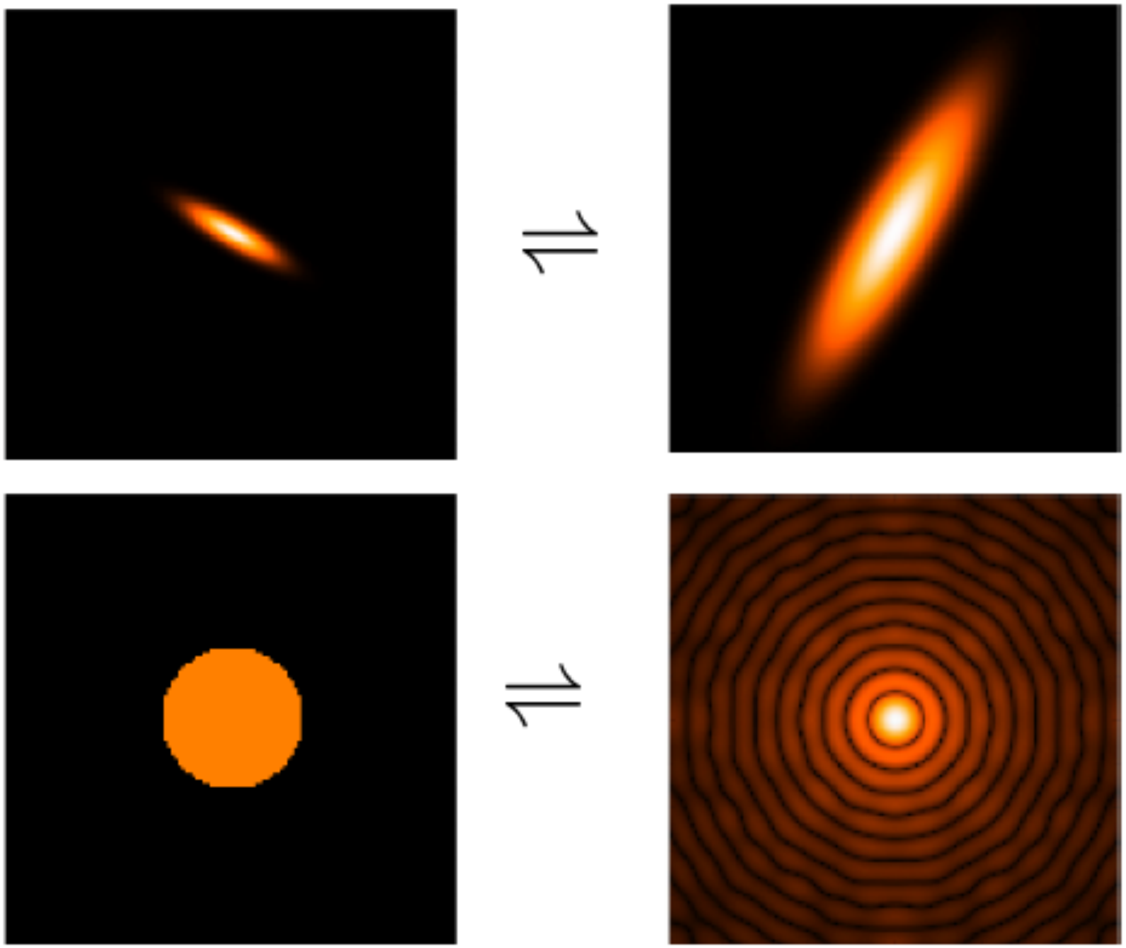


PUPIL (Telescope) ⇔ IMAGE (PSFs)



PUPIL (Telescope) ⇔ IMAGE (PSFs) (2)

Amp{V(u,v)}



elliptical Gaussian

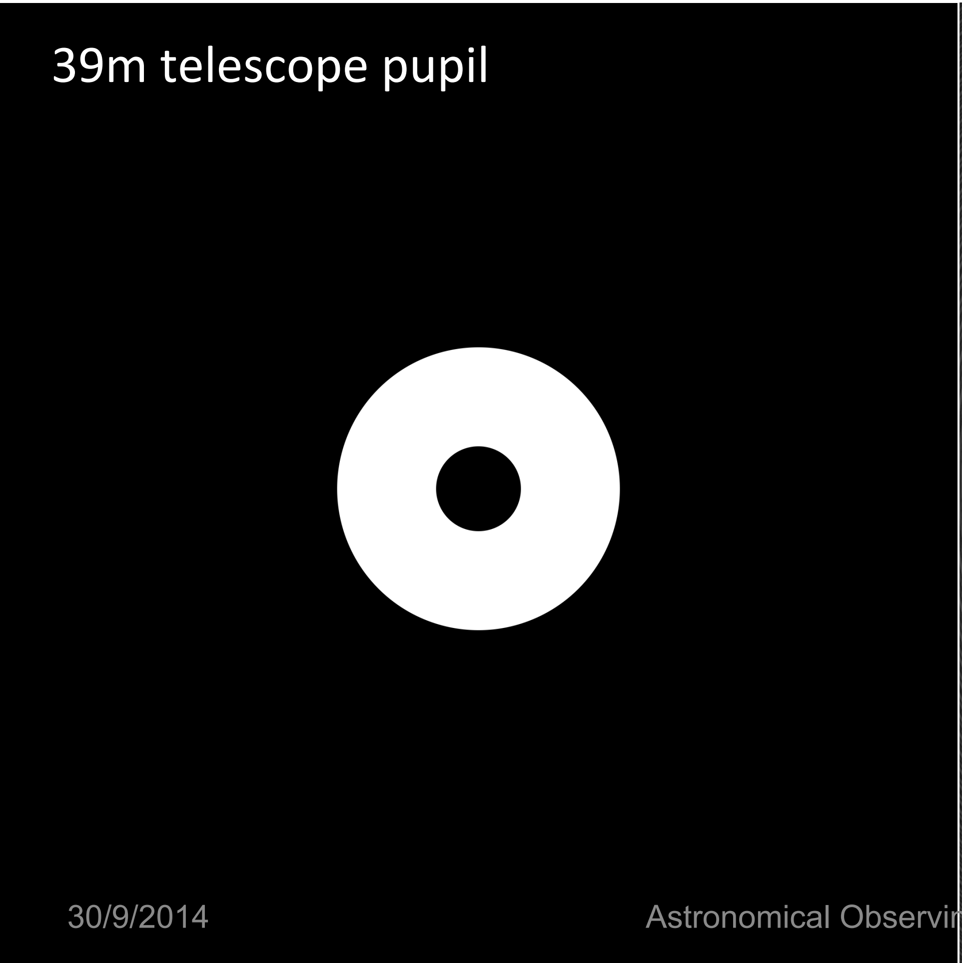
elliptical Gaussian

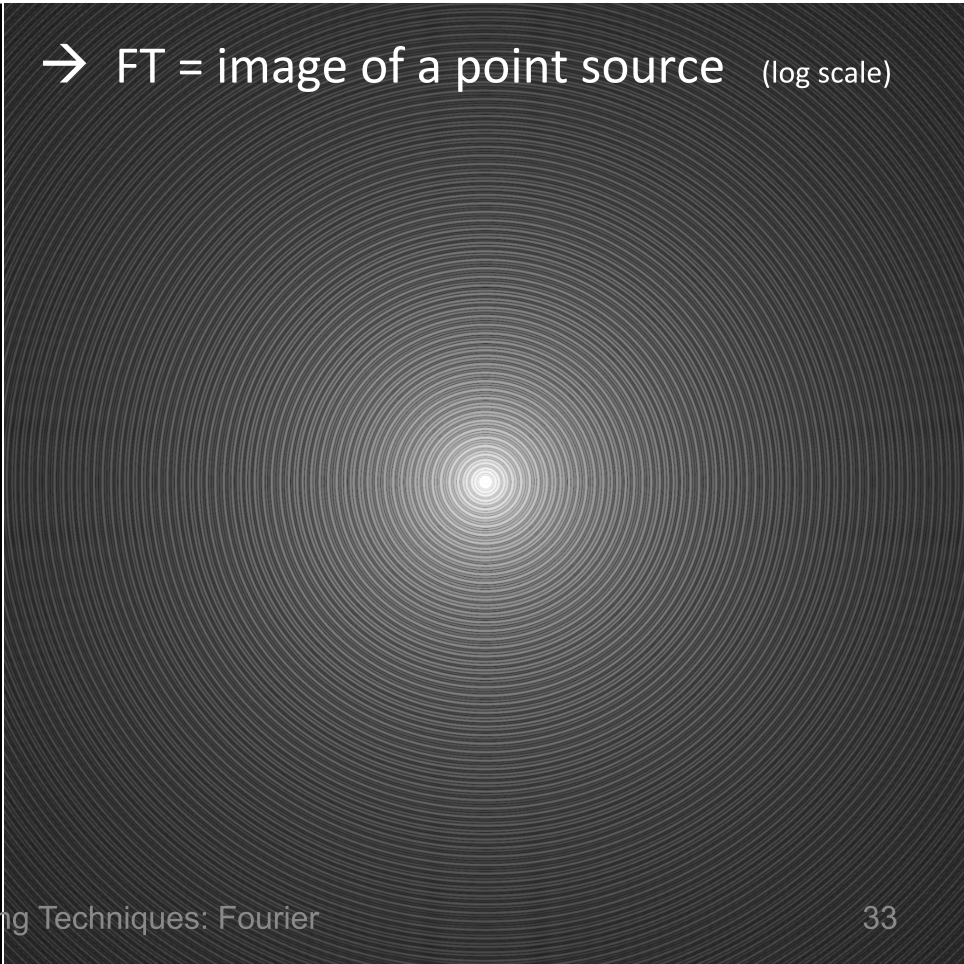
Disk

Bessel

Example 1:

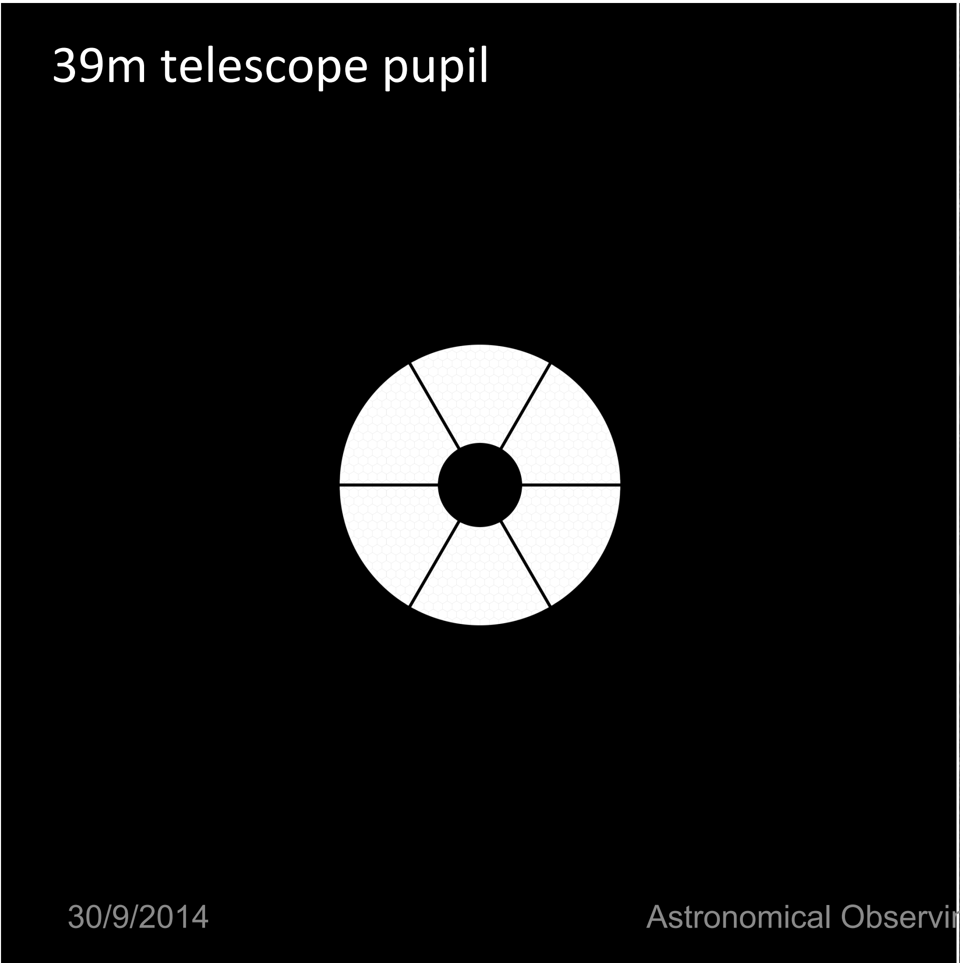
central obscuration, monolithic mirror (pupil) no support-spiders

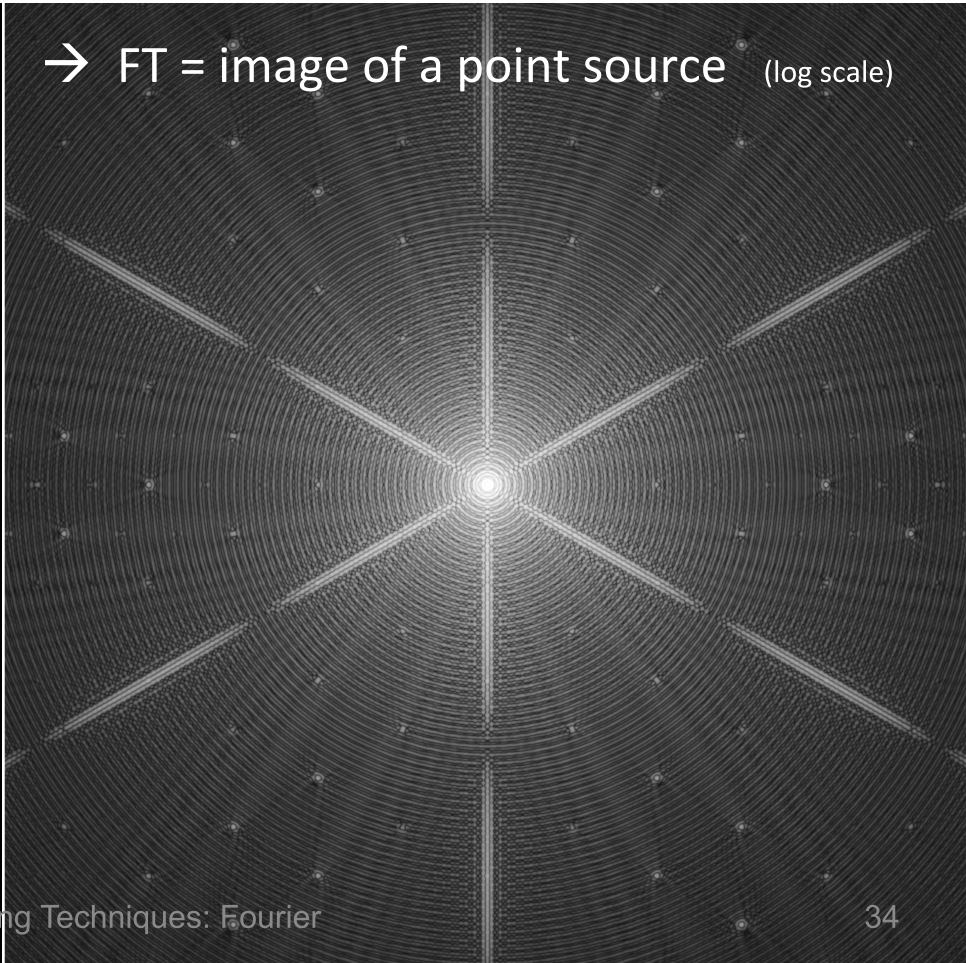




Example 2:

central obscuration, monolithic mirror (pupil) with 6 support-spiders





Example 3:

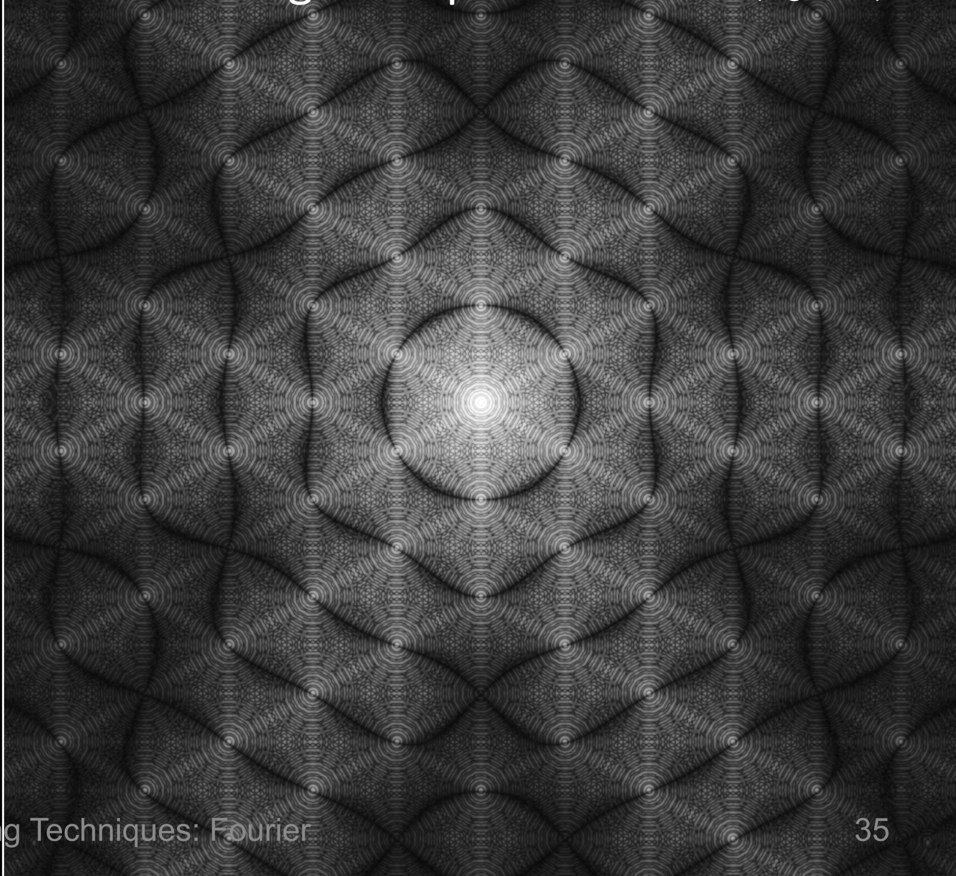
central obscuration,

segmented mirror (pupil)

no support-spiders

39m telescope pupil

→ FT = image of a point source (log scale)

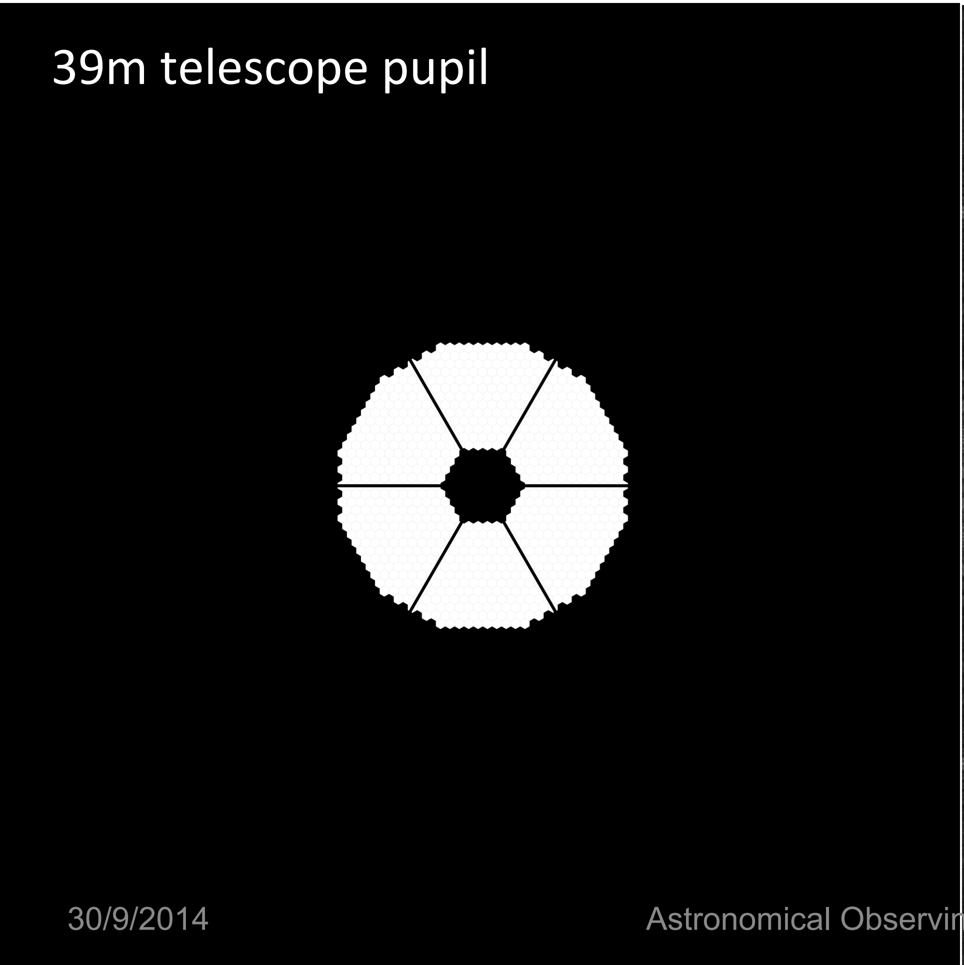


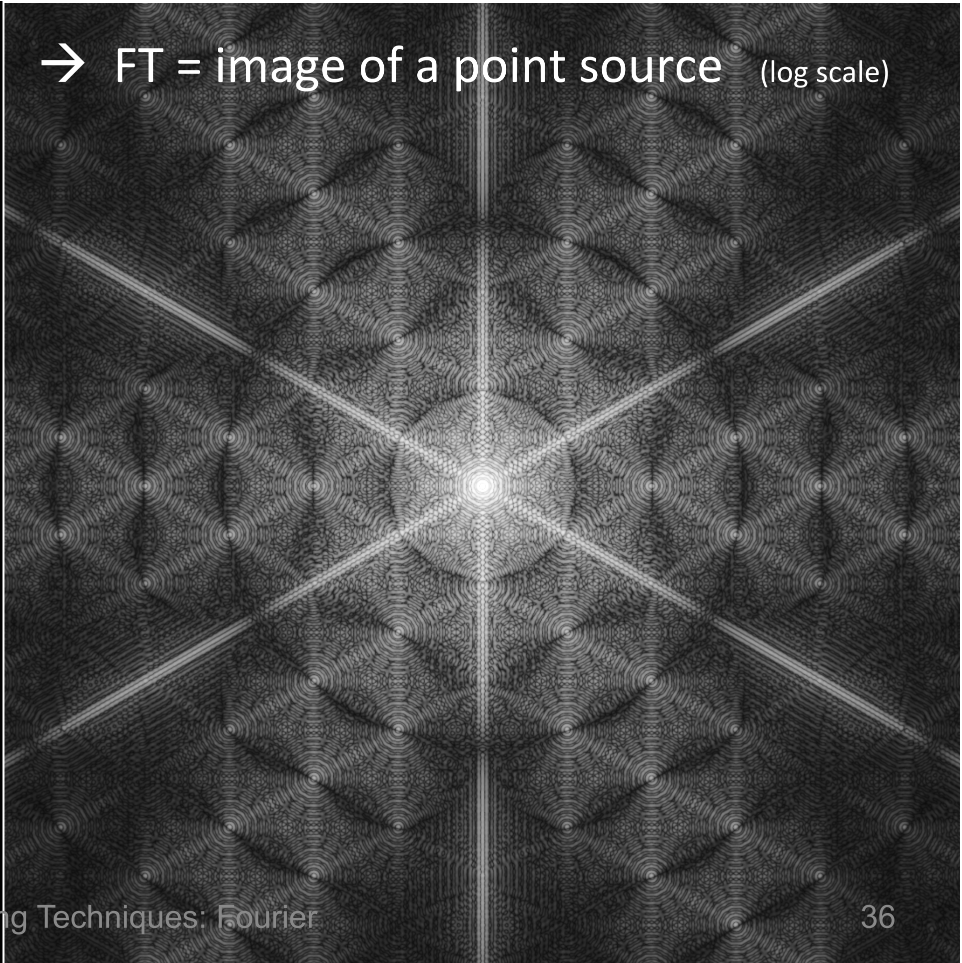
Example 4:

central obscuration,

segmented mirror (pupil)

with 6 support-spiders





Convolution

Convolution of two functions, f*g, is integral of product of functions after one is reversed and shifted:

$$h(x) = f(x) * g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x-u)\, du$$

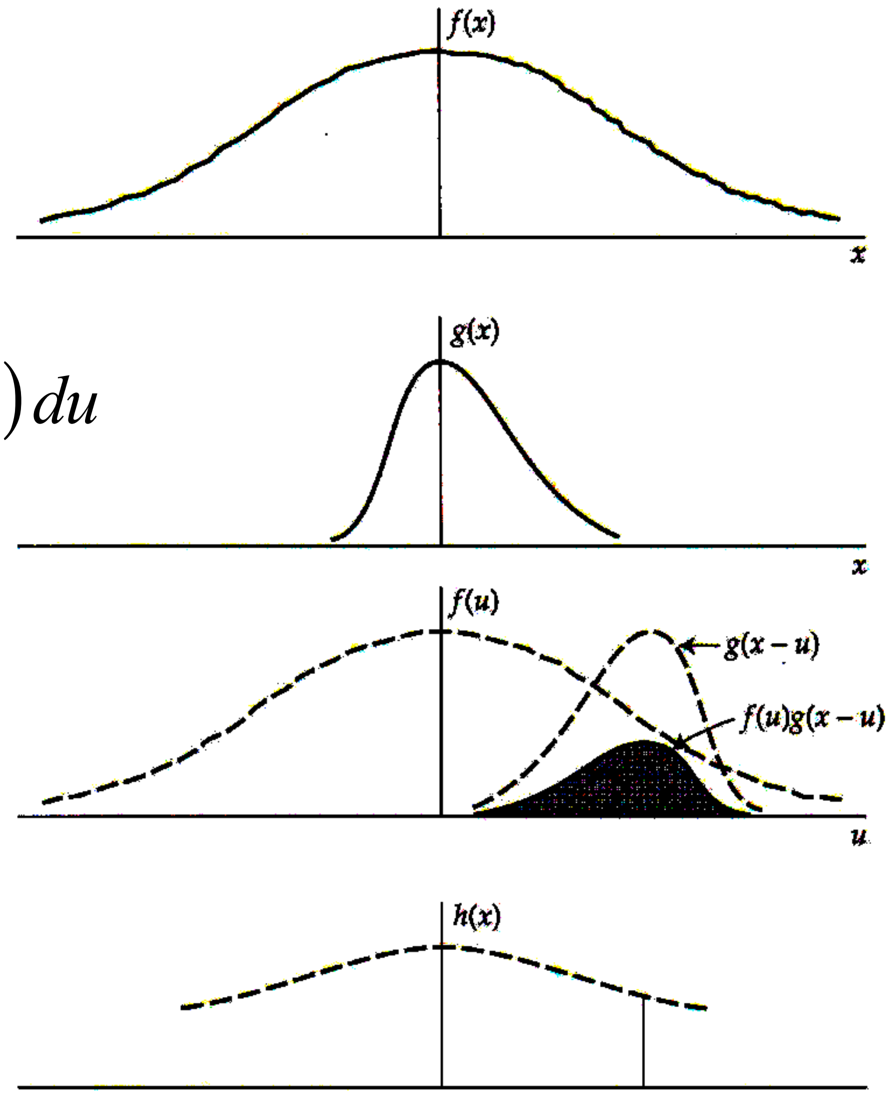
$$f(x) \Leftrightarrow F(s)$$

$$g(x) \Leftrightarrow G(s)$$

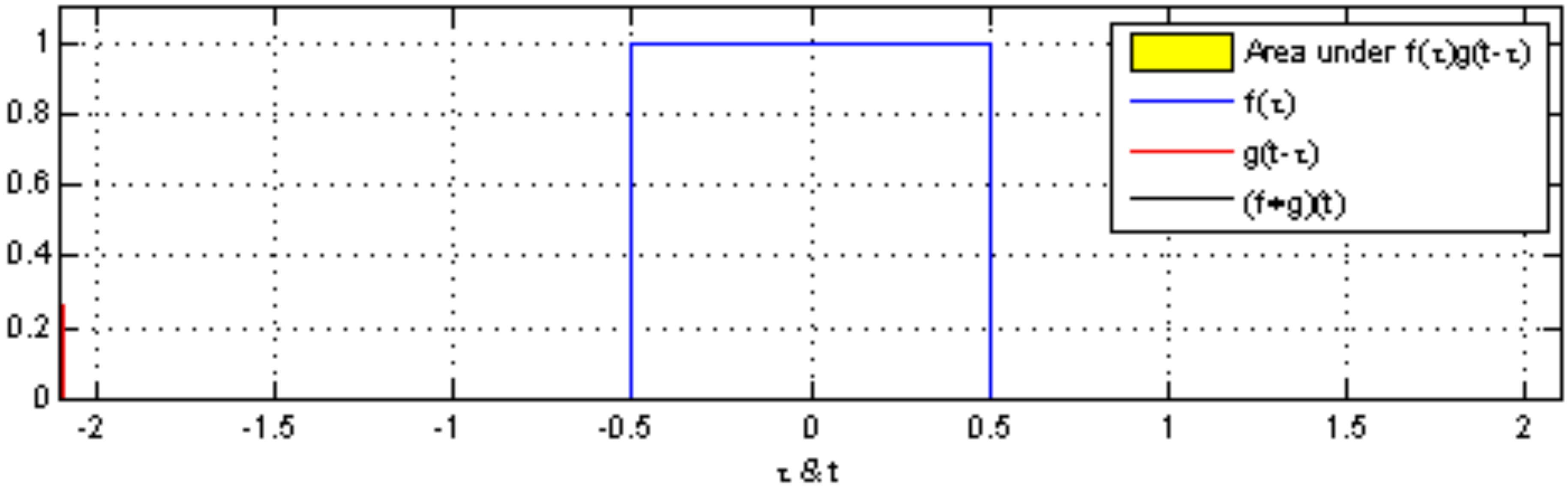
$$h(x) = f(x) * g(x)$$

$$\Leftrightarrow$$

$$F(s) \cdot G(s) = H(s)$$



Convolution: Example



Convolution: Applications

Example:

f(x) : star

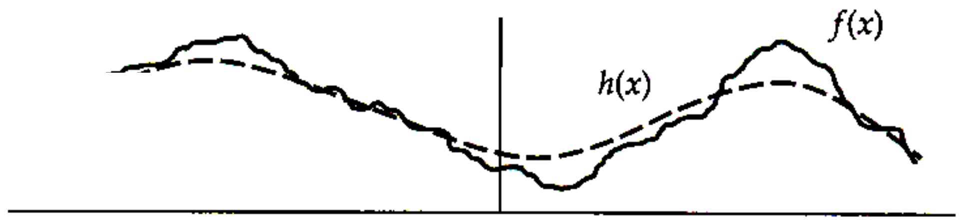
$$f(x) * g(x) = h(x)$$

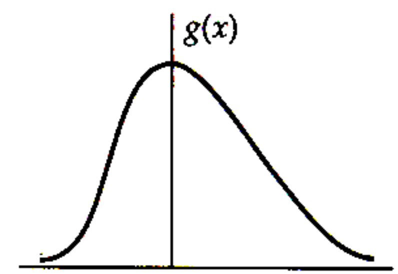
g(x) : telescope transfer function

Then *h(x)* is the point spread function (PSF) of the system

Example:

Convolution of *f(x)* with a smooth kernel g(x) can be used to smoothen *f(x)*

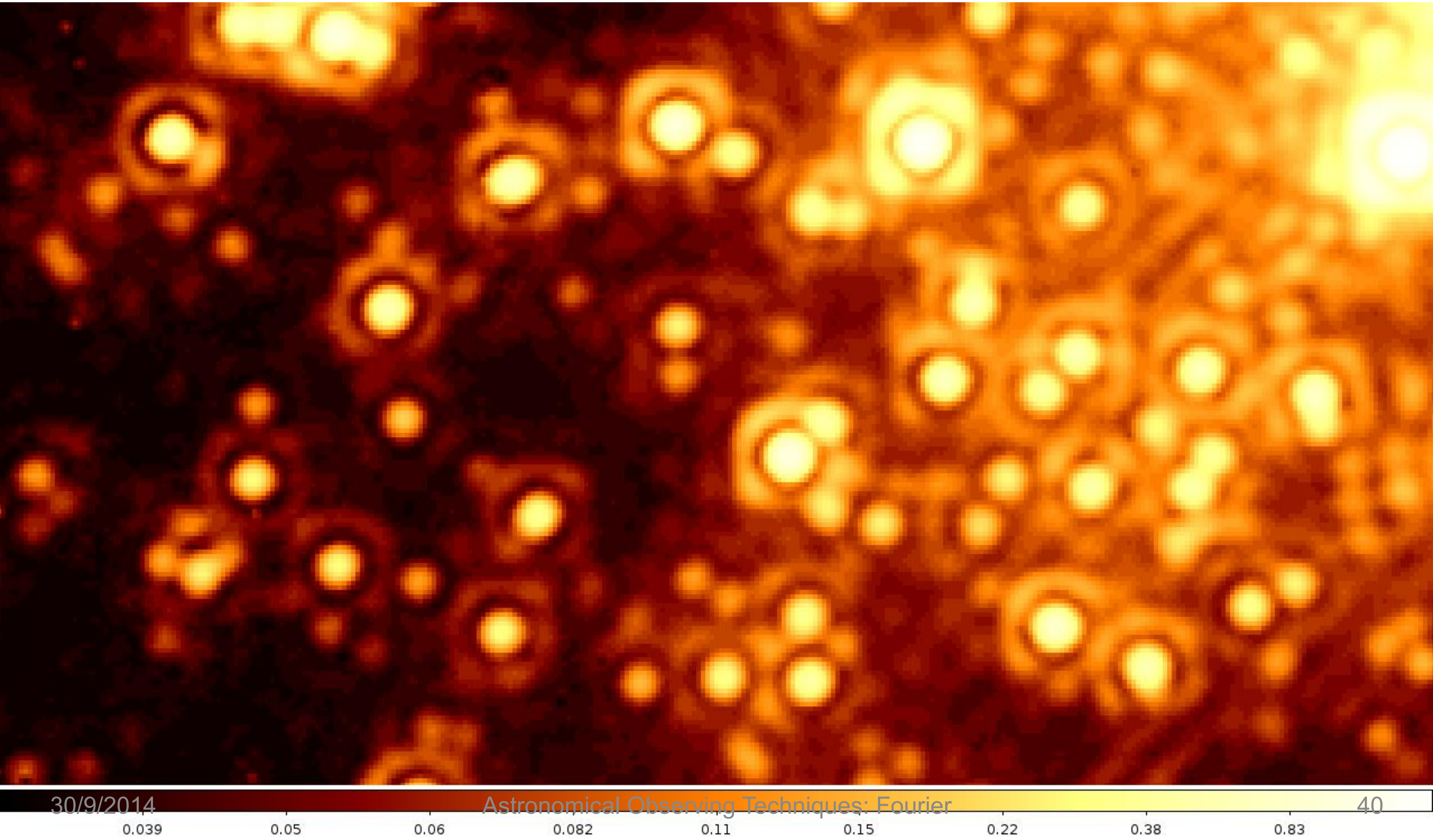




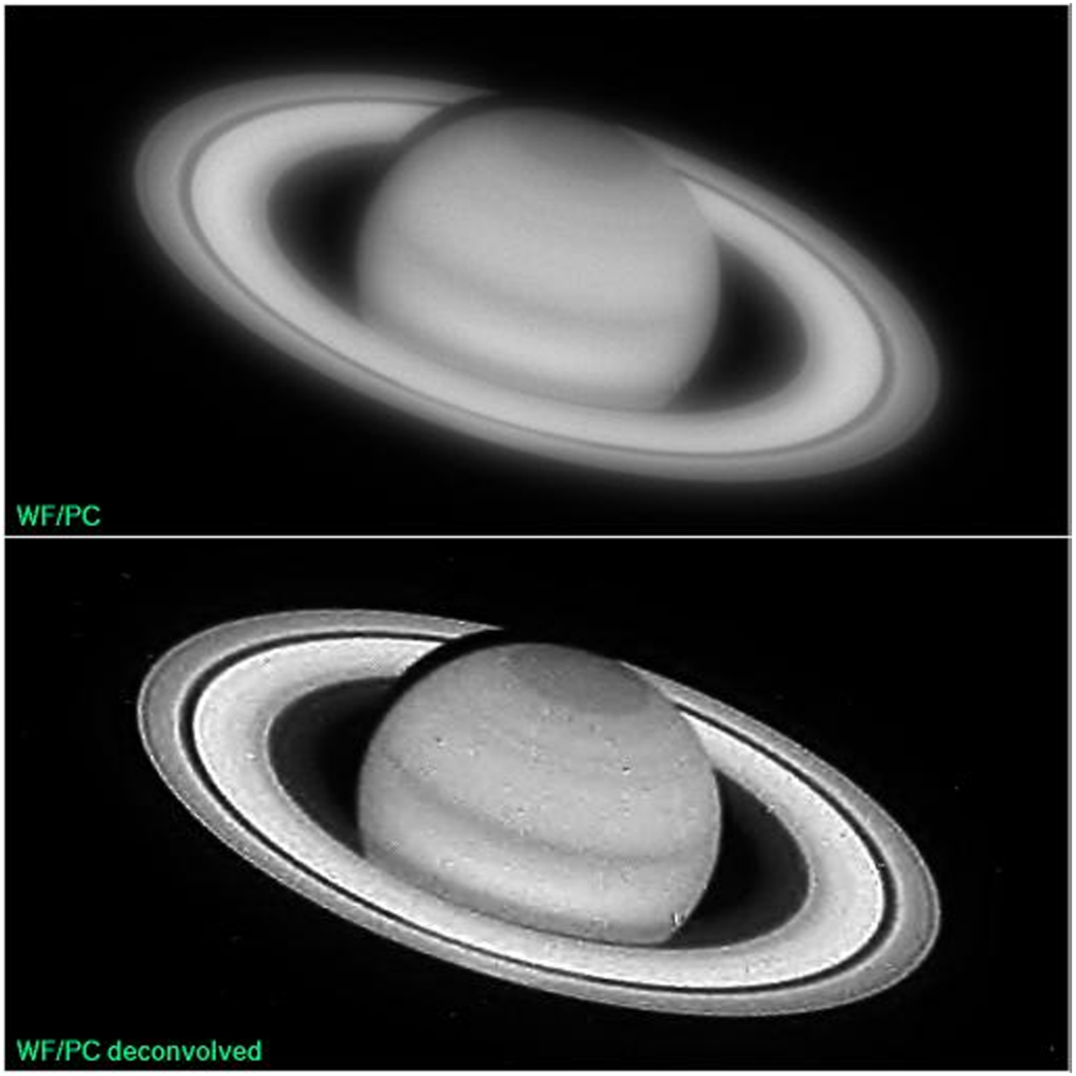
Example:

The inverse step (deconvolution) can be used to “disentangle” two components, e.g., removing the spherical aberration of a telescope.

Star cluster observed with HST/NICMOS



Deconvolution Example



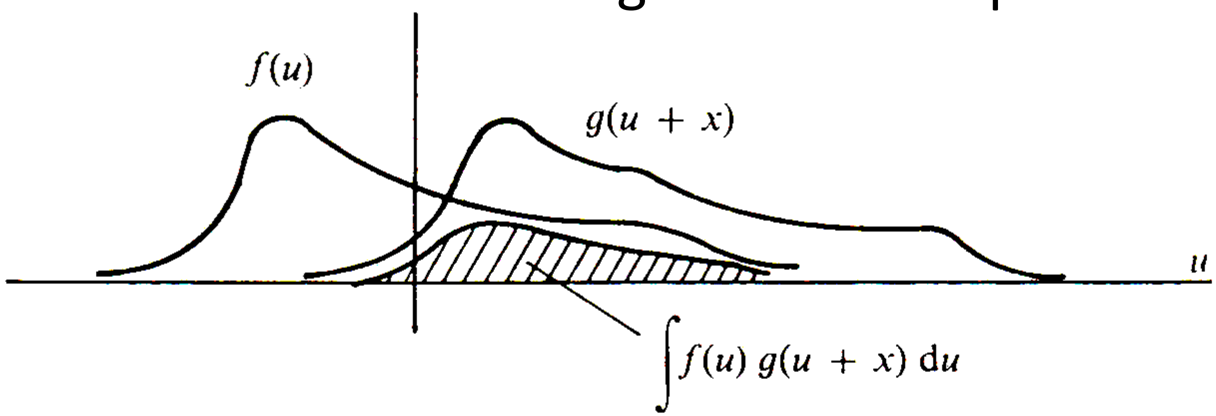
Cross-Correlation

Cross-correlation (or covariance) is measure of similarity of two waveforms as function of time-lag between them.

$$k(x) = f(x) \otimes g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x+u)\, du$$

Difference between cross-correlation and convolution:

- Convolution reverses the signal ('-' sign)
- Cross-correlation shifts the signal and multiplies it with another



Interpretation: By how much (x) must *g(u)* be shifted to match *f(u)*? Answer given by maximum of *k(x)*

Cross-Correlation in Fourier Space

$$f(x) \Leftrightarrow F(s)$$

$$g(x) \Leftrightarrow G(s)$$

$$h(x) = f(x) \otimes g(x) \Leftrightarrow F(s) \cdot G^*(s) = H(s)$$

In contrast to convolution, in general

$$f \otimes g \neq g \otimes f$$

Convolution and Cross-Correlation

The cross-correlation is a measure of similarity of two waveforms as a function of an offset (e.g., a time-lag) between them.

$$k(x) = f(x) \otimes g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x+u)\, du$$

Example: search a long duration signal for a shorter, known feature.

The convolution is similar in nature to the cross-correlation but the convolution first reverses the signal ("mirrors the function") prior to calculating the overlap.

$$h(x) = f(x) * g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x-u)\, du$$

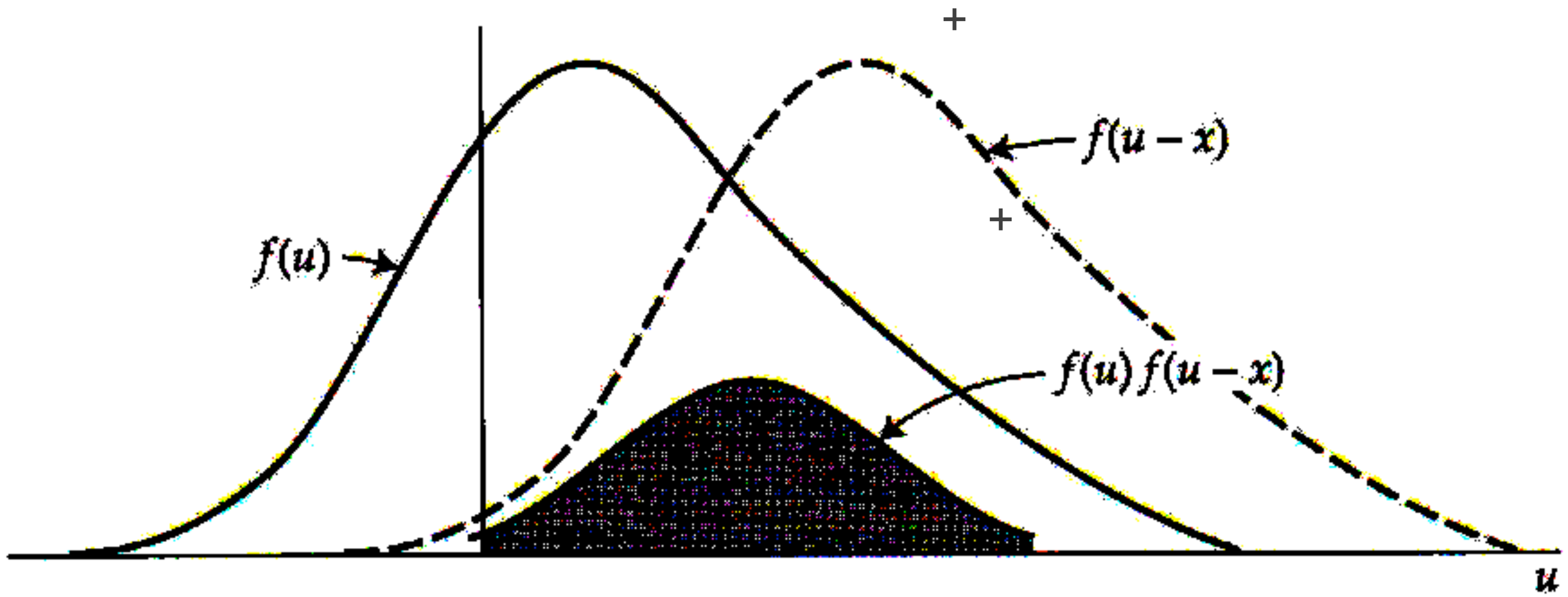
Example: the measured signal is the intrinsic signal convolved with the response function

Whereas convolution involves reversing a signal, then shifting it and multiplying by another signal, correlation only involves shifting it and multiplying (no reversing).

Auto-Correlation Theorem

Auto-correlation is cross-correlation of function with itself:

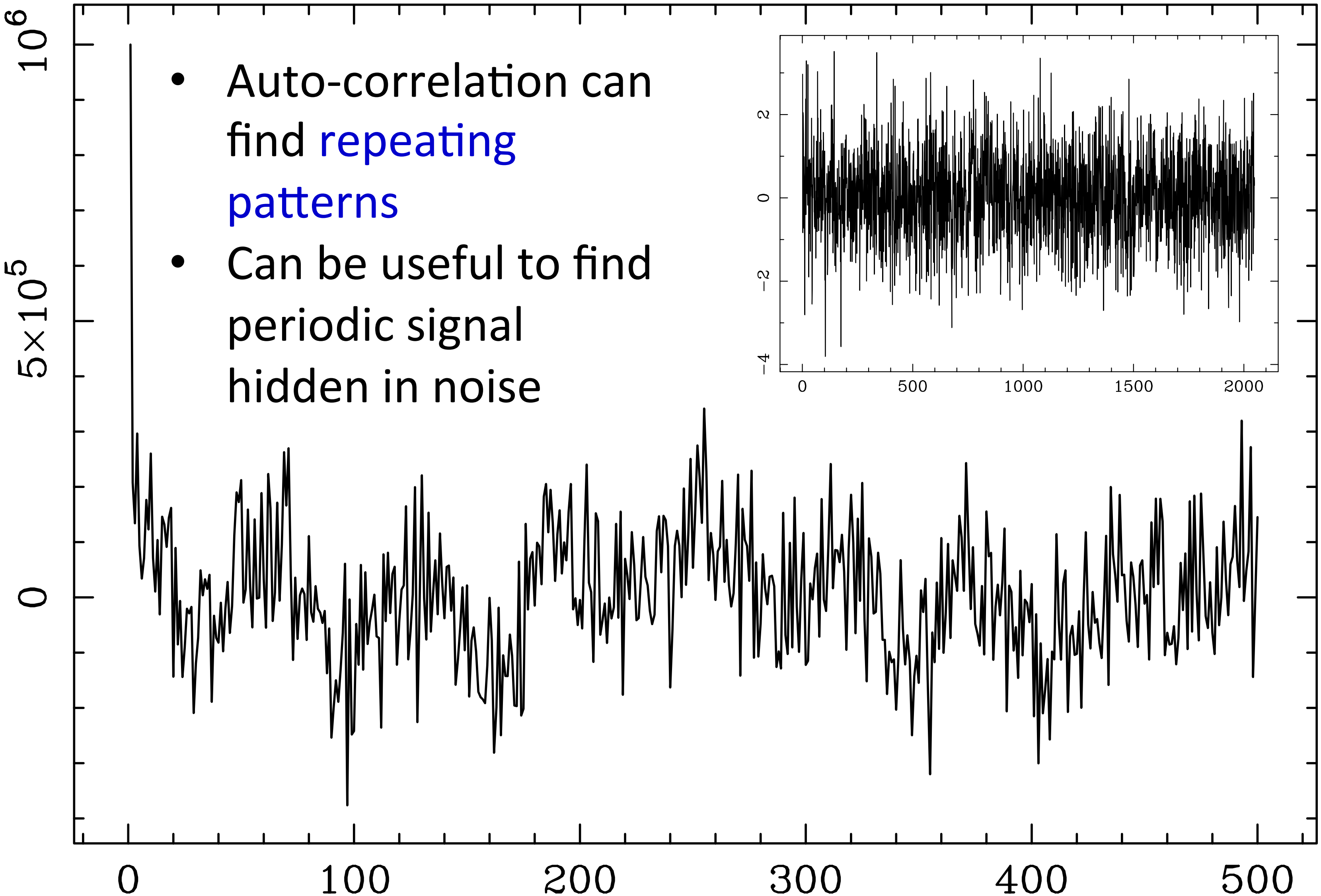
$$k(x) = f(x) \otimes f(x) = \int_{-\infty}^{+\infty} f(u) \cdot f(x+u)\, du$$



$$f(x) \otimes f(x) \Leftrightarrow F(s) F^{*}(s) = |F(s)|^{2}$$

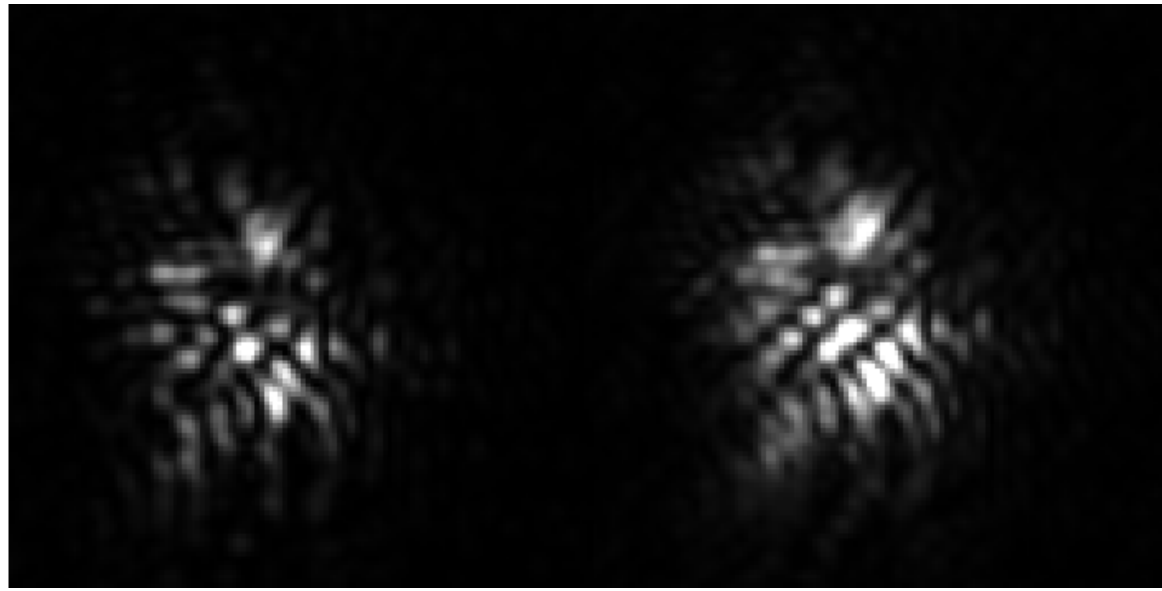
Auto-Correlation: Application

- Auto-correlation can find repeating patterns
- Can be useful to find periodic signal hidden in noise



Speckle Interferometry

- average auto-correlation of short-exposure images
- preserves high-resolution information



average cross-correlation

perfect image

Power Spectrum

Power Spectrum S_f of $f(x)$ (or the Power Spectral Density, PSD) describes how the power of a signal is distributed with frequency.

Power is often defined as squared value of signal:

$$S_f(s) = |F(s)|^2$$

Power spectrum is Fourier transform of autocorrelation and indicates what frequencies carry most of the energy.

Total energy of a signal is: $\int_{-\infty}^{+\infty} S_f(s)\,ds$

Applications: spectrum analyzers, calorimeters of light sources, …

Parseval's Theorem

Parseval's theorem (or Rayleigh's Energy Theorem) states that the sum of the square of a function is the same as the sum of the square of the Fourier transform:

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |F(s)|^2 ds$$

<u>Interpretation:</u> Total energy contained in signal *f(x)*, summed over all x is equal to total energy of signal's Fourier transform *F(s)* summed over all frequencies *s*.

Wiener-Khinchin Theorem

Wiener–Khinchin theorem states that the power spectral density S_f of a function *f(x)* is the Fourier transform of its auto-correlation function:

$$\left|F(s)\right|^2 = FT\{f(x)\otimes f(x)\}$$

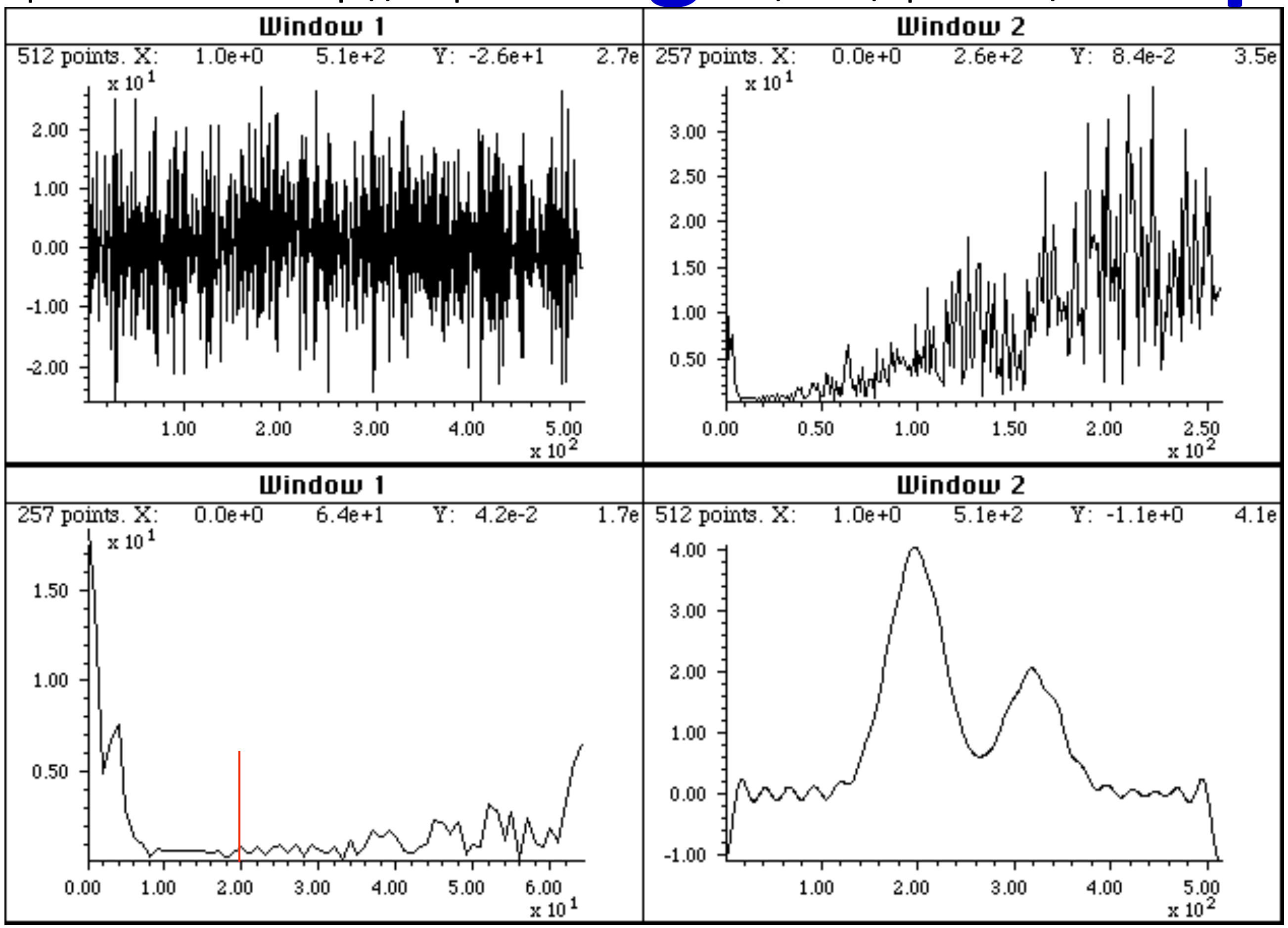
$$\updownarrow$$

$$F(s)\cdot F^*(s)$$

<u>Applications:</u> E.g. in the analysis of linear time-invariant systems, when the inputs and outputs are not square integrable, i.e. their Fourier transforms do not exist.

Fourier Filtering – an Example

Example taken from http://terpconnect.umd.edu/~toh/spectrum/FourierFilter.html



Top left: signal – is I just random noise?
Top right: power spectrum: high-frequency components dominate the signal
Bottom left: power spectrum expanded in X and Y to emphasize the low-frequency region.
Then: use Fourier filter function to delete all harmonics higher than 20
Bottom right: reconstructed signal → signal contains two bands at x=200 and x=300.

Fourier Relation Summary

Convolution	$h(x)=f(x)*g(x)=\int_{-\infty}^{+\infty} f(u)\cdot g(x-u)\,du$
Cross-correlation	$k(x)=f(x)\otimes g(x)=\int_{-\infty}^{+\infty} f(u)\cdot g(x+u)\,du$
Auto-correlation	$k(x)=f(x)\otimes f(x)=\int_{-\infty}^{+\infty} f(u)\cdot f(x+u)\,du$
Power spectrum	$S_f(s)=\left\lvert F(s)\right\rvert^2$
Parseval's theorem	$\int_{-\infty}^{+\infty}\left\lvert f(x)\right\rvert^2 dx=\int_{-\infty}^{+\infty}\left\lvert F(s)\right\rvert^2 ds$
Wiener-Khinchin theorem	$\left\lvert F(s)\right\rvert^2=FT\left\{f(x)\otimes f(x)\right\}=F(s)\cdot F^*(s)$