Astronomical Observing Techniques

Lecture 4: Signal & Noise

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Outline

1. Introduction
2. Statistics
3. Signal-to-Noise Ratio
4. Instrument Sensitivities

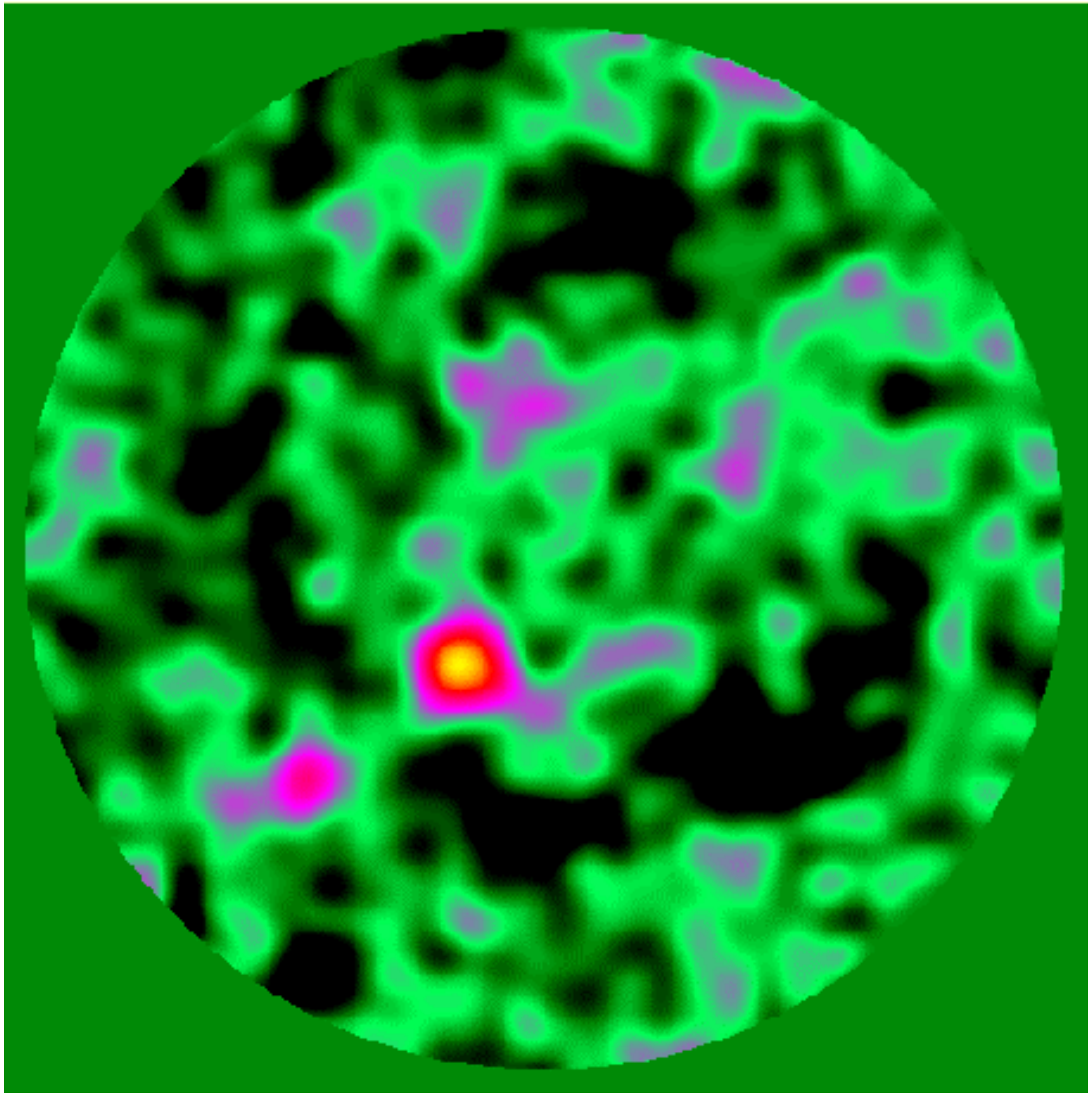
Noise

from Wikipedia:

- Common use: unwanted sound
- Signal processing: random unwanted data without meaning
- "Signal-to-noise ratio" is sometimes used to refer to the ratio of useful to irrelevant information

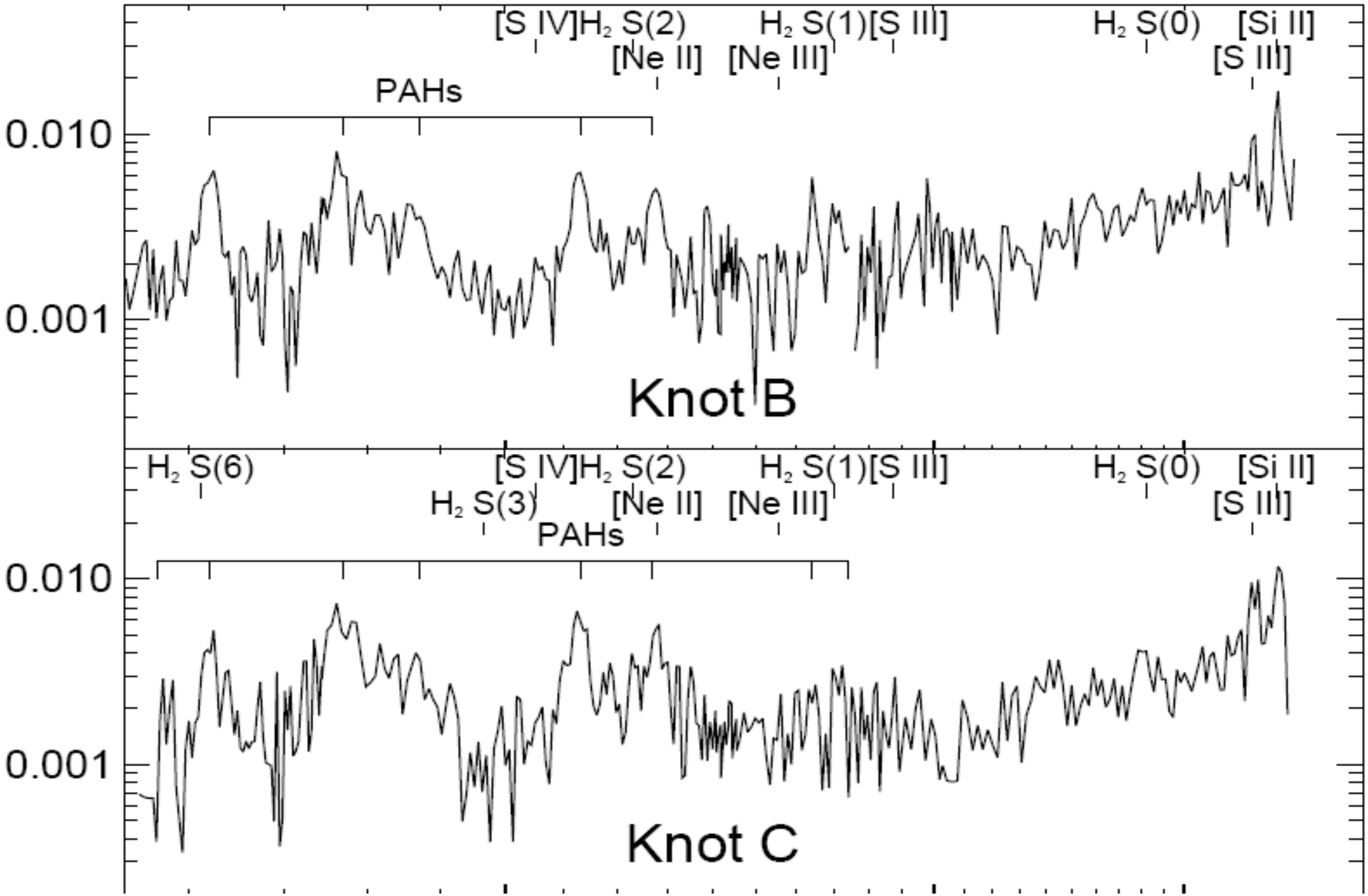
NASA researchers at Glenn Research Center conducting tests on aircraft engine noise in 1967

Signal?



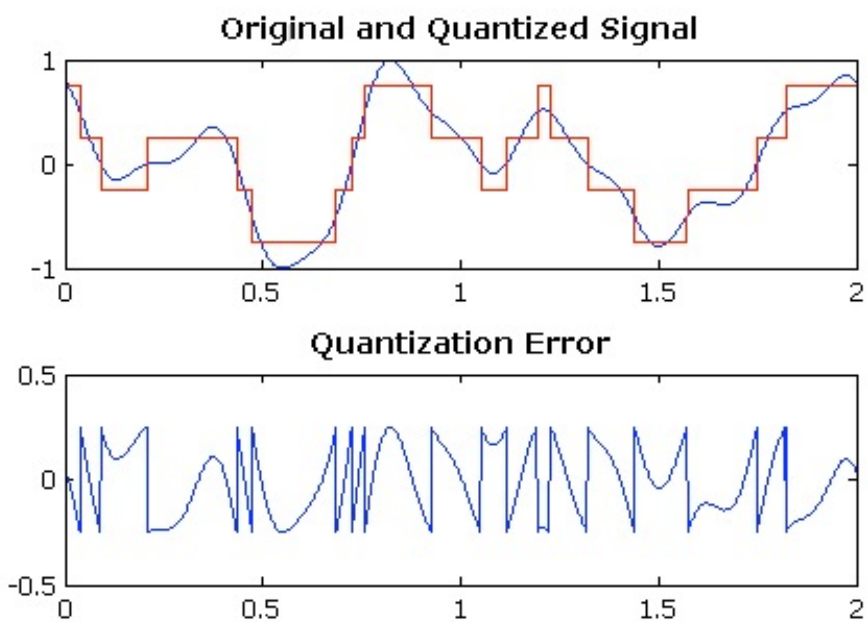
SCUBA 850μm map of the Hubble deep field

Noise or Signal?



Digitization/Quantization Noise

- Analog-to-Digital Signal Converter (ADC).
- Number of bits determines dynamic range of ADC
- Resolution: 12 bit 2^{12} = 4096 quantization levels
  16 bit 2^{16} = 65636 quantization levels
- Discrete, "artificial" steps in signal levels → noise



Some Noise Sources in Astronomical Data

Noise type	Signal	Background
Photon shot noise	X	X
Scintillation	X	
Cosmic rays		X
Image stability	X	
Read noise	X	X
Dark current noise	X	X
CTE (CCDs)	X	X
Flat fielding (non-linearity)	X	X
Digitization noise	X	X
Other calibration errors	X	X
Image subtraction	X	X

Distribution Functions

- for every *t*, *X(t)* is distributed according to cumulative distribution function

$$F(x;t) = \mathbf{P}\{X(t) \le x\}$$

- indicates probability that outcome at *t* will not exceed *x*
- probability density function (PDF) of *X(t)* defined by

$$f(x;t) \equiv \frac{\partial F(x;t)}{\partial x}$$

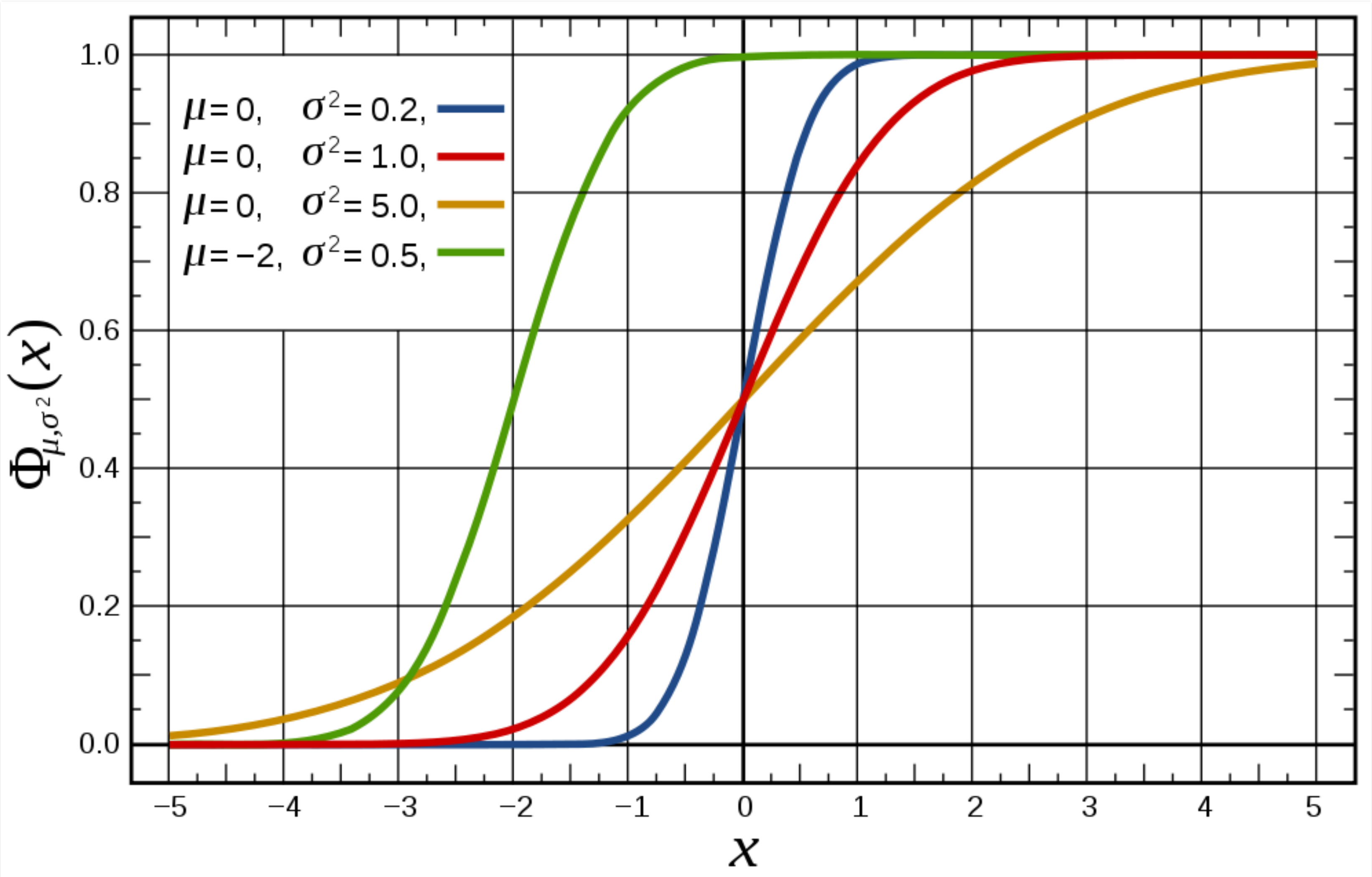
- probability density function often

binomial: $\binom{n}{k} p^k (1-p)^{n-k}$

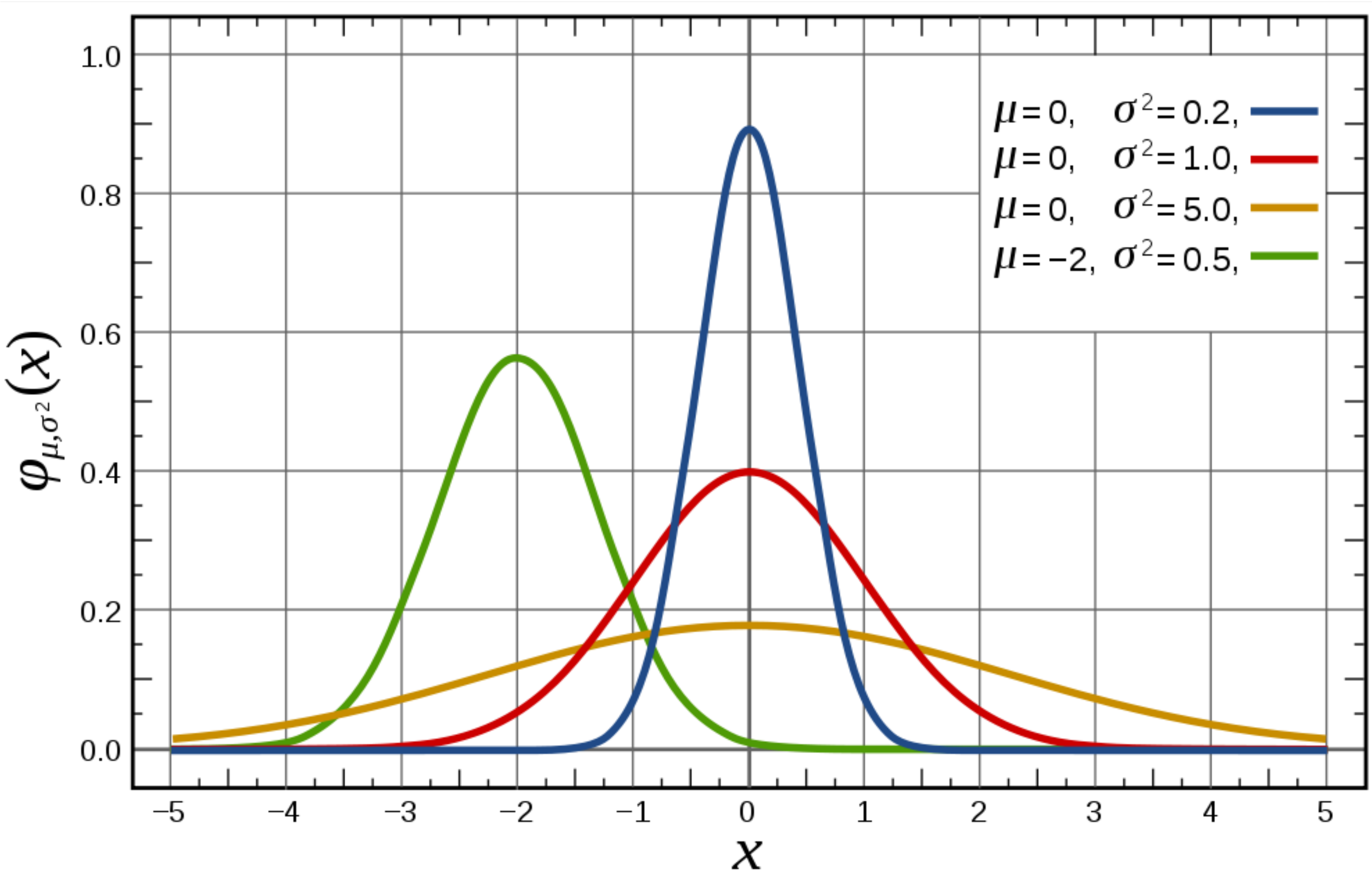
Poisson: $f(k;\lambda) = \dfrac{\lambda^k e^{-\lambda}}{k!}$

Gaussian (normal): $\dfrac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Normal Cumulative Distribution

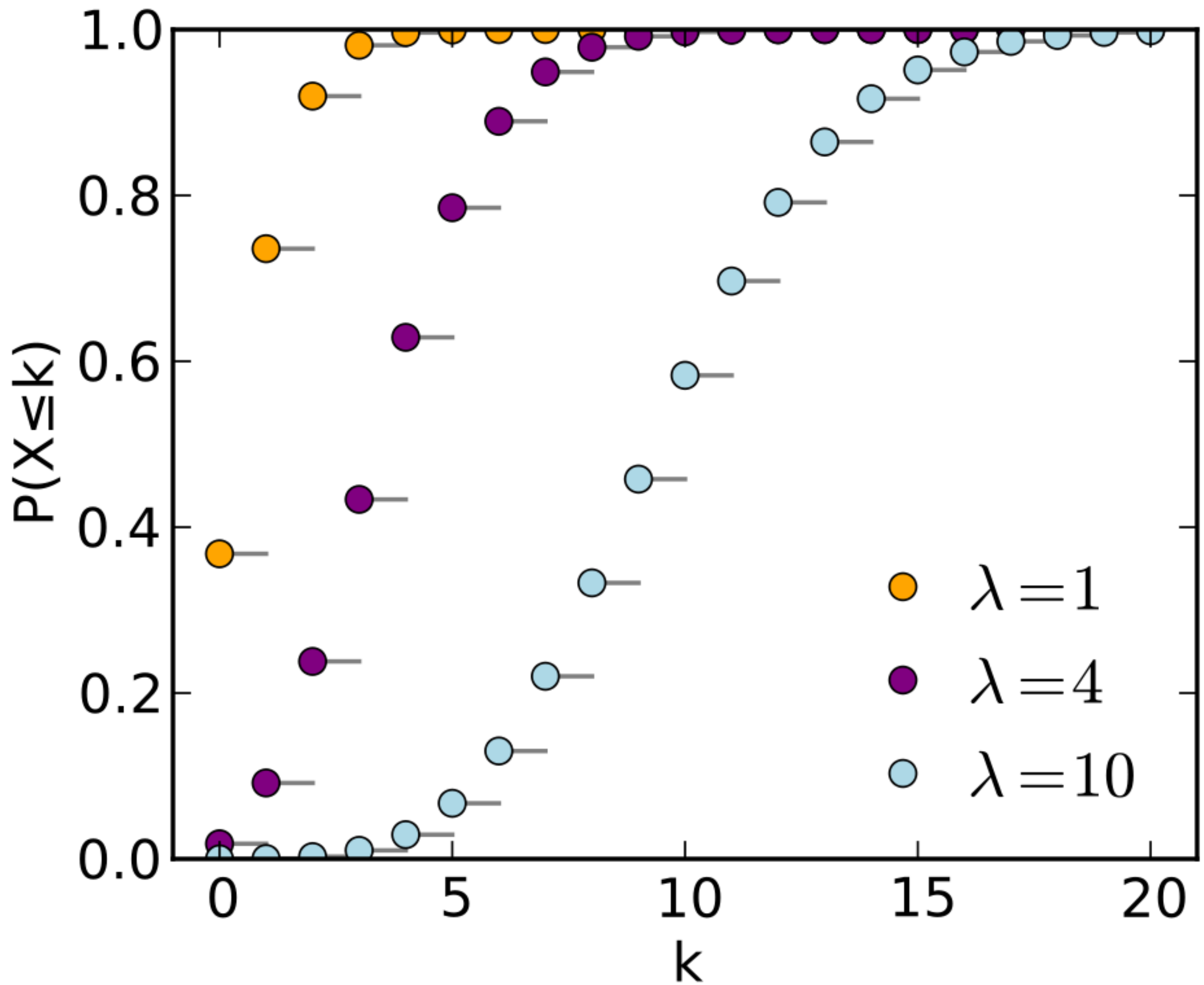


Normal PDF

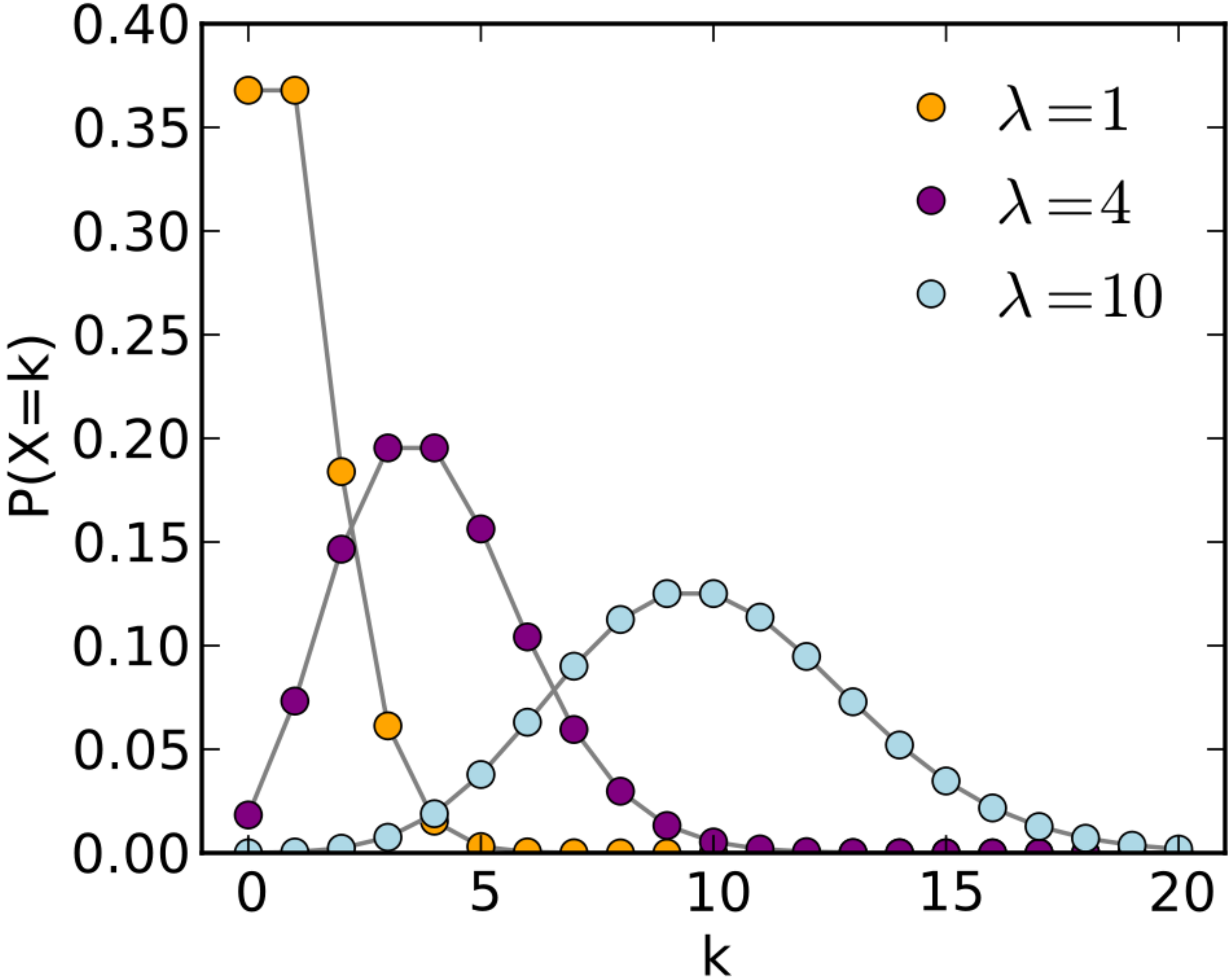


$\mu = 0, \quad \sigma^2 = 0.2,$
$\mu = 0, \quad \sigma^2 = 1.0,$
$\mu = 0, \quad \sigma^2 = 5.0,$
$\mu = -2, \quad \sigma^2 = 0.5,$
$\varphi_{\mu,\sigma^2}(x)$
x
1.0
0.8
0.6
0.4
0.2
0.0
−5
−4
−3
−2
−1
0
1
2
3
4
5

Poisson Cumulative Distribution Function



Poisson PDF



Mean, Variance, RMS

- properties of distributions often described by a few parameters, often moments of distribution
- mean or average *μ(t)* of *X(t)*: expected value of *X(t)*

$$\mu(t) = \mathbf{E}\{X(t)\} = \int_{-\infty}^{+\infty} x\, f(x;t)\, dx$$

- variance of X(t): expected value of the square of the difference of *X(t)* and *μ(t)*

$$\sigma^2(t) = \mathbf{E}\{(X(t) - \mu(t))^2\} = \mathbf{E}\{X^2(t)\} - \mu^2(t)$$

- variance is square of standard deviation or root mean square (RMS)

Noise Propagation

- same as error propagation
- function *f(u,v,...)* depends on variables *u,v, ...*
- estimate variance of *f* knowing variances σ_u^2, σ_v^2,... of variables *u,v, ...*

$$\sigma_f^{\ 2} \equiv \lim_{N\to\infty} \sum_{i=1}^{N} (f_i - \overline{f})^2$$

- make assumption / approximately that average of *f* is well approximated by value of *f* for averages of variables:

$$\overline{f} = f(\overline{u}, \overline{v}, \ldots)$$

Noise Propagation (cont.)

- Taylor expansion of *f* around average:

$$f_i - \overline{f} \simeq (u_i - \overline{u})\frac{\partial f}{\partial u} + (v_i - \overline{v})\frac{\partial f}{\partial v} + \dots$$

- variance in f:

$$\sigma_f{}^2 \simeq \lim_{N\to\infty}\sum_{i=1}^{N}\left[(u_i - \overline{u})\frac{\partial f}{\partial u} + (v_i - \overline{v})\frac{\partial f}{\partial v} + \dots\right]^2$$

$$= \lim_{N\to\infty}\sum_{i=1}^{N}\left[(u_i - \overline{u})^2\left(\frac{\partial f}{\partial u}\right)^2 + (v_i - \overline{v})^2\left(\frac{\partial f}{\partial v}\right)^2 + 2(u_i - \overline{u})(v_i - \overline{v})\frac{\partial f}{\partial u}\frac{\partial f}{\partial v} + \dots\right]$$

Noise Propagation (cont.)

- variances of *u* and *v*

$$\sigma_u^{\ 2} \equiv \lim_{N\to\infty} \sum_{i=1}^{N} (u_i - \overline{u})^2; \qquad \sigma_v^{\ 2} \equiv \lim_{N\to\infty} \sum_{i=1}^{N} (v_i - \overline{v})^2$$

- covariance of *u* and *v*

$$\sigma_{uv}^{\ 2} \equiv \lim_{N\to\infty} \sum_{i=1}^{N} (u_i - \overline{u})(v_i - \overline{v})$$

- combine Taylor expansion and these definitions

$$\sigma_f^{\ 2} = \sigma_u^{\ 2} \left(\frac{\partial f}{\partial u} \right)^2 + \sigma_v^{\ 2} \left(\frac{\partial f}{\partial v} \right)^2 + 2\sigma_{uv}^{\ 2} \frac{\partial f}{\partial u} \frac{\partial f}{\partial v} + \ldots$$

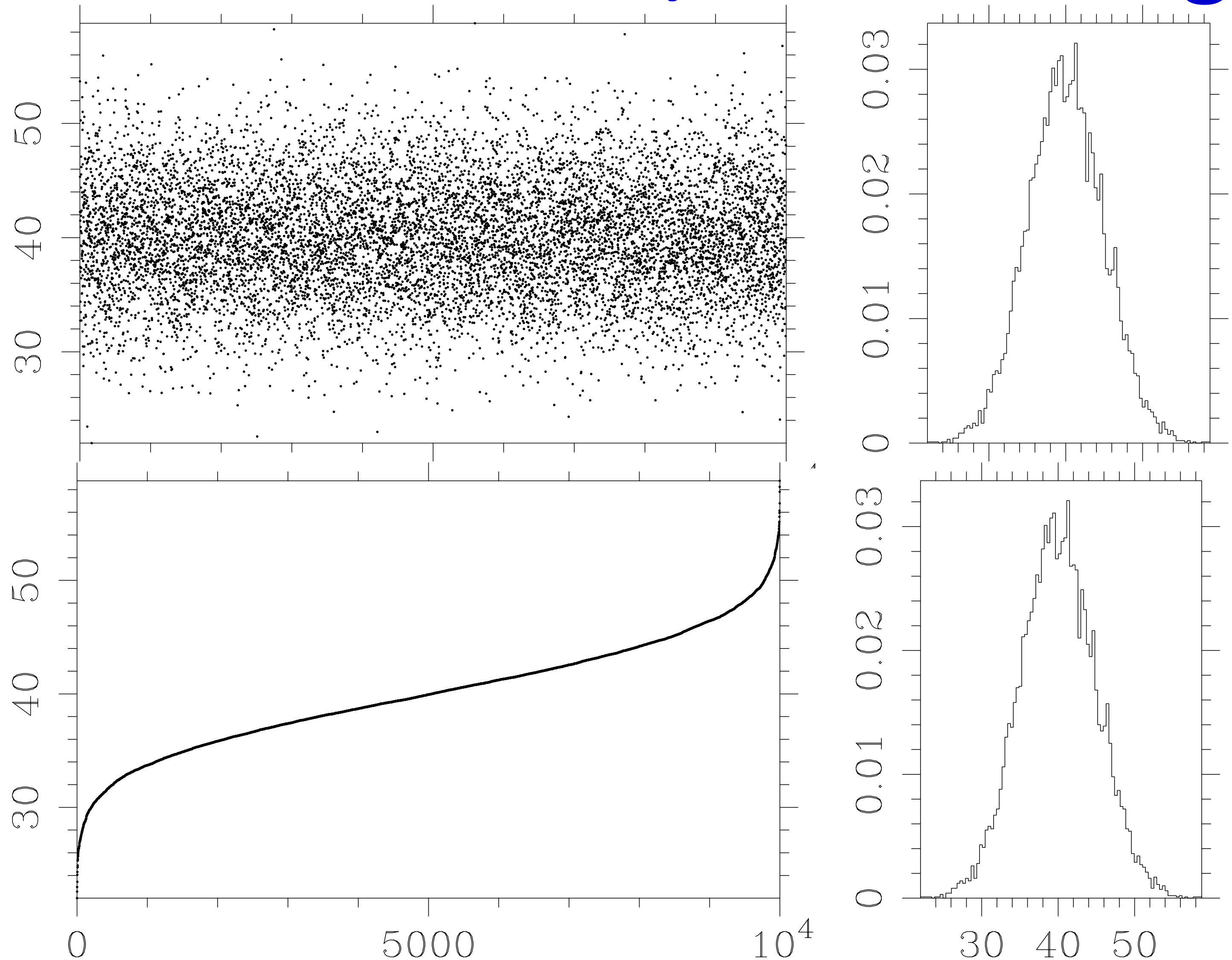
Noise Propagation (cont.)

- from before

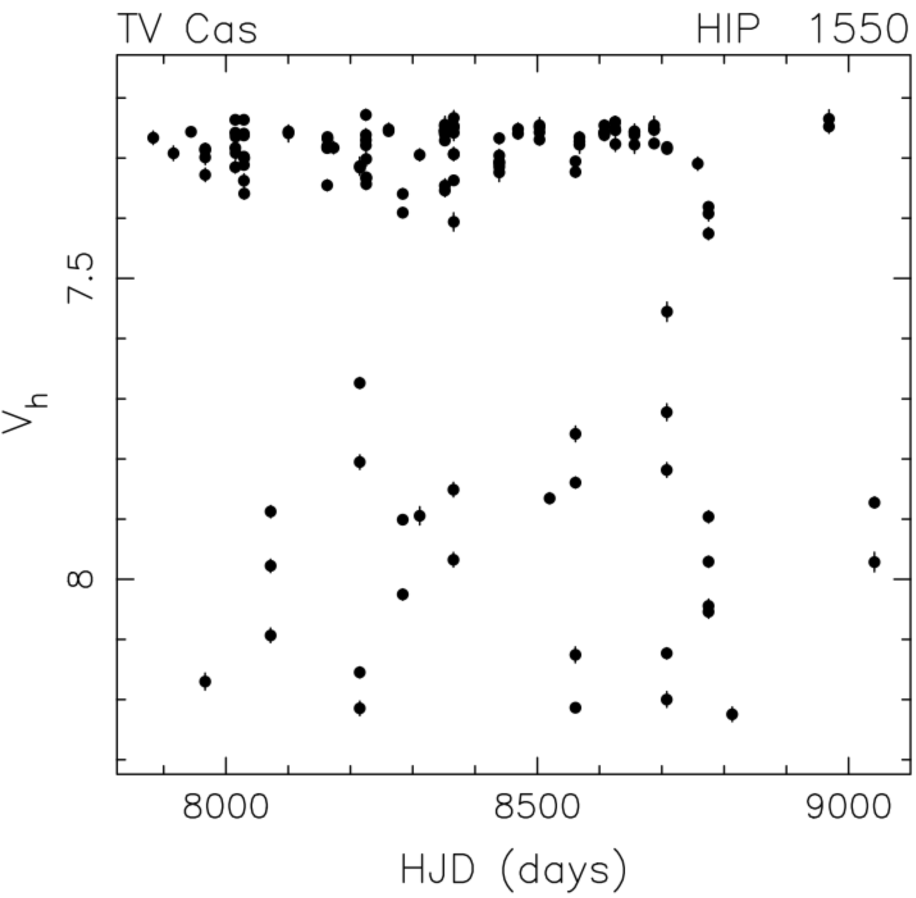
$$\sigma_f^{\,2} = \sigma_u^{\,2}\left(\frac{\partial f}{\partial u}\right)^2 + \sigma_v^{\,2}\left(\frac{\partial f}{\partial v}\right)^2 + 2\sigma_{uv}^{\,2}\frac{\partial f}{\partial u}\frac{\partial f}{\partial v} + \ldots$$

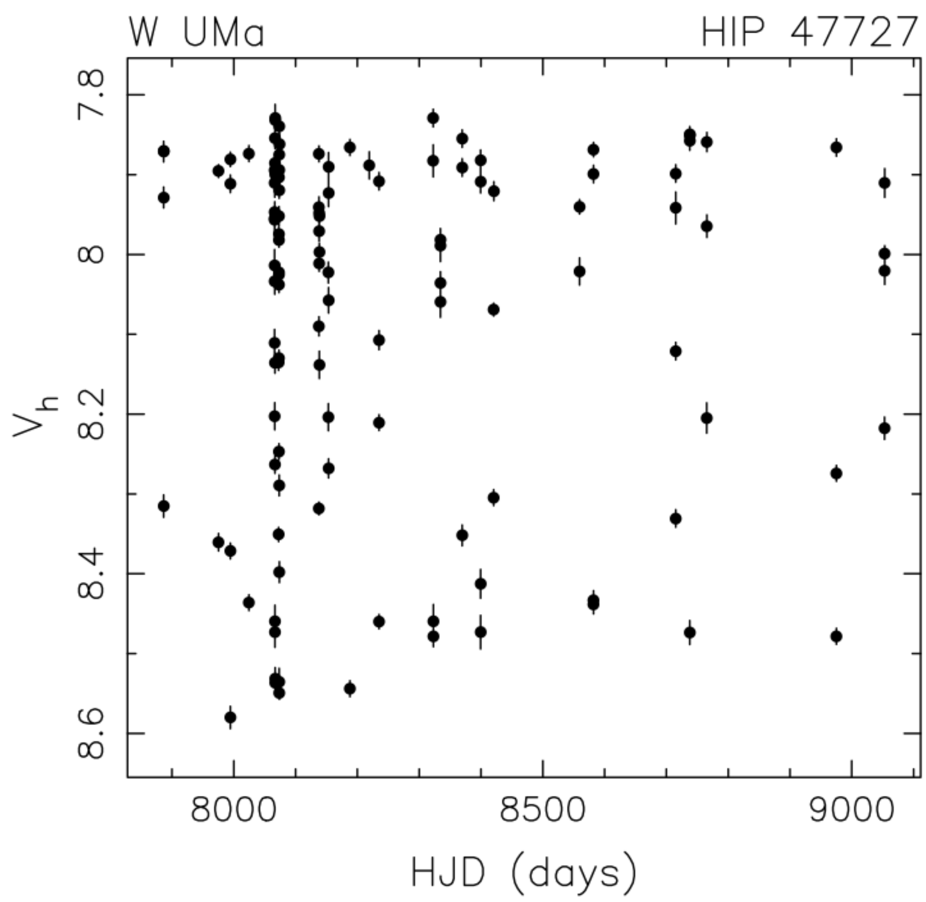
- if differences $(u_i - \bar{u})$ and $(v_i - \bar{v})$ not correlated ➔ sign of product as often positive as negative ➔ covariance small compared to other terms
- if differences are correlated ➔ most products $(u_i - \bar{u})(v_i - \bar{v})$ have the same sign ➔ cross-correlation term can be large

Same Distribution, Different Signals

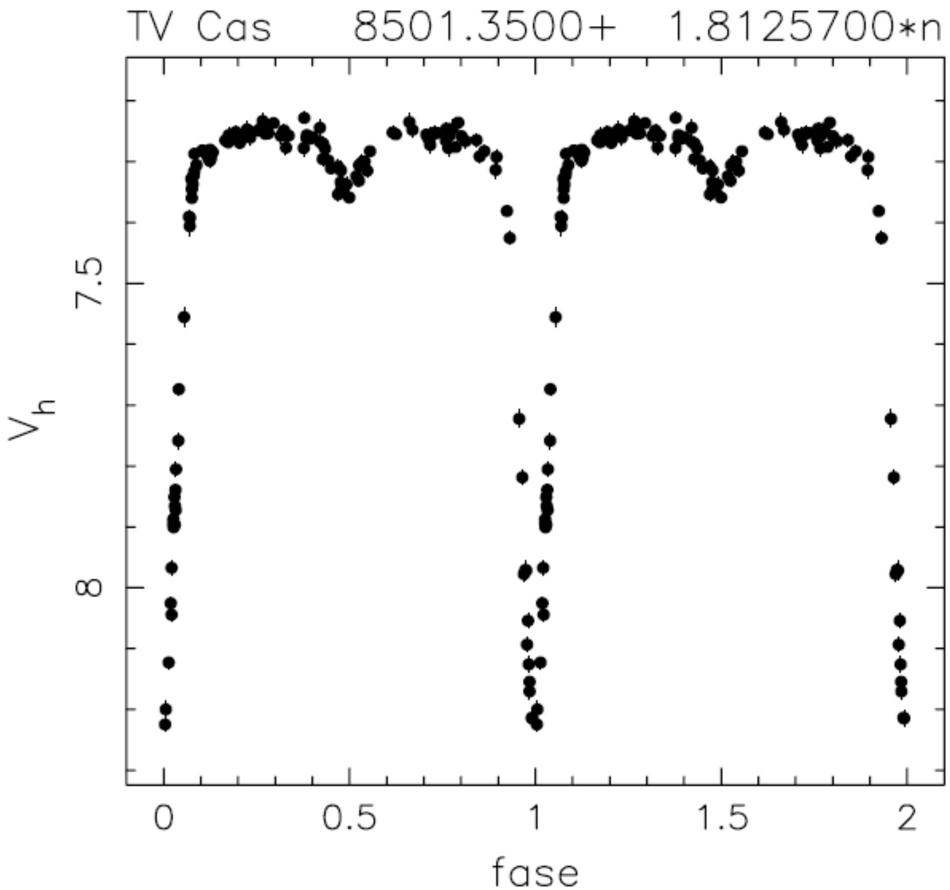


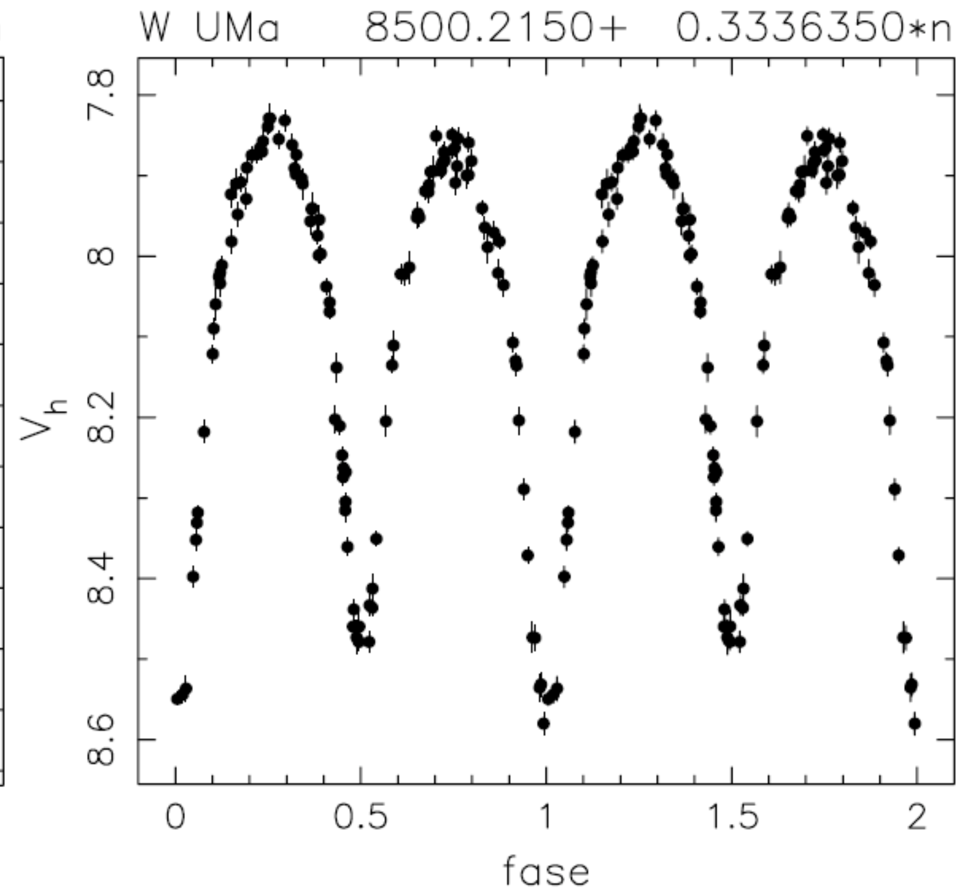
Time Series of Two Stars





Two Stars: Differently Sorted





Gaussian Noise

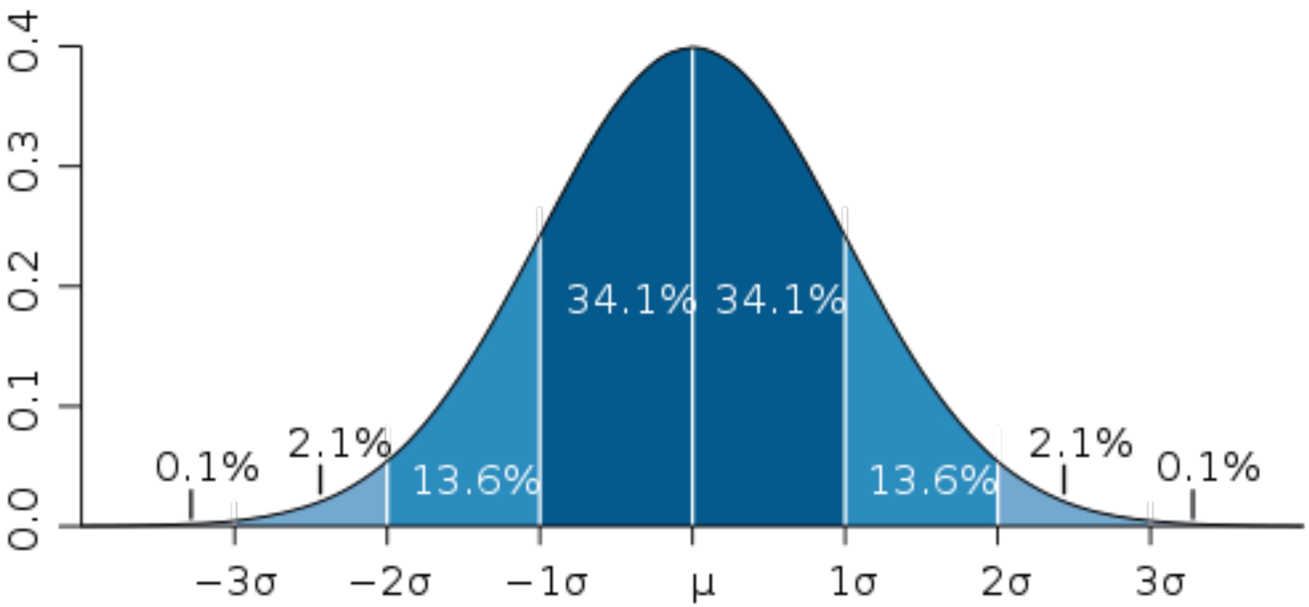
- Gaussian noise has Gaussian (normal) distribution
- Sometimes (incorrectly) called white noise (uncorrelated noise)

$$S = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2} \right]$$

x is the actual value

μ is the mean of the distribution

s is the standard deviation of the distribution



1-σ ~ 68%

2-σ ~ 95%

3-σ ~ 99.7%

Astronomers usually consider S/N > 3σ as significant.

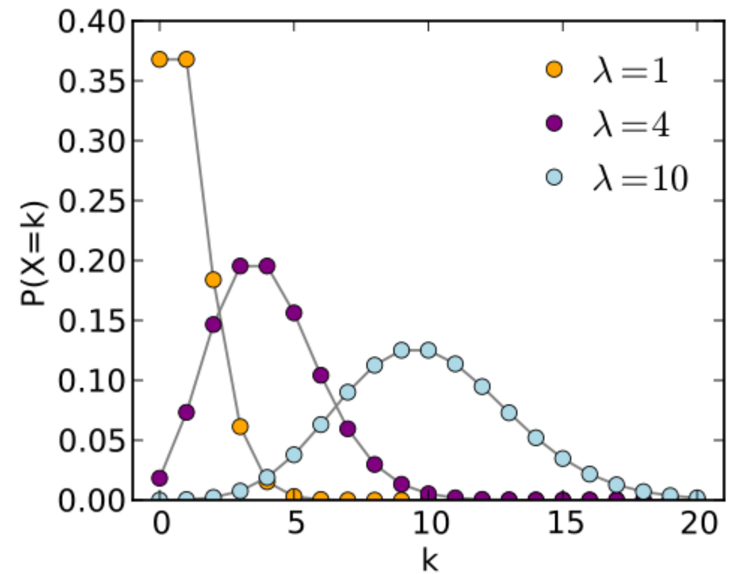
Poisson Noise

- Poisson noise has Poisson distribution
- probability of number of events occurring in constant interval of time/space if events occur with known *average rate* and *independently* of each other.

$$P(k,\lambda)=\frac{\lambda^k e^{-\lambda}}{k!}$$

k is the number of occurrences of an event (probability)

λ is the *expected* number of occurrences



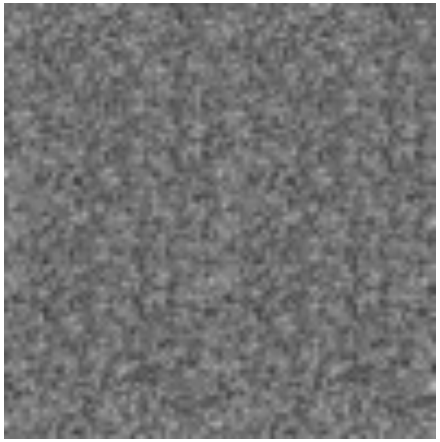
- mean of *P(k,λ)* is *λ*
- standard deviation of *P(k,λ)* is square root of *λ*
- example: fluctuations in photon flux in finite time intervals *Δt*. Chance to detect *k* photons with average flux of λ photons

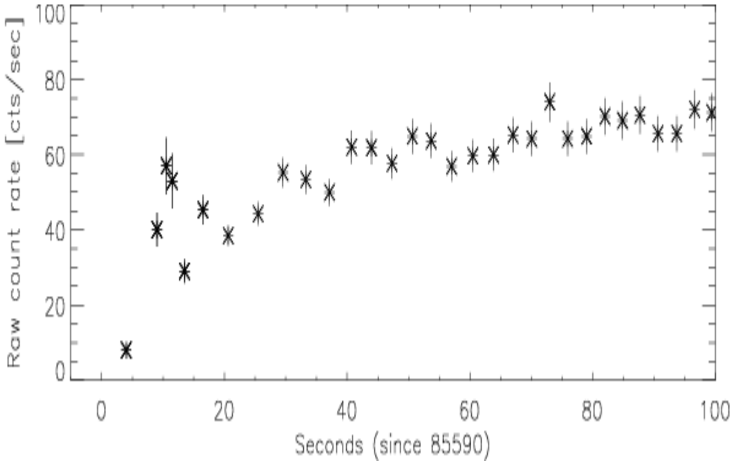
Noise Measurement

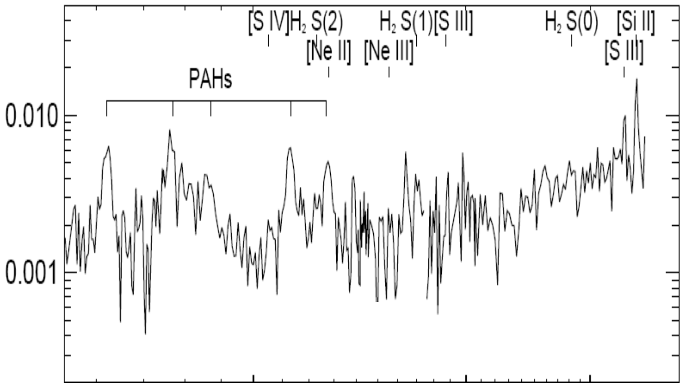
Assume purely Gaussian or Poissonian noise distribution, no other systematic noise, no correlations

Then the spatial distribution (neighbouring pixels) of the noise is equivalent to the temporal distribution (successive measurements with one pixel)

This is analogous to throwing 5 dices once versus throwing one dice







Case 1: Spatial noise (detector pixels)

Case 2: Repeated measurements in time (time series)

Case 3: Spectrum (dispersed information)

Poisson Noise and Integration Time

- Integrate light from uniformly extended source on CCD
- In finite time interval Δt, expect average of λ photons
- Statistical nature of photon arrival rate ➔ some pixels will detect more, some less than λ photons.
- Noise of average signal λ (i.e., between pixels) is $\sqrt{\lambda}$
- Integrate for $2\times\Delta t$ → expect average of $2\times\lambda$ photons
- Noise of that signal is now $\sqrt{(2\times\lambda)}$, i.e., increased by $\sqrt{2}$
- With respect to integration time t, noise will only increase $\sim\sqrt{t}$ while signal increases $\sim t$

Signal-to-Noise Ratio

from Wikipedia:

Signal-to-noise ratio (often abbreviated SNR or S/N) is a measure used in science and engineering that compares the level of a desired signal to the level of background noise.

Signal = S; Background = B; Noise = N;

$$\sigma = \frac{\text{Signal}}{\text{Noise}}$$

← measured as ($S+B$) - mean{B}

← total noise = $\sqrt{\sum (N_i)^2}$ (if statistically independent)

Both S and N should be in units of events (photons, electrons, data numbers) per unit area (pixel, PSF size, arcsec^2)

S/N and Integration Time

Assuming the signal suffers from Poisson shot noise. Let's calculate the dependence on integration time t_{int}:

Integrating t_{int}: $$\sigma = \frac{S}{N}$$

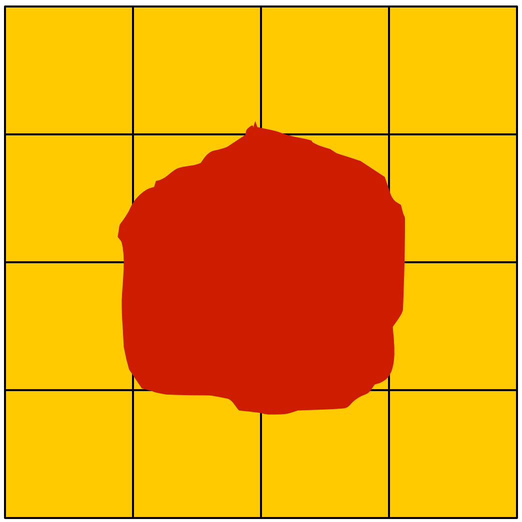
Integrating $n \times t_{int}$: $$\sigma = \frac{n \cdot S}{\sqrt{n \cdot B}} \overset{N=\sqrt{B}}{=} \sqrt{n}\,\frac{S}{N} \quad \Rightarrow \quad \frac{S}{N} \propto \sqrt{t_{\text{int}}}$$

Need to integrate four times as long to get twice the S/N

3 Cases

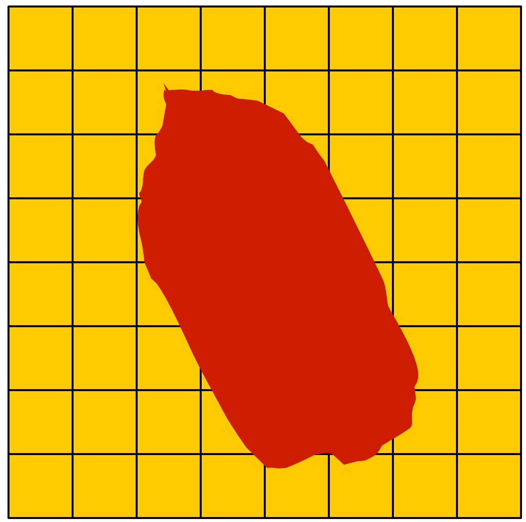
Background (=noise)

Target



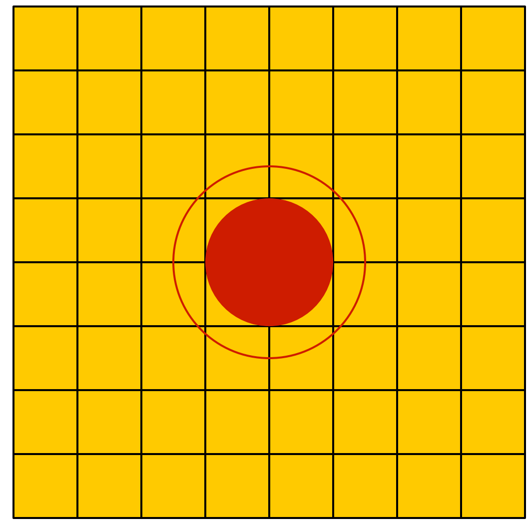
Seeing-limited point source

- pixel size ~ seeing
- PSF ≠ f(D)



Diffraction-limited, extended source

- pixel size ~ diff.lim
- PSF = f(D)
- target >> PSF



Diffraction-limited, point source

- pixel size ~ diff.lim
- PSF = f(D)
- target << PSF

Case 1: Seeing-limited "Point Source"

Signal = S; Background = B; Noise = N; Telescope diameter = D

θ_{seeing} ~ const

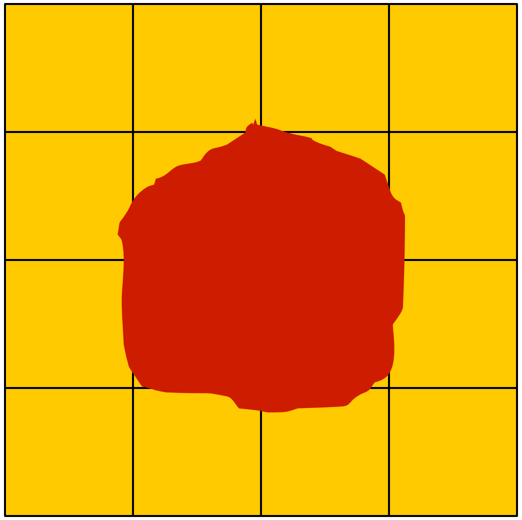
If detector is Nyquist-sampled to θ_{seeing} :

$S \sim D^2$ (area)

$B \sim D^2$ → $N \sim D$ (Poisson std.dev)

➔ $S/N \sim D$

➔ $t_{int} \sim D^{-2}$



Case 2: Diffraction-limited extended Source

Signal = S;   Background = B;   Noise = N;   Telescope diameter = D

"Diameter" of PSF ~ const

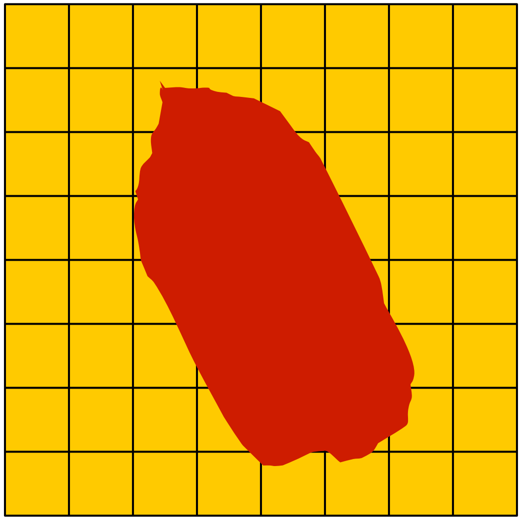
If detector Nyquist sampled to θ_{diff}: pixel ~ D^{-2} but $S \sim D^2$

D^2 (telescope size) and D^{-2} (pixel FOV) cancel each other → no change in signal

same for the background flux

➔ S/N ~ const → t_{int} ~ const

→ *no gain for larger telescopes!*

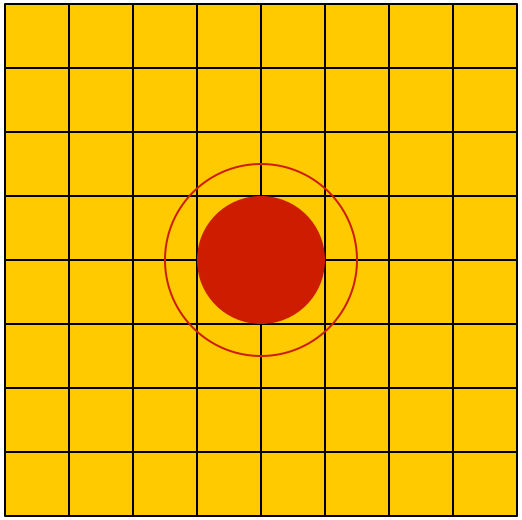


Case 2B: offline re-sampling by a factor x (makes θ_{diff} x-times larger)

since S/N ~ $\sqrt{n_{pix}}$ → S/N ~ $\sqrt{x^2} = x$ → $t_{int} \sim x^{-2}$

Case 3: Diffraction-limited "Point Source"

Signal = S; Background = B; Noise = N; Telescope diameter = D



"$S/N = (S/N)_{\text{light bucket}} \cdot (S/N)_{\text{pixel scale}}$"

(i) Effect of telescope aperture:

Signal $S \sim D^2$

Background $B \sim D^2 \rightarrow N \sim D$

$\rightarrow S/N \sim D$

(ii) Effect of pixel FOV (if Nyquist sampled to θ_{diff}):

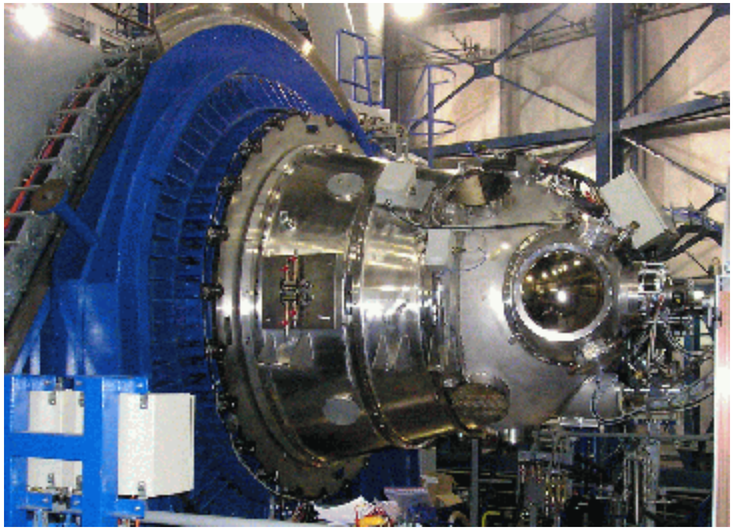
$S \sim$ const (pixel samples PSF = all source flux)

$B \sim D^{-2} \rightarrow N \sim D^{-1}$ $\rightarrow S/N \sim D$

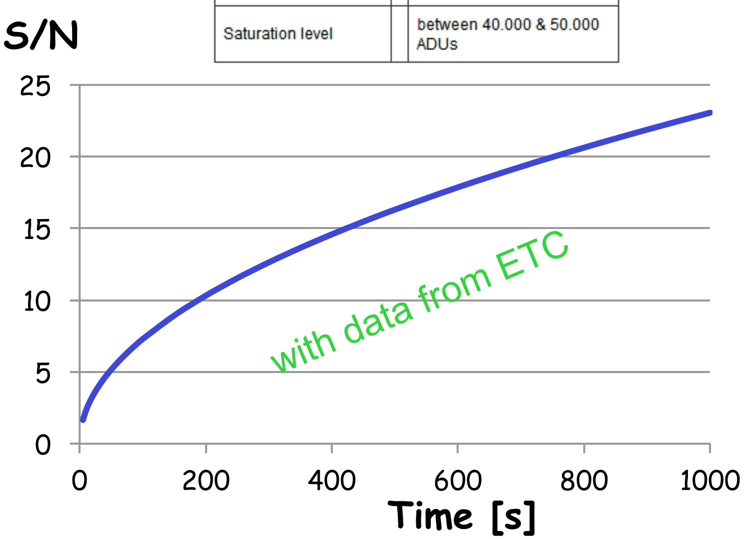
(i) and (ii) combined $S/N \sim D^2 \rightarrow t_{int} \sim D^{-4}$

→ *huge gain: 1hr ELT = 3 months VLT*

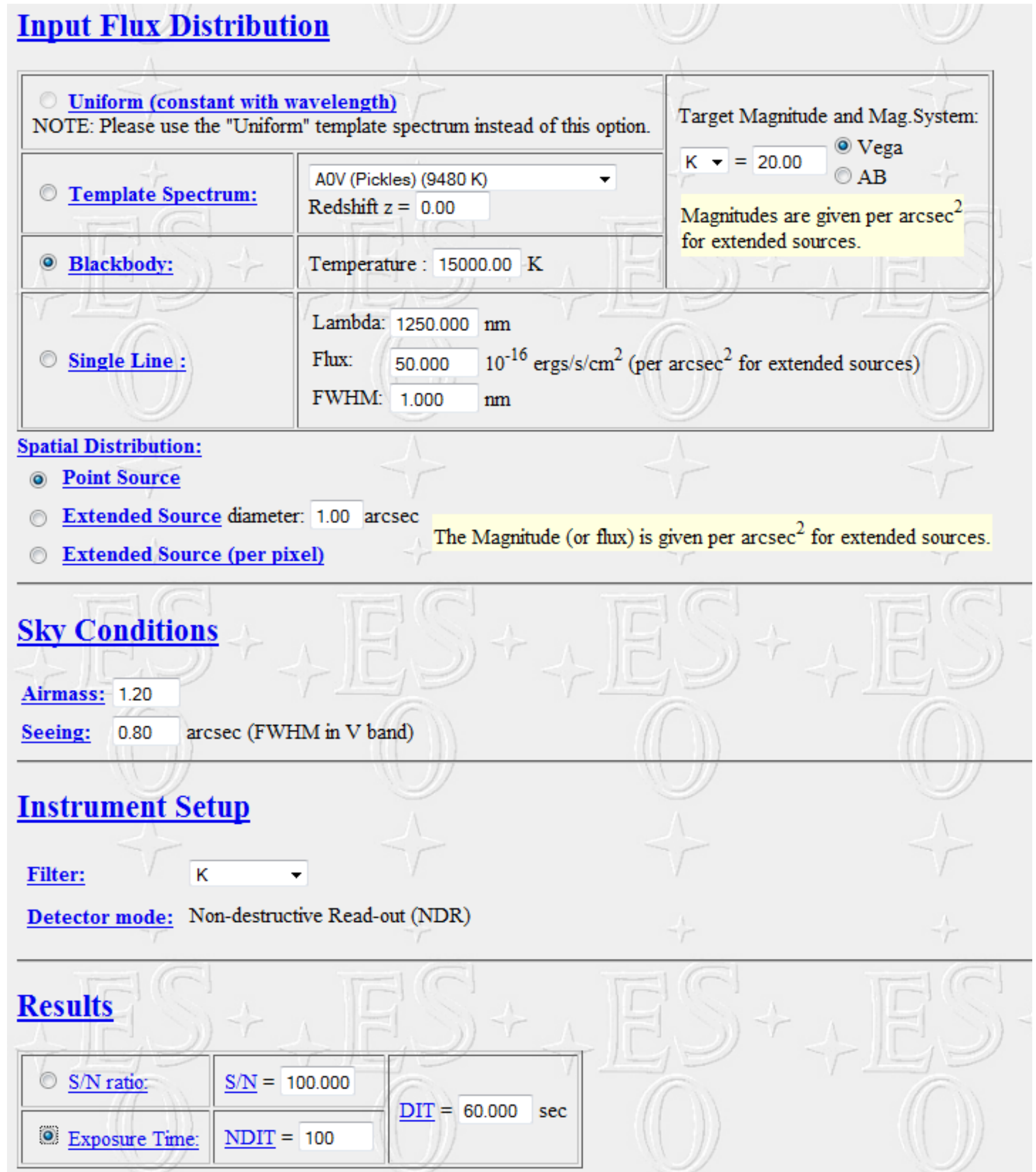
Instrument Sensitivity Example: HAWK-I



Operating temperature	75K, controlled to 1mK
Dark current [e-/s] (at 75K)	between 0.10 & 0.15
Read noise* (DCR)	~ 12 e-
Read noise* (NDR)	~ 5 e-
Linear range (1%)	60.000e- (~30.000 ADUs)
Saturation level	between 40.000 & 50.000 ADUs

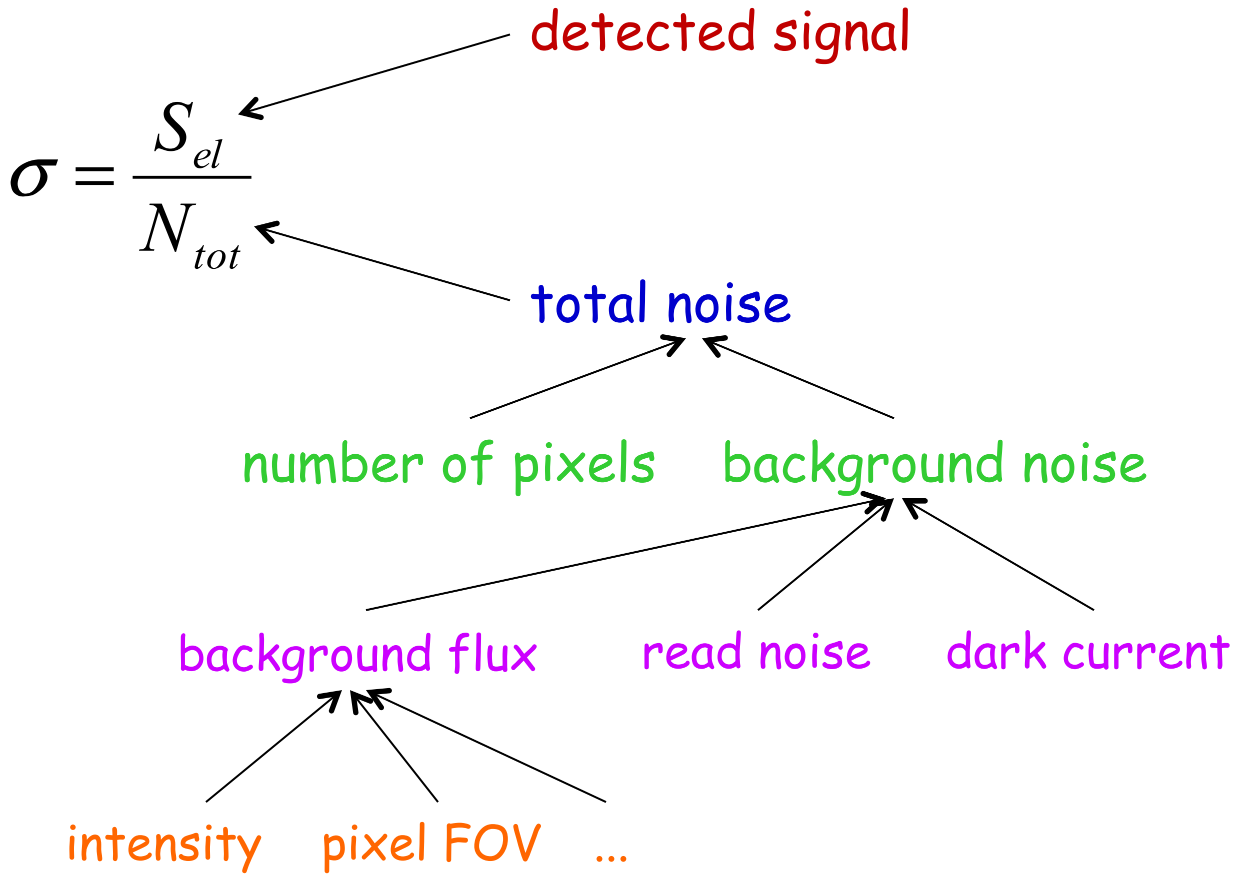


http://www.eso.org/observing/etc/bin/ut4/hawki/script/hawkisimu



Instrument Sensitivity: Example

$$\sigma = \frac{S_{el}}{N_{tot}}$$



Detected Signal

Detected signal S_{el} depends on:

- source flux density S_{src} [photons s^{-1} cm^{-2} μm^{-1}]
- integration time t_{int} [s]
- telescope aperture A_{tel} [m^2]
- transmission of the atmosphere η_{atm}
- total throughput of the system η_{tot}, which includes:
  - reflectivity of all telescope mirrors
  - reflectivity (or transmission) of all instrument components, such as mirrors, lenses, filters, beam splitters, grating efficiencies, slit losses, etc.
- Strehl ratio SR (ratio of actual to theoretical maximum intensity)
- detector responsivity $\eta_D G$
- spectral bandwidth $\Delta\lambda$ [µm]

$$S_{el} = S_{src} \cdot SR \cdot \Delta\lambda \cdot A_{tel} \cdot \eta_D G \cdot \eta_{atm} \cdot \eta_{tot} \cdot t_{\mathrm{int}}$$

Total Noise

Total noise N_{tot} depends on:

- number of pixels n_{pix} of one resolution element
- background noise per pixel N_{back}

$$N_{tot} = N_{back}\sqrt{n_{pix}}$$

total background noise N_{back} depends on:

- background flux density S_{back}
- integration time t_{int}
- detector dark current I_d
- pixel read noise (N) and detector frames (n)

$$N_{back} = \sqrt{S_{back}\cdot t_{\text{int}} + I_d \cdot t_{\text{int}} + N_{read}^2 \cdot n}$$

Background Flux

Background flux density S_{back} depends on:

- the total background intensity $B_{tot} = (B_T + B_A) \cdot \eta_{tot}$
  where B_T and B_A are the thermal emissions from telescope and atmosphere, approximated by black body emission

$$B_{T,A} = \frac{2hc^2}{\lambda^5}\left[\frac{\varepsilon}{\exp\left[\frac{hc}{kT\lambda}\right]-1}\right]$$

- the spectral bandwidth $\Delta\lambda$
- the pixel field of view $A \times \Omega = 2\pi\left(1-\cos\left(\arctan\left(\frac{1}{2F\#}\right)\right)\right)D^2{}_{pix}$
- the detector responsivity $\eta_D G$, and
- the photon energy hc/λ

$$S_{back} = B_{tot} \cdot A \times \Omega \cdot \frac{\eta_D G \lambda}{hc} \cdot \Delta\lambda$$

Instrument Sensitivity

Putting it all together, the minimum detectable source signal is:

$$\sigma = \frac{S_{el}}{N_{tot}} = \frac{S_{src} \cdot SR \cdot \Delta\lambda \cdot A_{tel} \cdot \eta_D G \cdot \eta_{atm} \cdot \eta_{tot} \cdot t_{\mathrm{int}}}{N_{back}\sqrt{n_{pix}}}$$

$$\Rightarrow \; S_{src} = \frac{\sigma \cdot \sqrt{S_{back} \cdot t_{\mathrm{int}} + I_d \cdot t_{\mathrm{int}} + N_{read}^2 \cdot n} \cdot \sqrt{n_{pix}}}{SR \cdot \Delta\lambda \cdot A_{tel} \cdot \eta_D G \cdot \eta_{atm} \cdot \eta_{tot} \cdot t_{\mathrm{int}}}$$