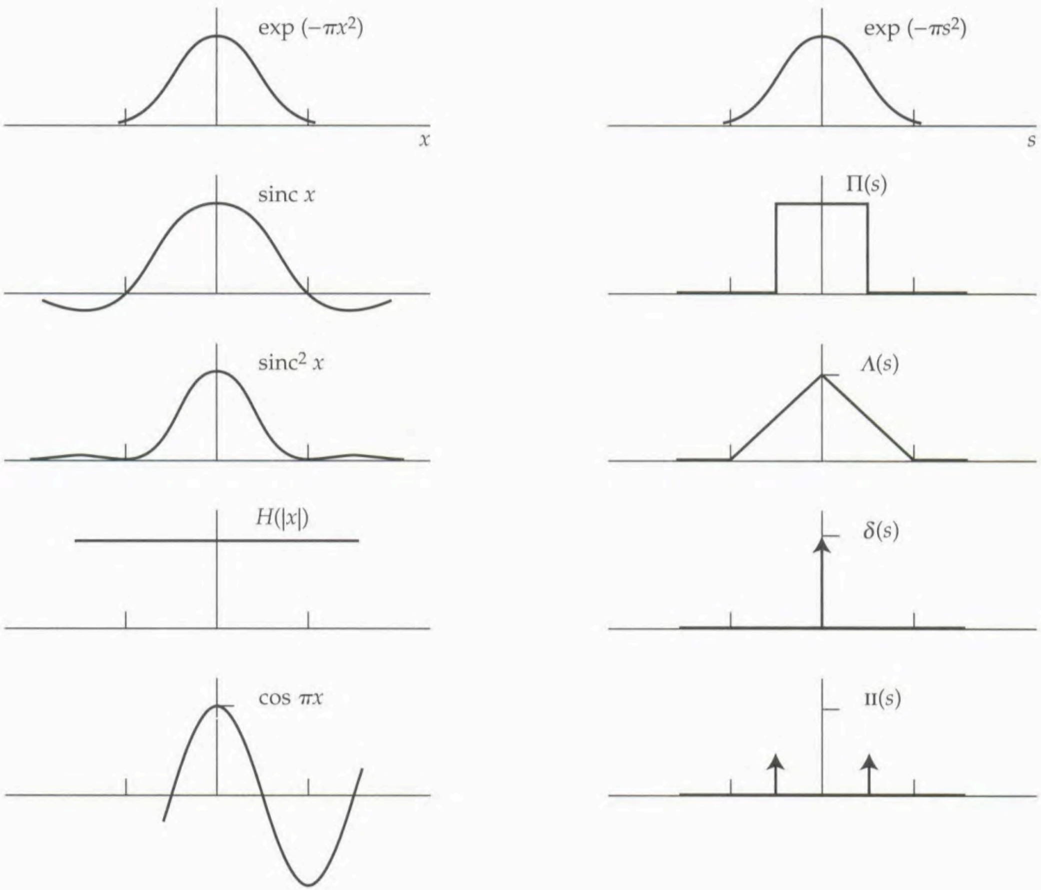
Astronomische Waarneemtechnieken
(Astronomical Observing Techniques)

based on lecture by Bernhard Brandl



<u>Lecture 5: Fourier</u>

1. Fourier Series
2. Fourier Transform
3. FT Examples in
   - 1D
   - 2D
4. Telescope ⇔ PSF
5. Important Theorems

Jean Baptiste Joseph Fourier

From Wikipedia:
Jean Baptiste Joseph Fourier (21 March 1768 – 16 May 1830), French mathematician and physicist best known for initiating the investigation of Fourier series and their applications to problems of heat transfer and vibrations.

Fourier series decomposes any periodic function or signal into sum of sines and cosines (or complex exponentials).

Application: harmonic analysis of functions to study spatial or temporal frequencies.

Fourier Series

Fourier analysis = decomposition using sines and cosines as orthonormal basis set.

Consider periodic function: $f(x) = f(x+P)$

Fourier series: $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nx}{P}\right) + b_n \sin\left(\frac{2\pi nx}{P}\right)\right]$$

Fourier coefficients: $$a_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\cos\left(\frac{2\pi nx}{P}\right)dx$$

$$b_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\sin\left(\frac{2\pi nx}{P}\right)dx$$

Period: P

Frequency: 1/P

Angular frequency: 2π/P

Orthonormal Basis Set

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx = \begin{cases} 1 \text{ for } n = m \\ 0 \text{ for } n \neq m \end{cases}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sin(mx)\cos(nx)\,dx = 0$$

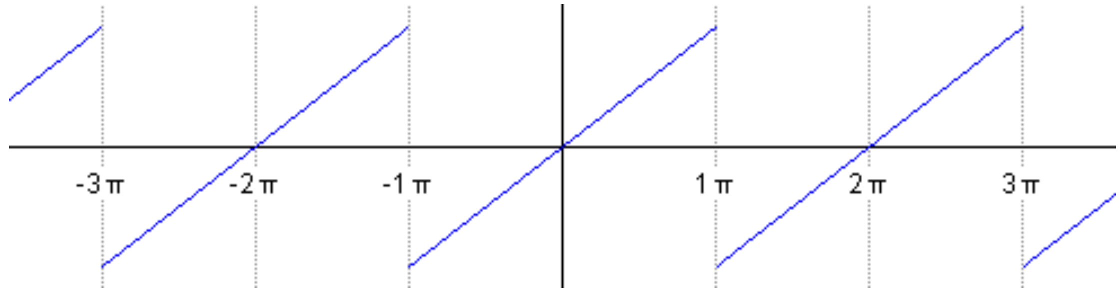
$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sin(mx)\sin(nx)\,dx = \begin{cases} 1 \text{ for } n = m \\ 0 \text{ for } n \neq m \end{cases}$$

Example: Sawtooth Function

Sawtooth function:

$$f(x) = x \qquad \text{for} \quad -\pi < x < \pi$$

$$f(x+2\pi) = f(x)$$



Fourier coefficients are:

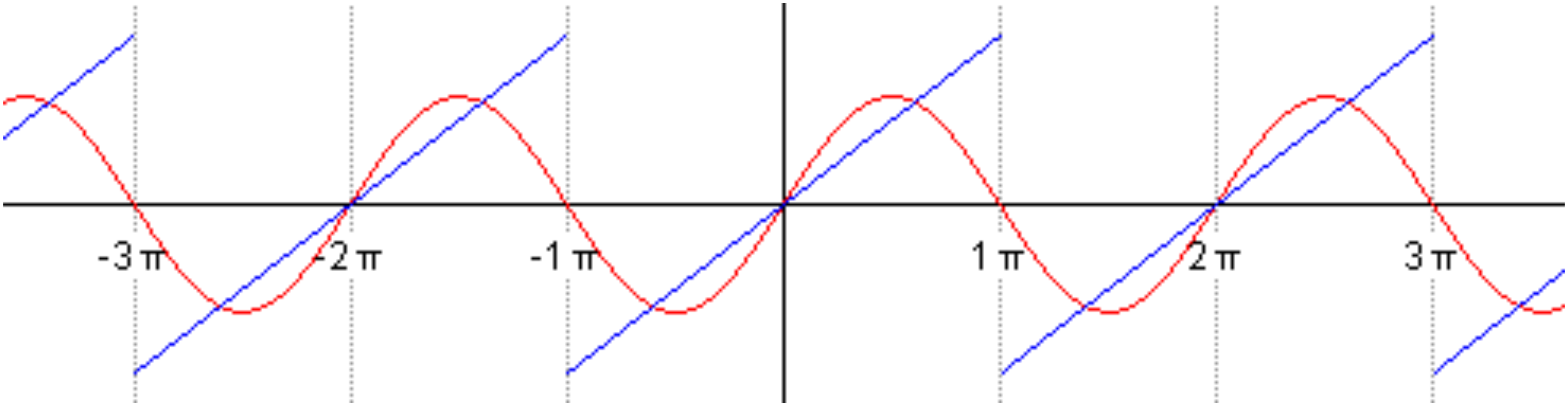
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx \overset{!}{=} 0 \qquad \text{(cos() is symmetric around 0)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = 2\frac{(-1)^{n+1}}{n}$$

and hence: $f(x) = \cancel{\frac{a_0}{2}} + \sum_{n=1}^{\infty}\left[\cancel{a_n\cos(nx)} + b_n\sin(nx)\right] = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$

Example: Sawtooth Function (2)

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

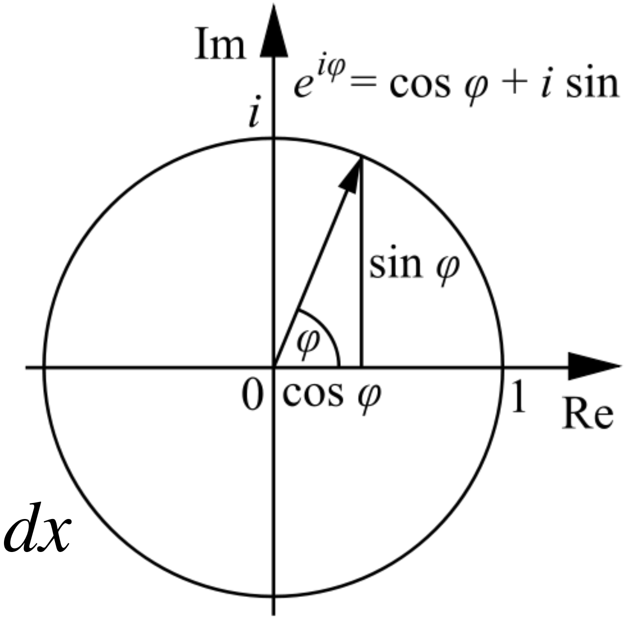


Side note: Euler's Formula

Wikipedia: Leonhard Euler (1707 – 1783), pioneering Swiss mathematician and physicist, made important discoveries in fields as diverse as infinitesimal calculus and graph theory. Introduced much of the modern mathematical terminology and notation.

Euler's formula: relation between trigonometric functions and complex exponential function:

$$e^{i2\pi\theta} = \cos(2\pi\theta) + i\sin(2\pi\theta)$$



Rewrite Fourier series in terms of waves with amplitudes and phases:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi nx}{P}} \qquad c_n = \frac{1}{P}\int_{-P/2}^{P/2} f(x) e^{-i\frac{2\pi nx}{P}} dx$$

Fourier Transform

Functions *f(x)* and *F(s)* are Fourier pairs if:

$$\boxed{F(s)=\int_{-\infty}^{+\infty} f(x)\cdot e^{-i2\pi xs}\,dx}$$

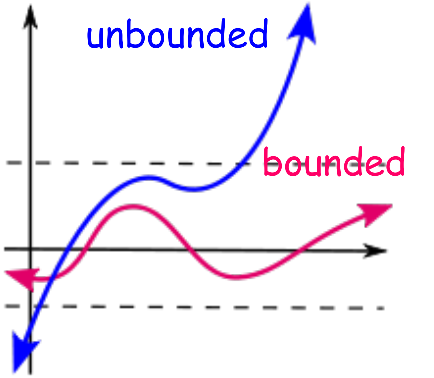
Here scalar x, but can be generalized to more dimensions.

Fourier transform is reciprocal, back-transformation is:

$$f(x)=\int_{-\infty}^{+\infty} F(s)\cdot e^{i2\pi xs}\,ds$$

Requirements:

- f(x) is bounded
- f(x) is square-integrable $\int_{-\infty}^{+\infty} |f(x)|^2\,dx$
- f(x) has a finite number of extremas and discontinuities



Fourier Transform Properties: Symmetry

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x)$$

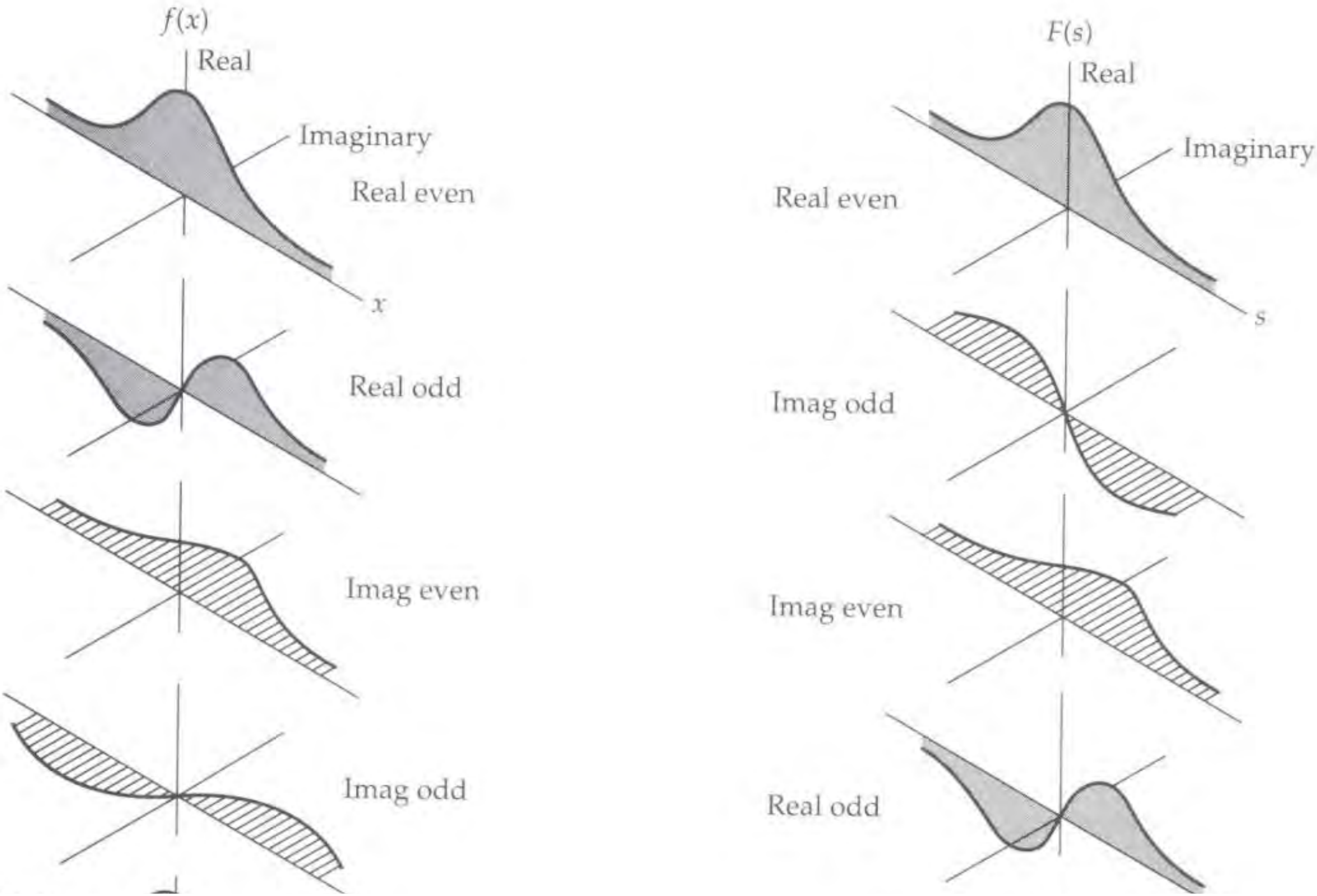
$$f_{\text{even}}(-x) = f_{\text{even}}(x) \quad f_{\text{odd}}(-x) = -f_{\text{odd}}(x)$$

$$\Rightarrow F(s) = 2\int_{0}^{+\infty} f_{\text{even}}(x)\cos(2\pi xs)dx$$

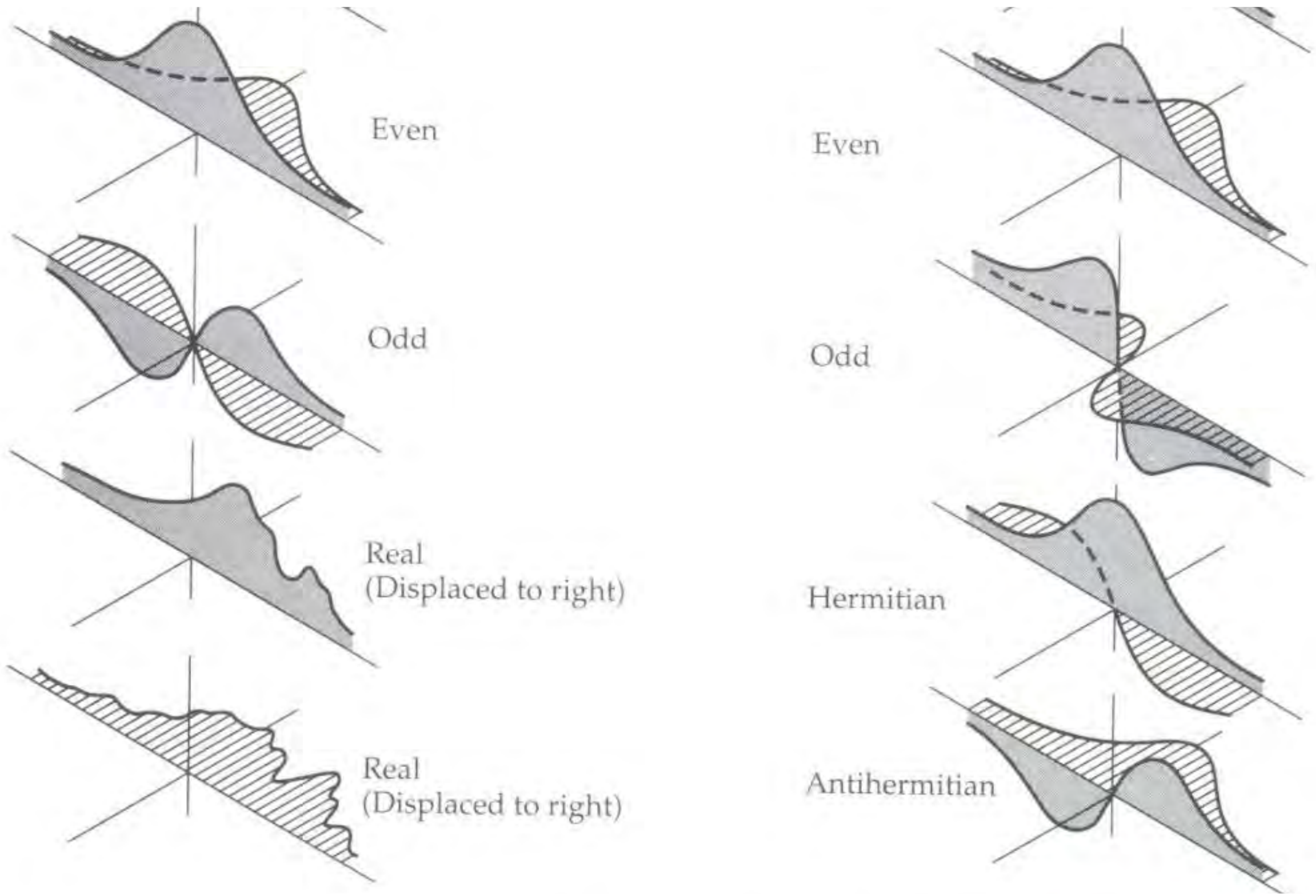
$$-i\ 2\int_{0}^{+\infty} f_{\text{odd}}(x)\sin(2\pi xs)dx$$

If f(x) is real, the even part of f(x) corresponds to the (even) real part of the Fourier transform F(s), and the odd part of f(x) corresponds to the (odd) imaginary part of F(s).

Fourier Transform Properties: Symmetry (2)



Fourier Transform Properties: Symmetry (3)



Fourier Transform Properties: Similarity

Expansion of function *f(x)* causes contraction of its transform *F(s)*:

$$f(x) \to f(ax) \Leftrightarrow \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

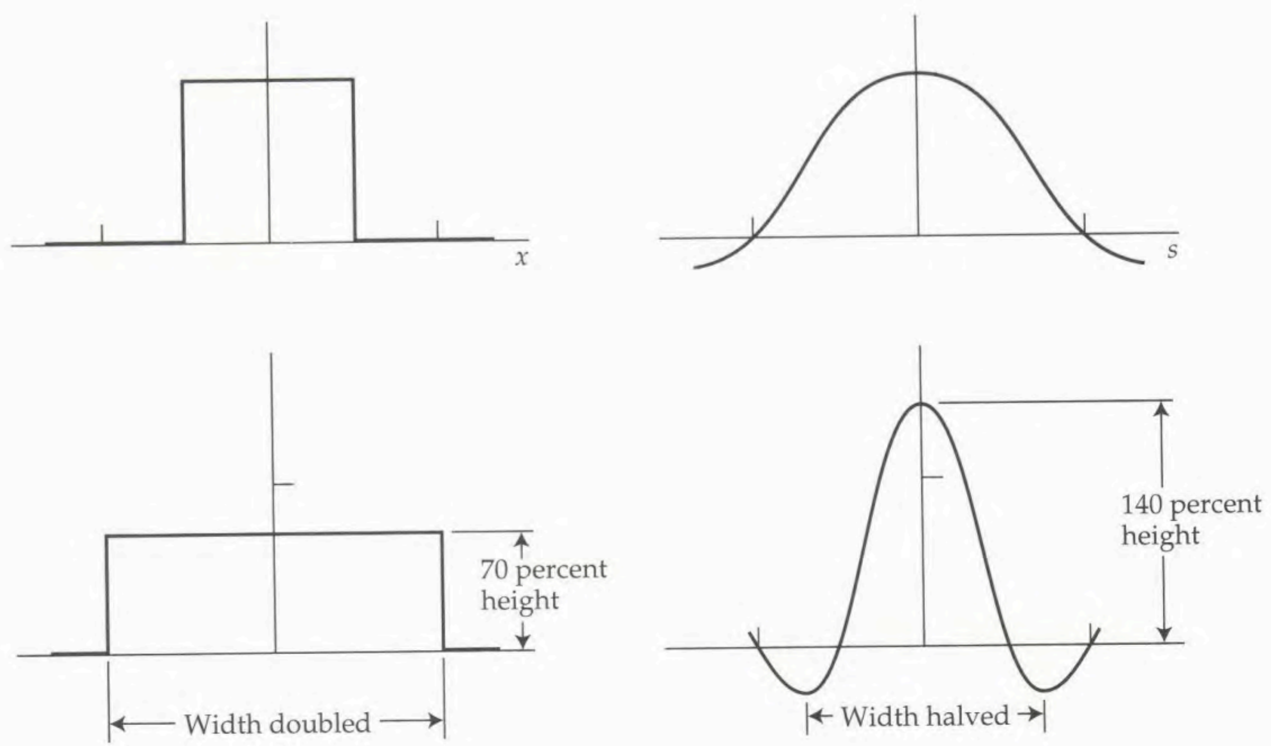


Fig. 6.4 **A symmetrical version of the similarity theorem.**

Other Fourier Transform Properties

LINEARITY: $F(as) = a \cdot F(s)$

TRANSLATION: $f(x-a) \Leftrightarrow e^{-i2\pi as} F(s)$

DERIVATIVE: $\dfrac{\partial^n f(x)}{\partial x^n} \Leftrightarrow (i2\pi s)^n F(s)$

INTEGRAL:

$$\int f(x)\partial x \Leftrightarrow (i2\pi s)^{-1} F(s) + c\delta(s)$$

ADDITION:

$$f(x) + g(x) \Leftrightarrow F(s) + G(s)$$

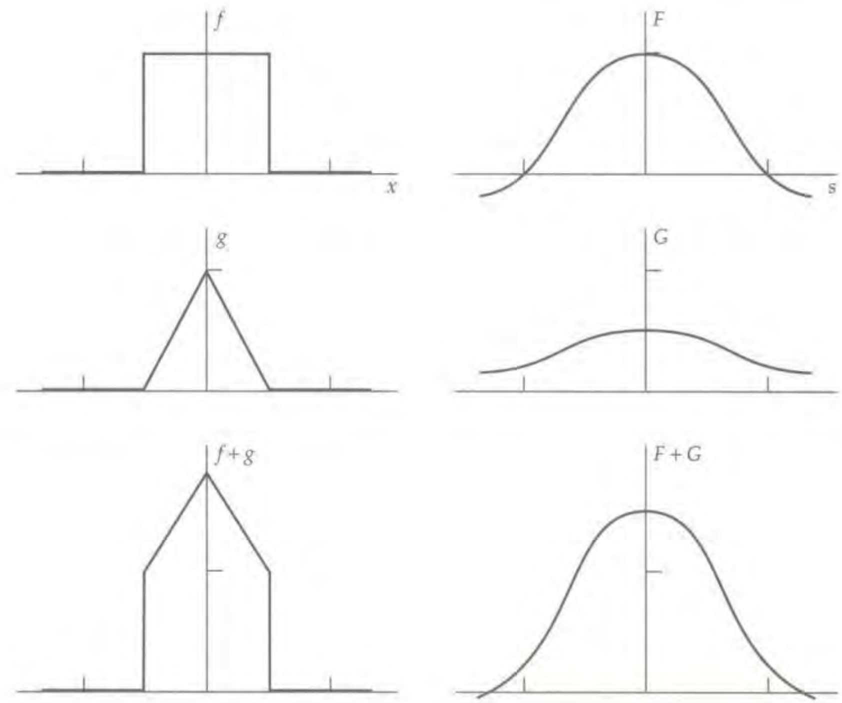
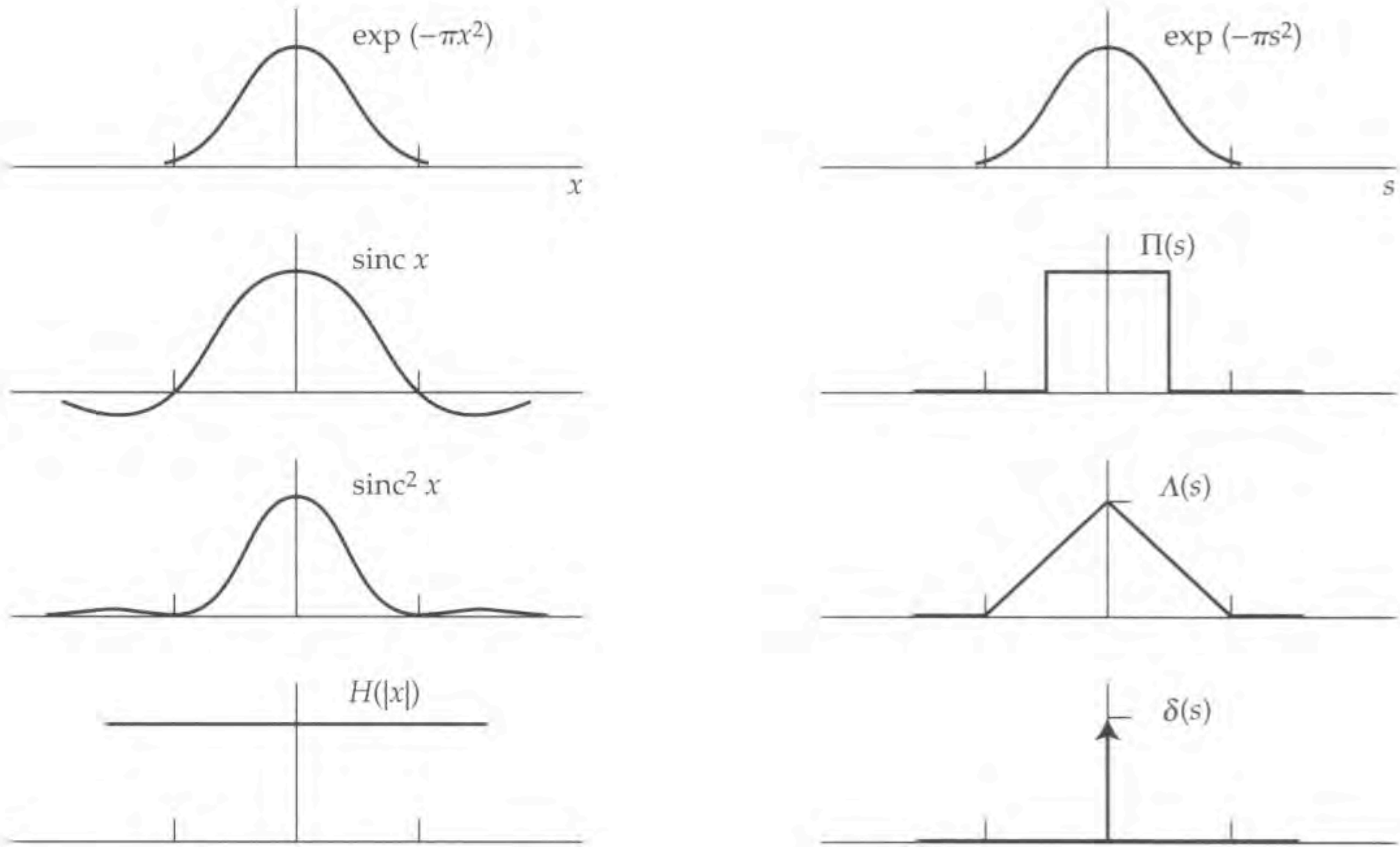
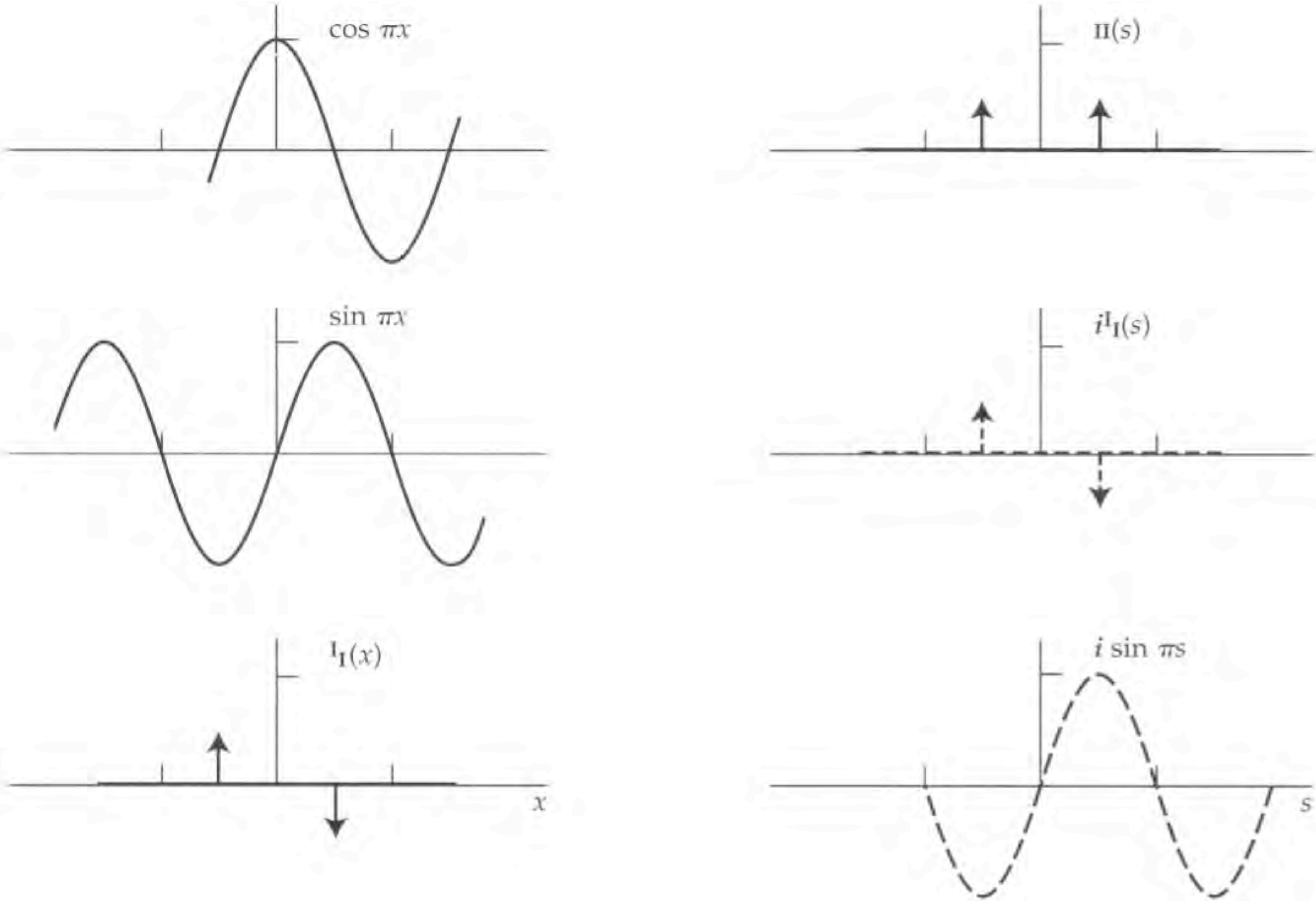


Fig. 6.5 The addition theorem $f + g \supset F + G$.

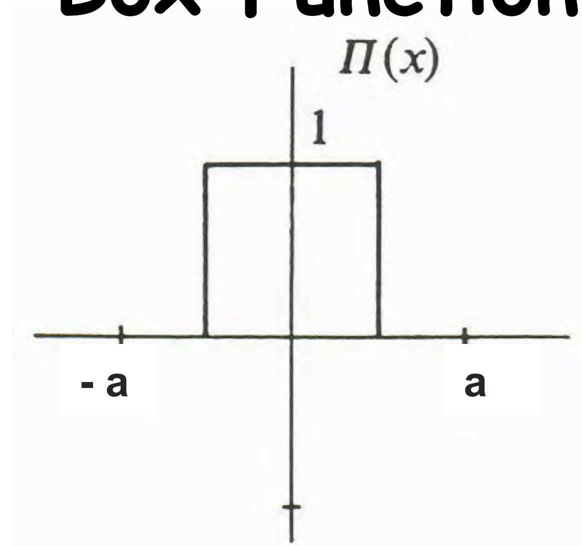
Important 1-D Fourier Pairs



Important 1-D Fourier Pairs (2)



Special 1-D Pairs (1): Box Function



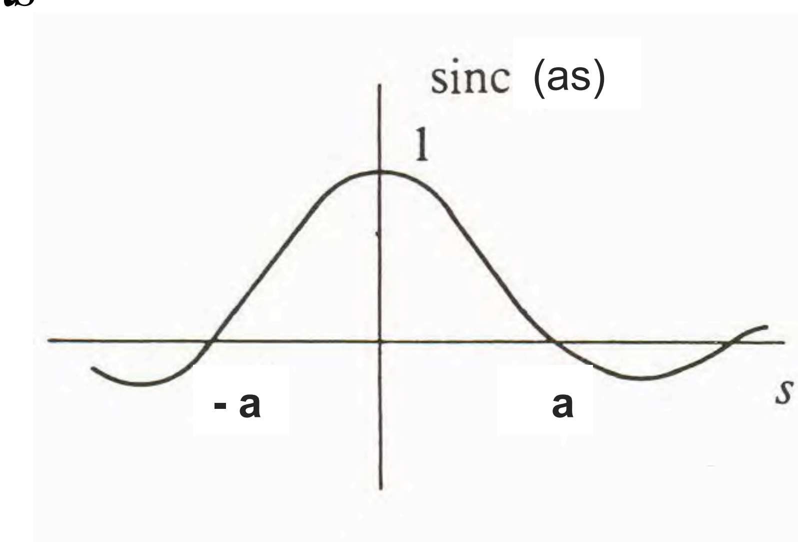
Box function:

$$\Pi\left(\frac{x}{a}\right)=\begin{cases}1 & \text{for } -\frac{a}{2}<x<\frac{a}{2}\\ 0 & \text{elsewhere}\end{cases}$$

With the Fourier pairs $\Pi(x)\Leftrightarrow\frac{\sin(\pi s)}{\pi s}\equiv\mathrm{sinc}\,(s)$

and using the similarity relation:

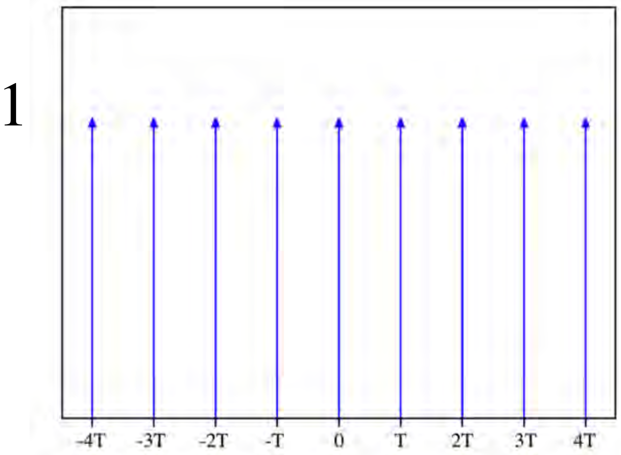
$$\Pi\left(\frac{x}{a}\right)\Leftrightarrow|a|\cdot\mathrm{sinc}\,(as)$$



Special 1-D Pairs (2): Dirac Comb

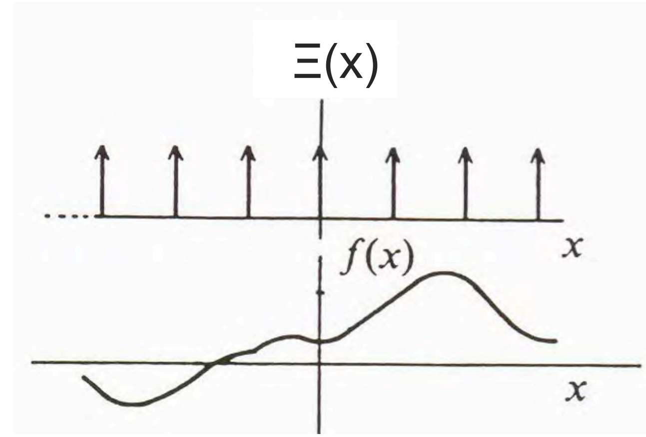
Dirac's delta "function":

$$f(x) = \delta(x) = \int_{-\infty}^{+\infty} e^{i2\pi sx}\, ds \quad \rightarrow \quad FT\{\delta(x)\} = 1$$

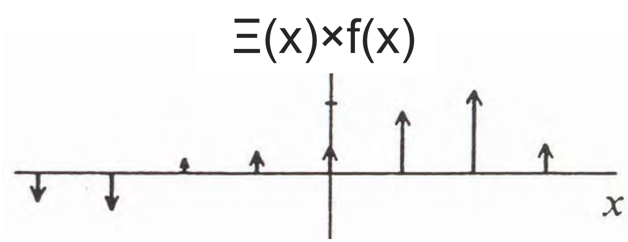


Dirac comb: infinite series of delta-functions spaced at intervals of T:

$$\Xi_T(x) = \sum_{k=-\infty}^{\infty} \delta(x - kT) \underset{series}{\overset{Fourier}{=}} \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i2\pi nx/T}$$



- Fourier transform of Dirac comb is also a Dirac comb
- Because of its shape, the Dirac comb is also called impulse train or sampling function.



Sampling (1)

Sampling: signal at discrete values of x: $f(x) \rightarrow f(x) \cdot Ш\left(\frac{x}{\Delta x}\right)$

Interval between two successive readings is sampling rate.

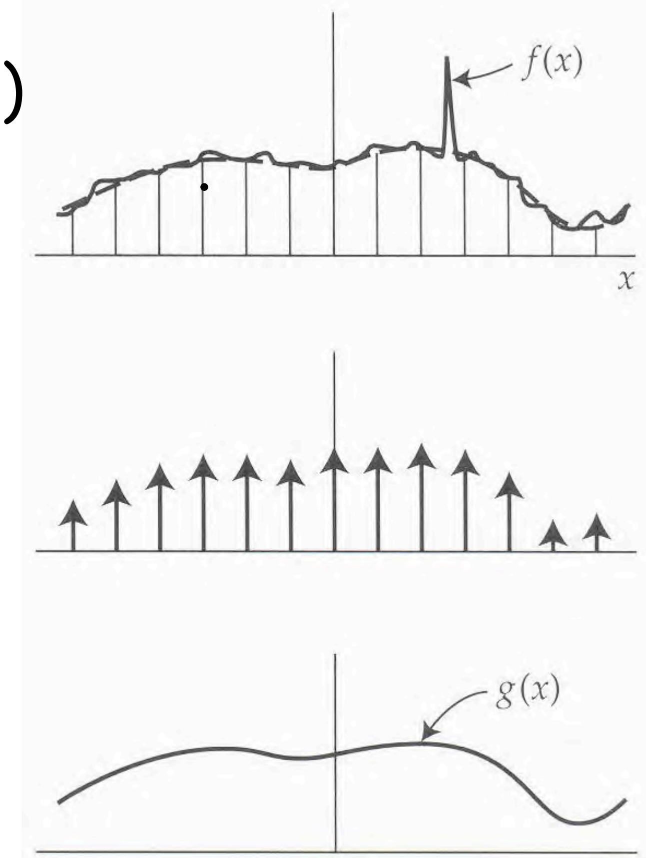
Critical sampling given by Nyquist-Shannon theorem:

Function f(x), its Fourier Transform F(s) with bounded support $[-s_{max}, +s_{max}]$

Sampled distribution of the form

$$g(x) = f(x) \cdot Ш\left(\frac{x}{\Delta x}\right)$$

with a sampling rate of $\Delta x = 1/(2s_{max})$ is enough to reconstruct *f(x)* for all *x*.



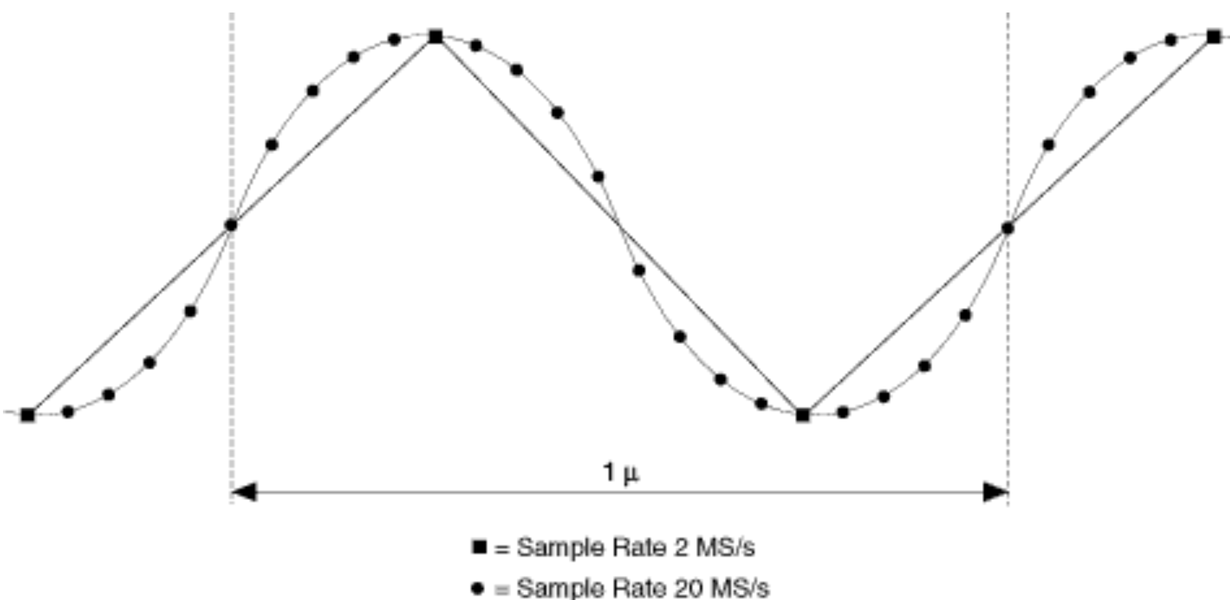
Sampling (2)

Sampling at any rate above or below the critical sampling is called oversampling or undersampling, respectively.

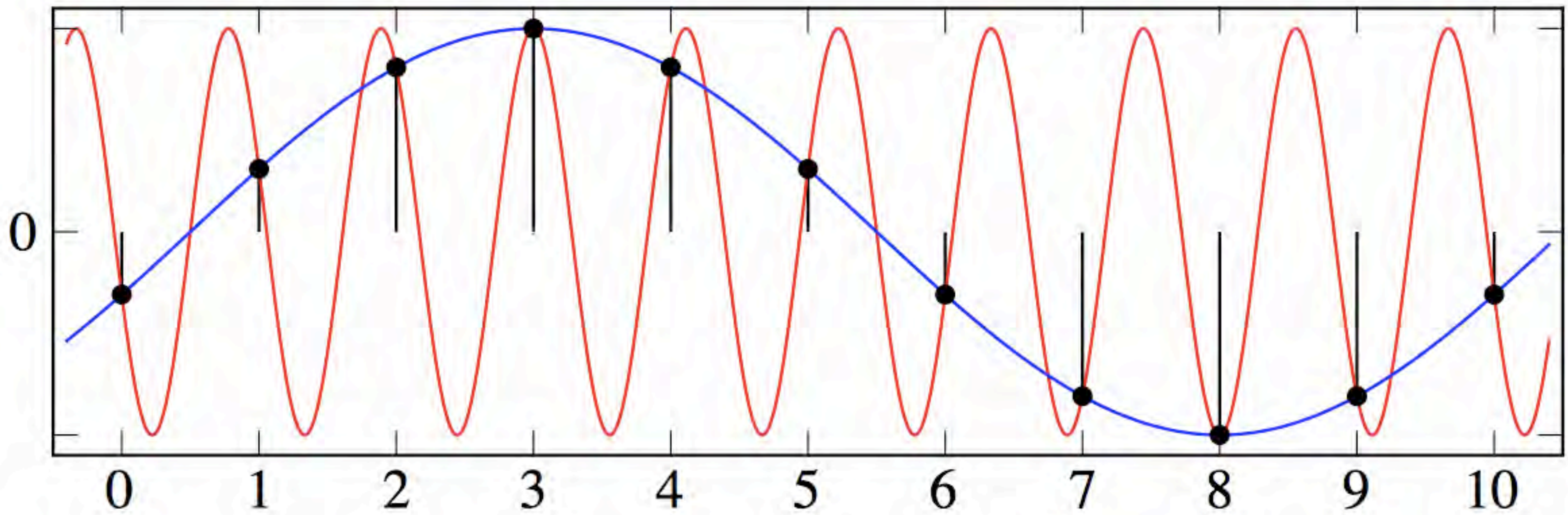
Oversampling: redundant measurements, often lowering the S/N

Undersampling: signal contains frequencies higher than $1/(2s_{max})$,

- source signal cannot be determined after sampling
- loss of fine details
- must apply low-pass filter before sampling



Aliasing



- unresolved, high frequencies beat with measured frequencies
- produce spurious components in frequency domain below Nyquist frequency
- may give rise to major problems and uncertainties in the determination of source function

Bessel Functions (1)

Friedrich Wilhelm Bessel (1784 – 1846), German mathematician, astronomer, and systematizer of the Bessel functions. "His" functions were first defined by the mathematician Daniel Bernoulli and then generalized by Friedrich Bessel.

Bessel functions are canonical solutions *y(x)* of Bessel's differential equation:

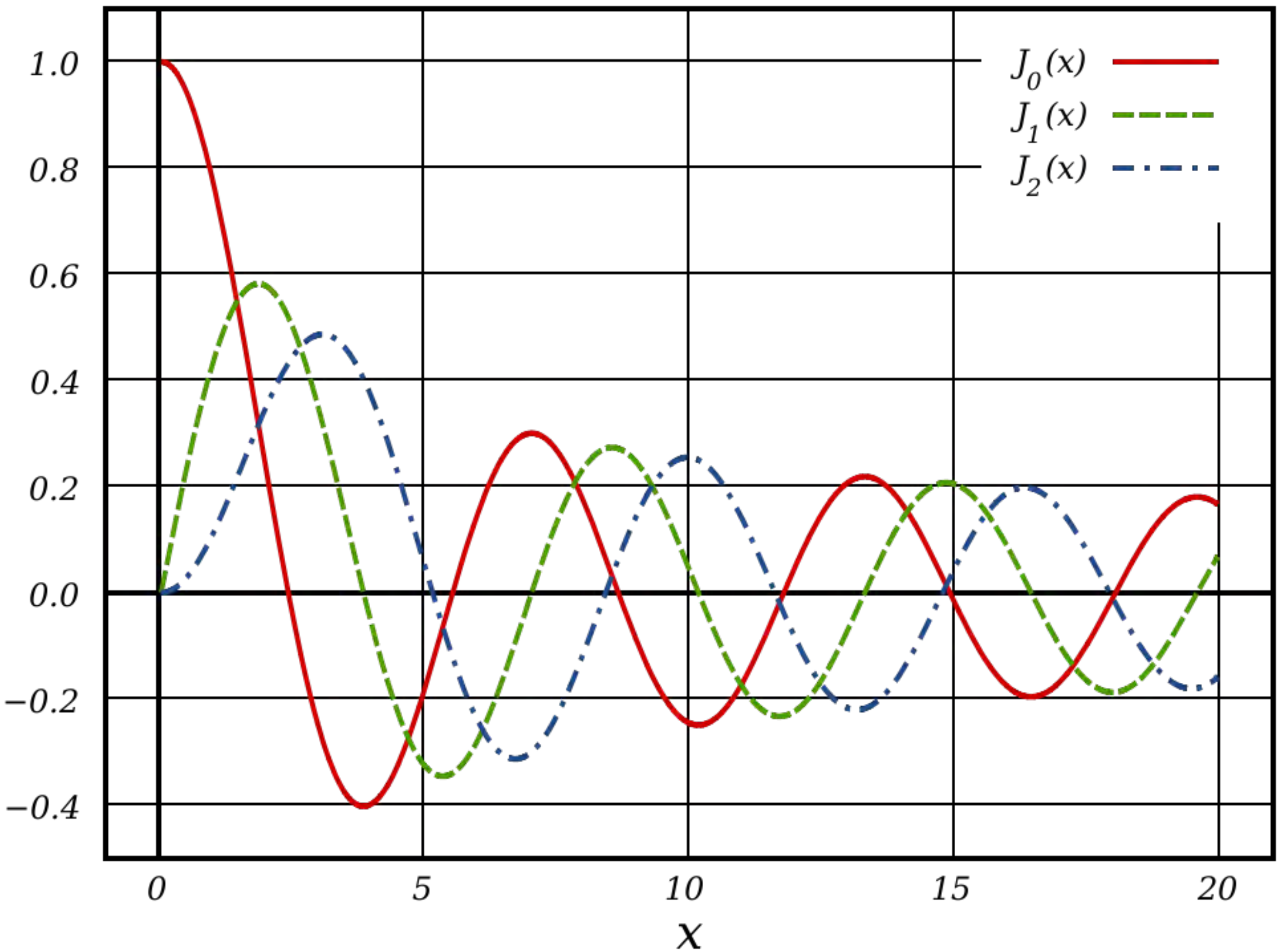
$$x^2 \frac{\partial^2 y}{\partial x^2} + x \frac{\partial y}{\partial x} + \left(x^2 - n^2\right) y = 0$$

for an arbitrary real or complex number *n*, the so-called order of the Bessel function.

These solutions are: $$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+n}}{k!(k+n)!}$$

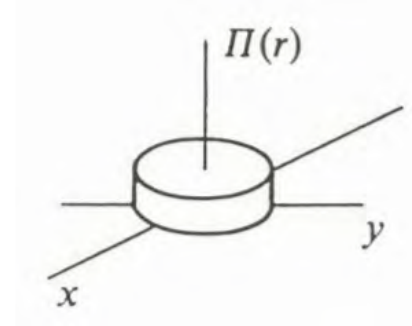
Bessel Functions (2)

Bessel functions are also known as cylinder functions or cylindrical harmonics because they are found in the solution to Laplace's equation in cylindrical coordinates.



Special 2-D Pairs (1): Box Function

Consider 2-D box function with $r^2 = x^2 + y^2$:

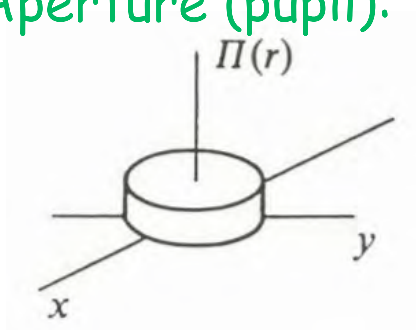


$$\Pi\left(\frac{r}{2}\right) = \begin{cases} 1 & \text{for } r < 1 \\ 0 & \text{for } r \geq 1 \end{cases}$$

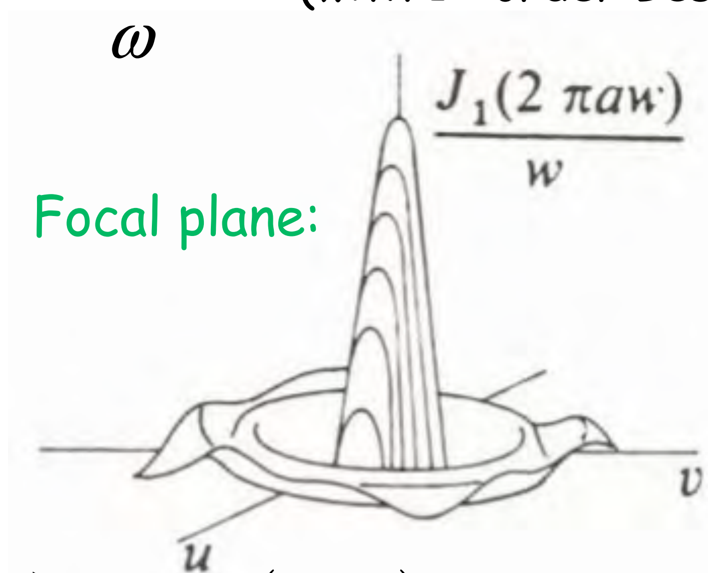
corresponding FT: $\Pi\left(\frac{r}{2}\right) \Leftrightarrow \frac{J_1(2\pi\omega)}{\omega}$ (with 1st order Bessel function J_1)

Example: optical telescope

Aperture (pupil):



Focal plane:

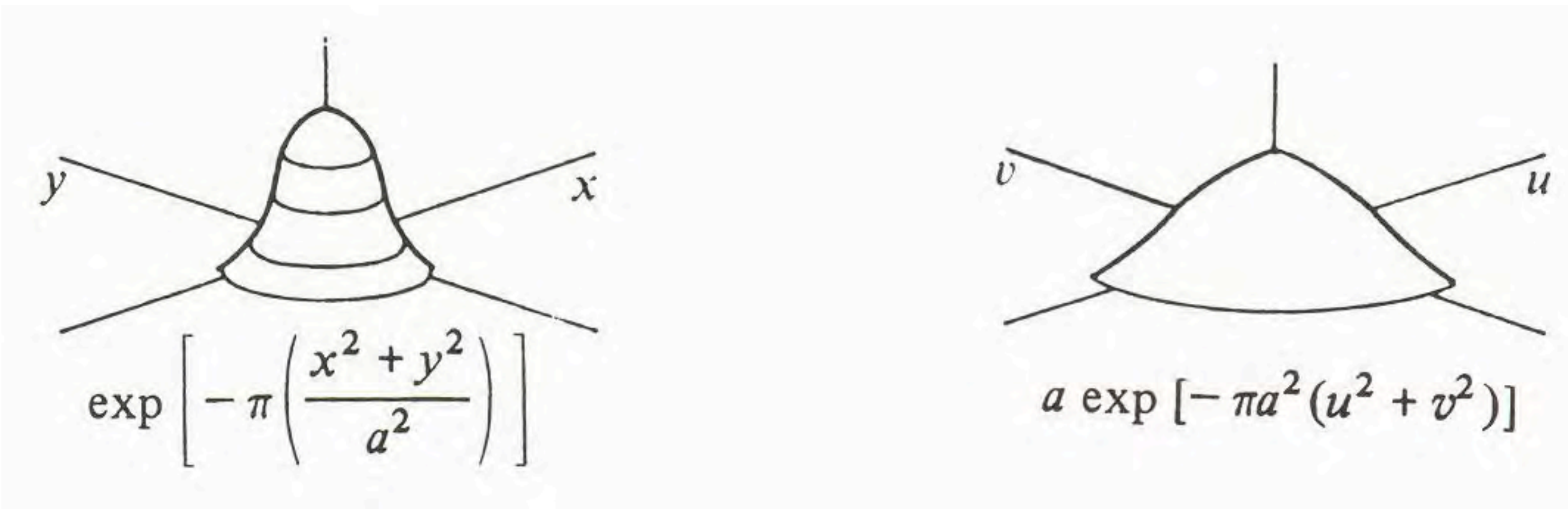


The similarity relation $\Pi\left(\frac{r}{2a}\right) \Leftrightarrow |a| \cdot \frac{J_1(2\pi a\omega)}{\omega}$ means that larger telescopes produce smaller Point Spread Functions (PSFs)!

Special 2-D Pairs (2): Gauss Function

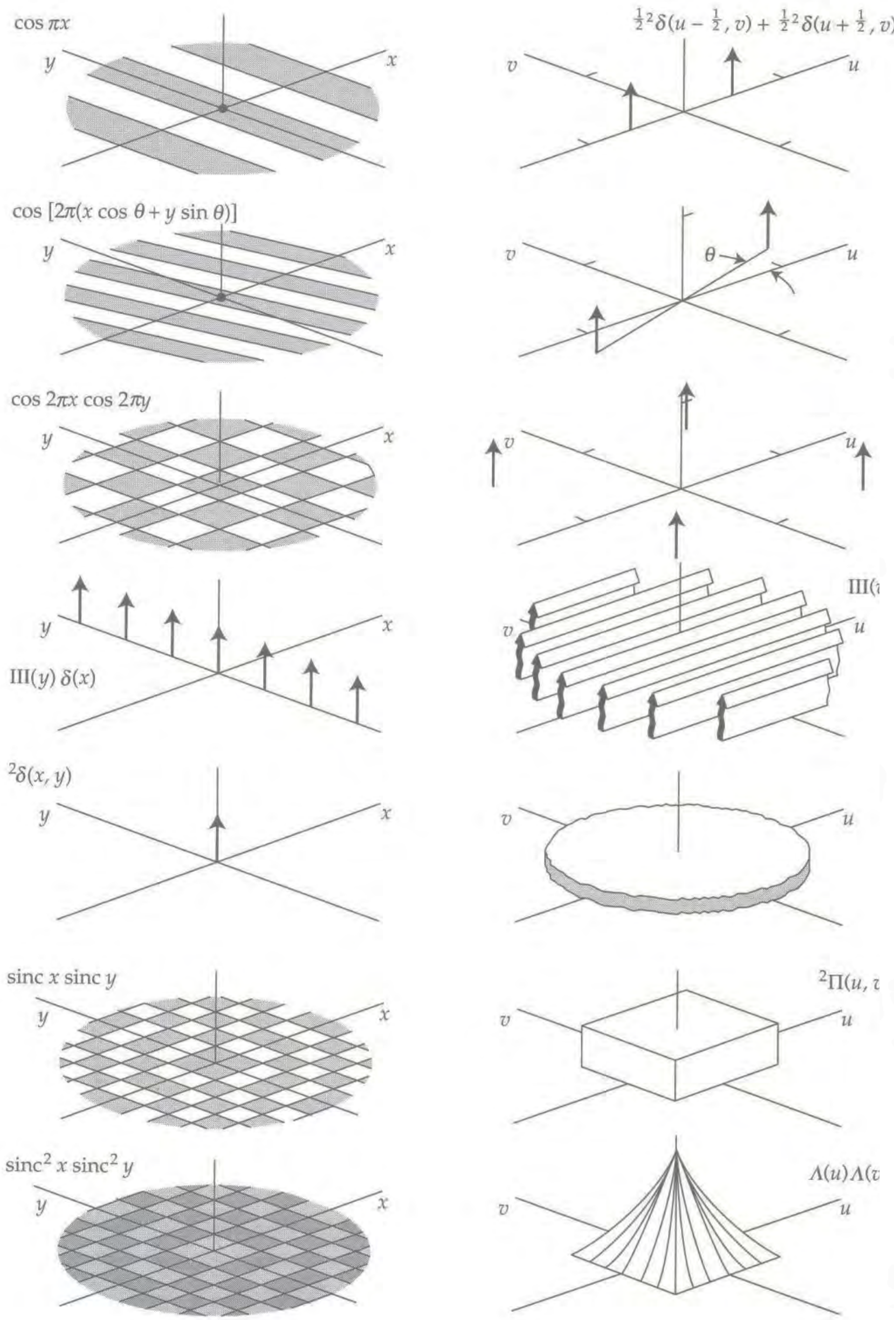
Consider a 2-D Gauss function with $r^2 = x^2 + y^2$:

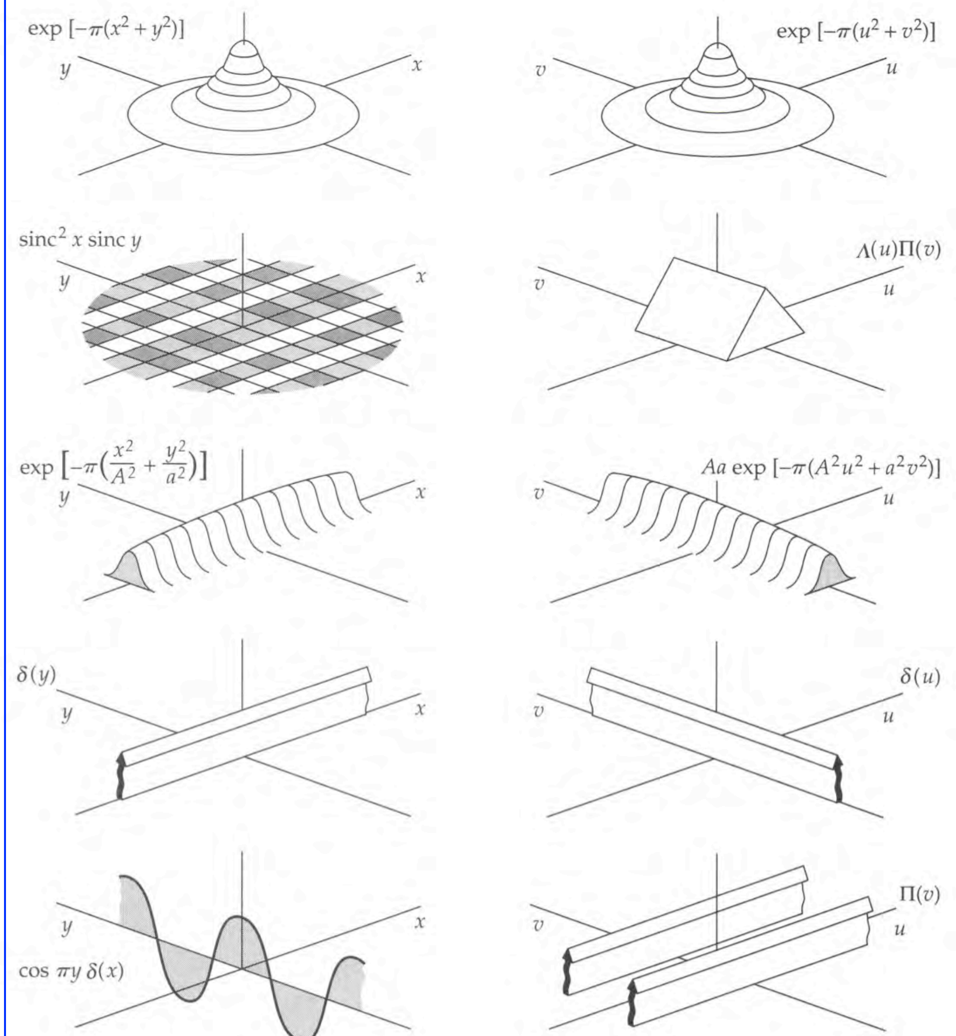
$$e^{-\pi r^2} \Leftrightarrow e^{-\pi\omega^2} \xrightarrow{\text{similarity}} e^{-\pi\left(\frac{r}{a}\right)^2} \Leftrightarrow |a| \cdot e^{-\pi(a\omega)^2}$$



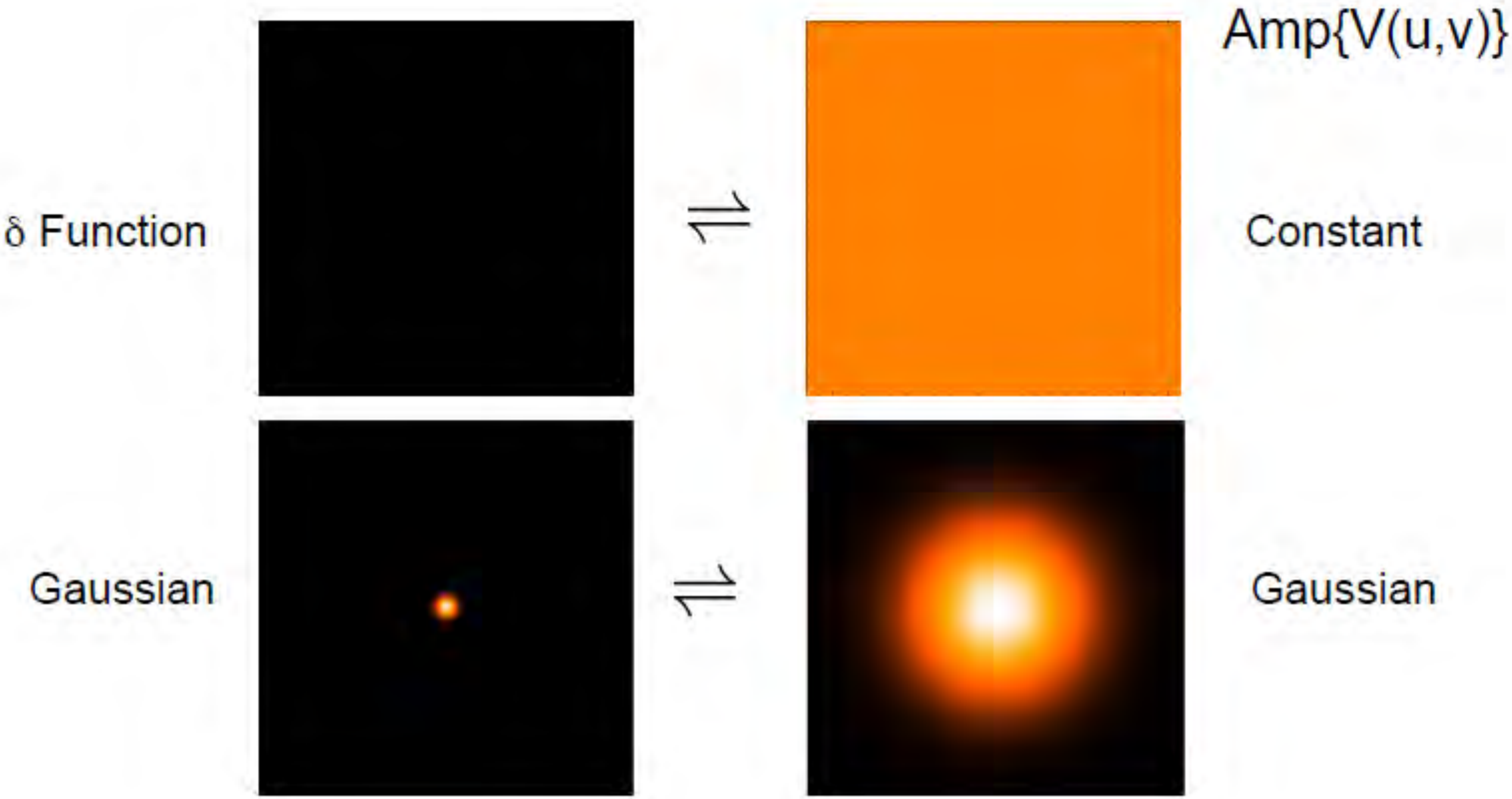
Gauss function is preserved under Fourier transform!

Important 2-D Fourier Pairs

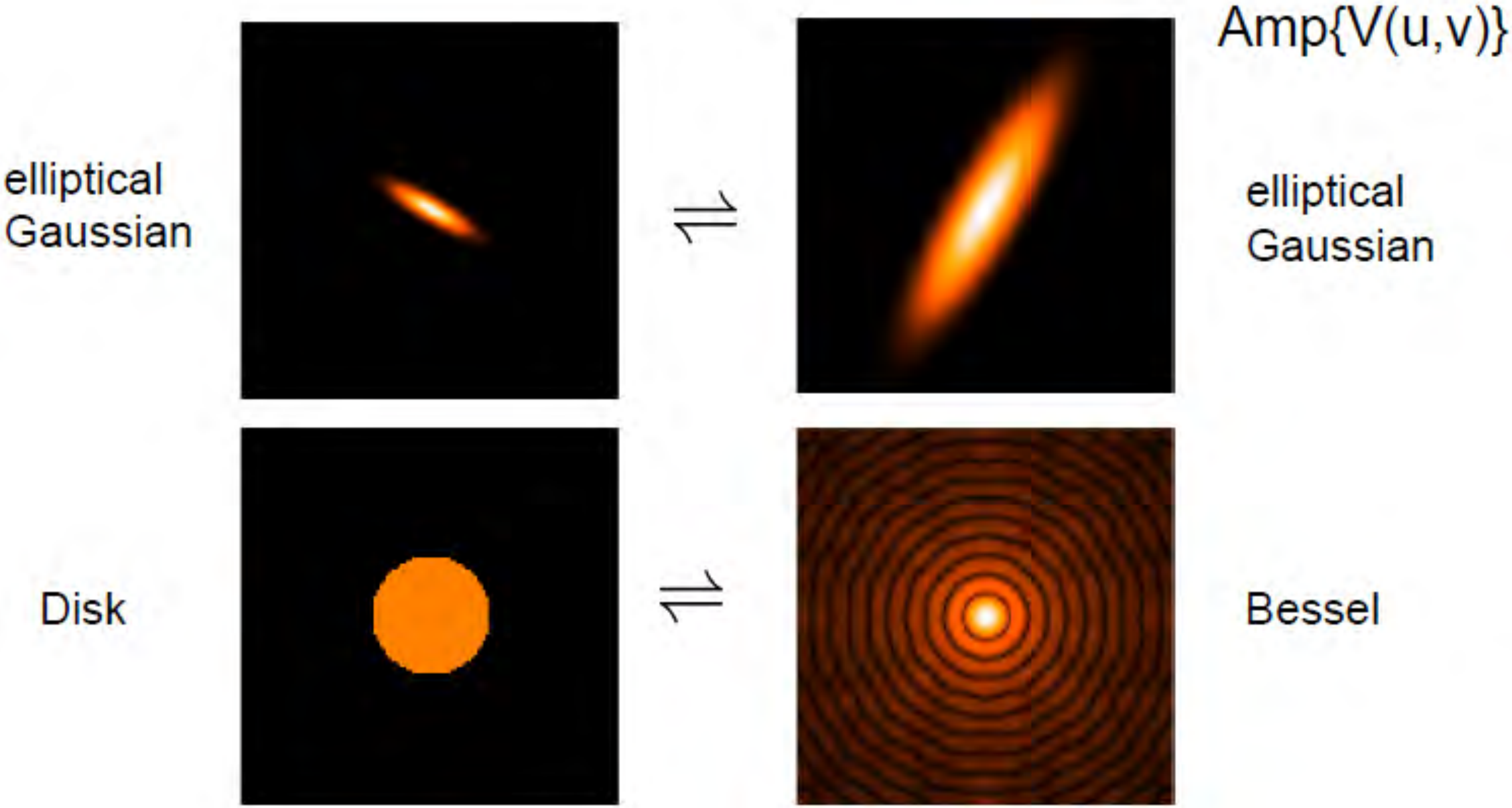




PUPIL (Telescope) ⇔ IMAGE (PSFs)



PUPIL (Telescope) ⇔ IMAGE (PSFs) (2)



Example 1:

central obscuration, monolithic mirror (pupil) no support-spiders

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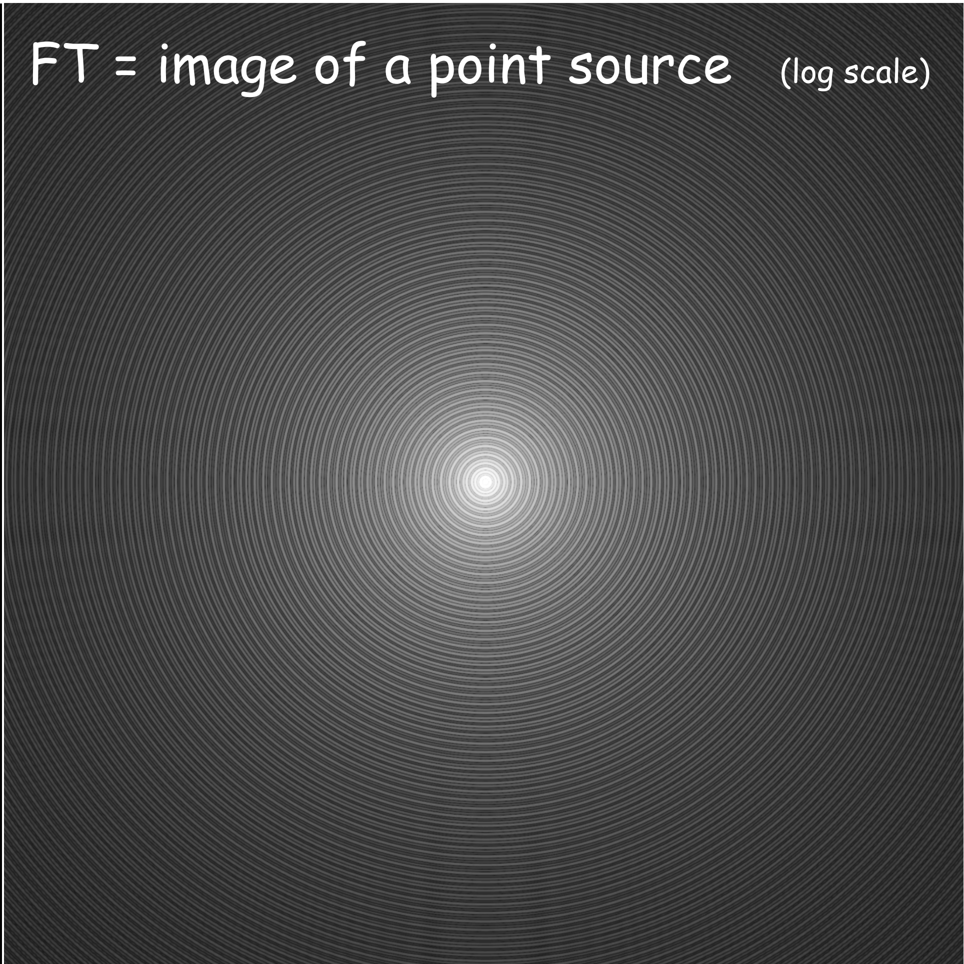
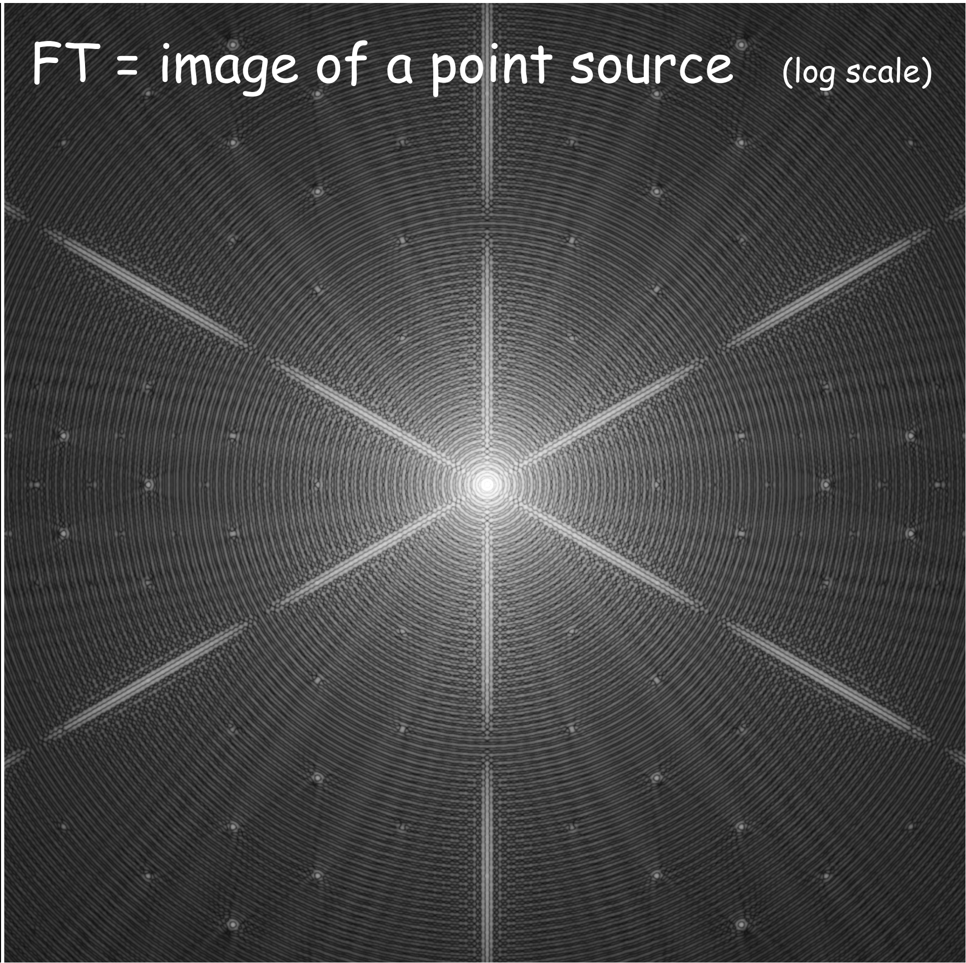


Example 2:

central obscuration, monolithic mirror (pupil) with 6 support-spiders

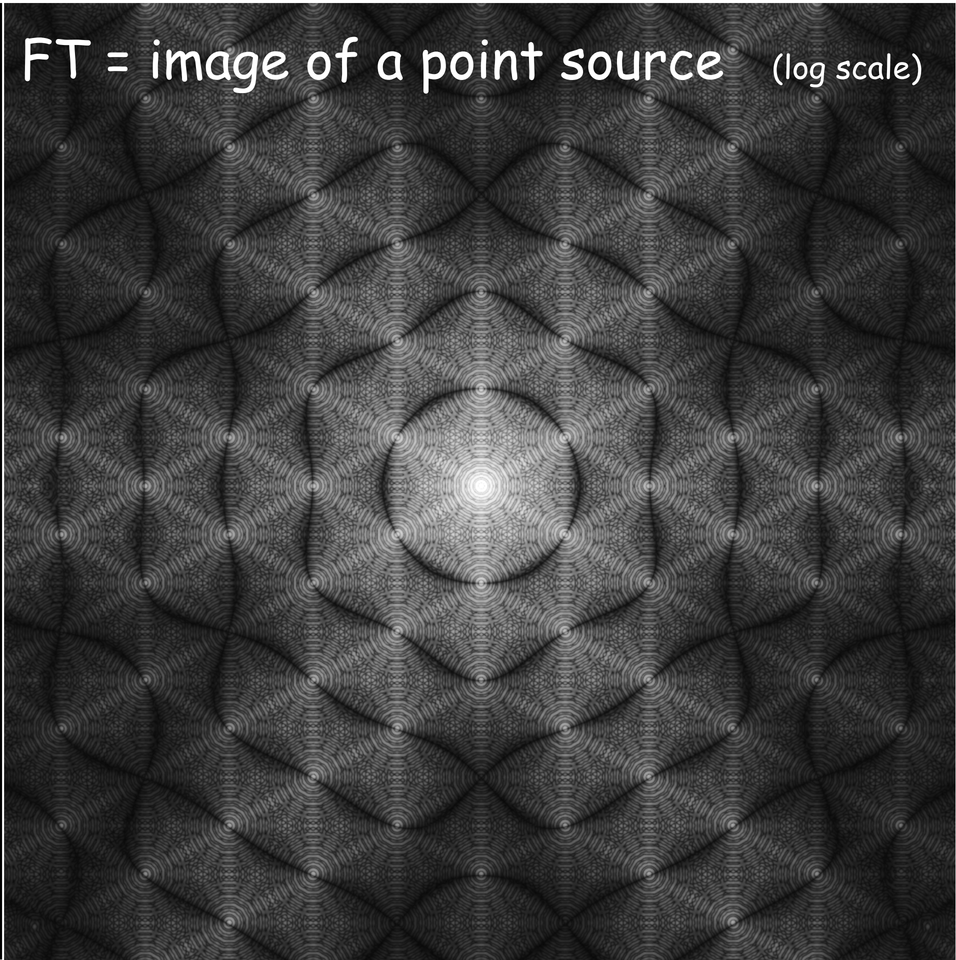


Example 3:

central obscuration,

segmented mirror (pupil)

no support-spiders

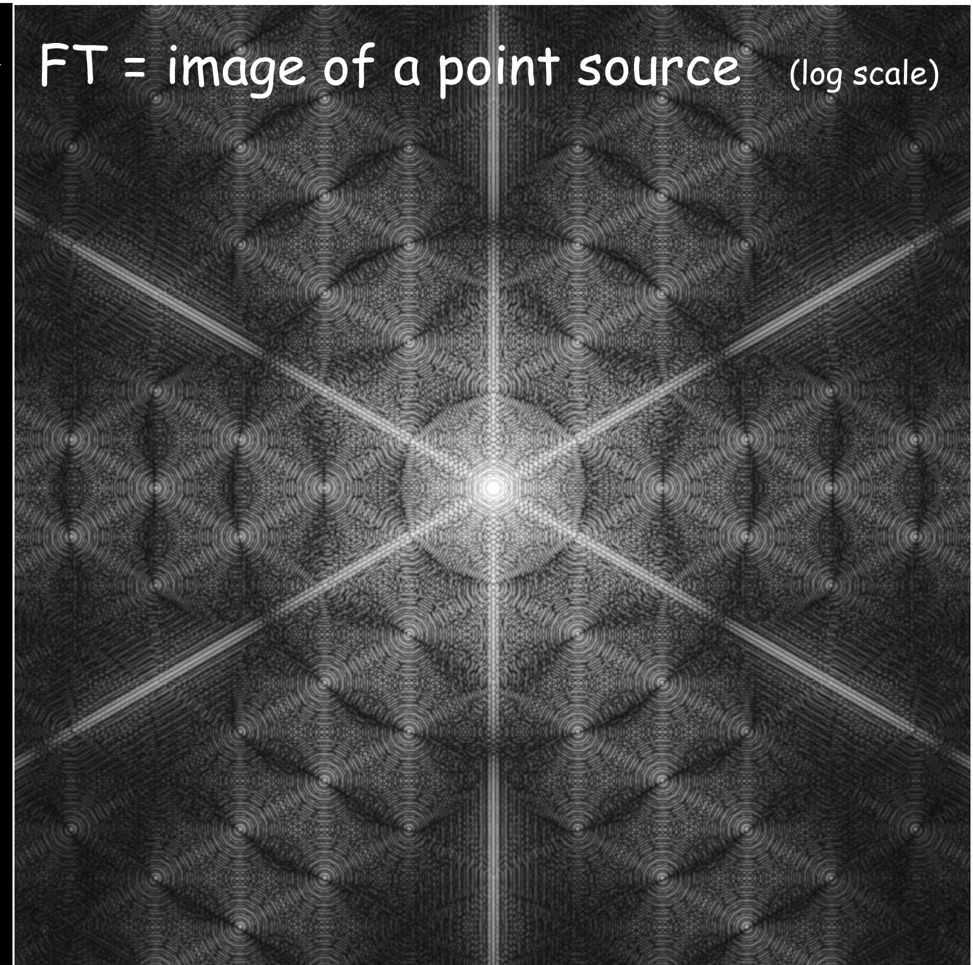


Example 4:

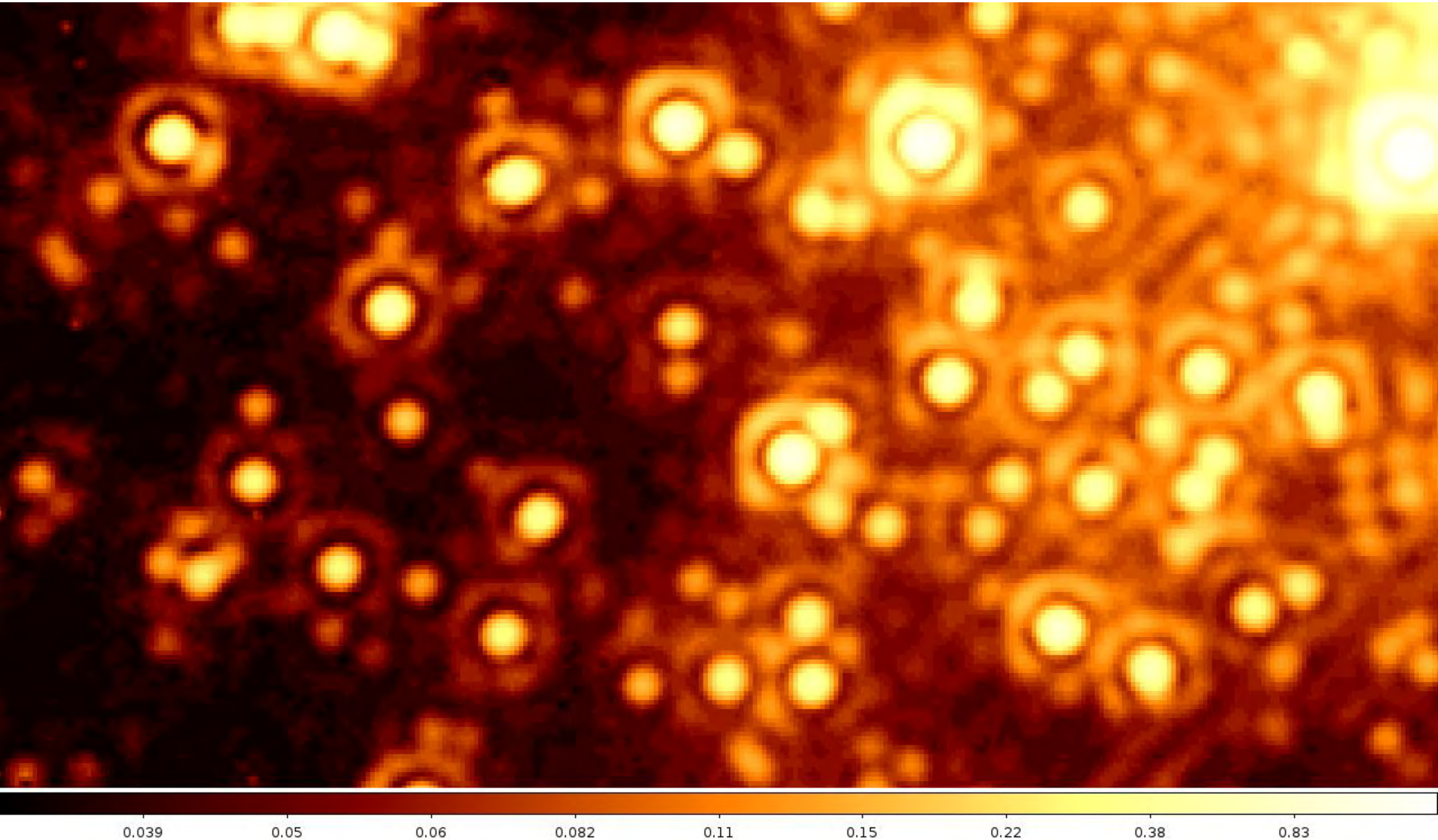
central obscuration,

segmented mirror (pupil)

with 6 support-spiders



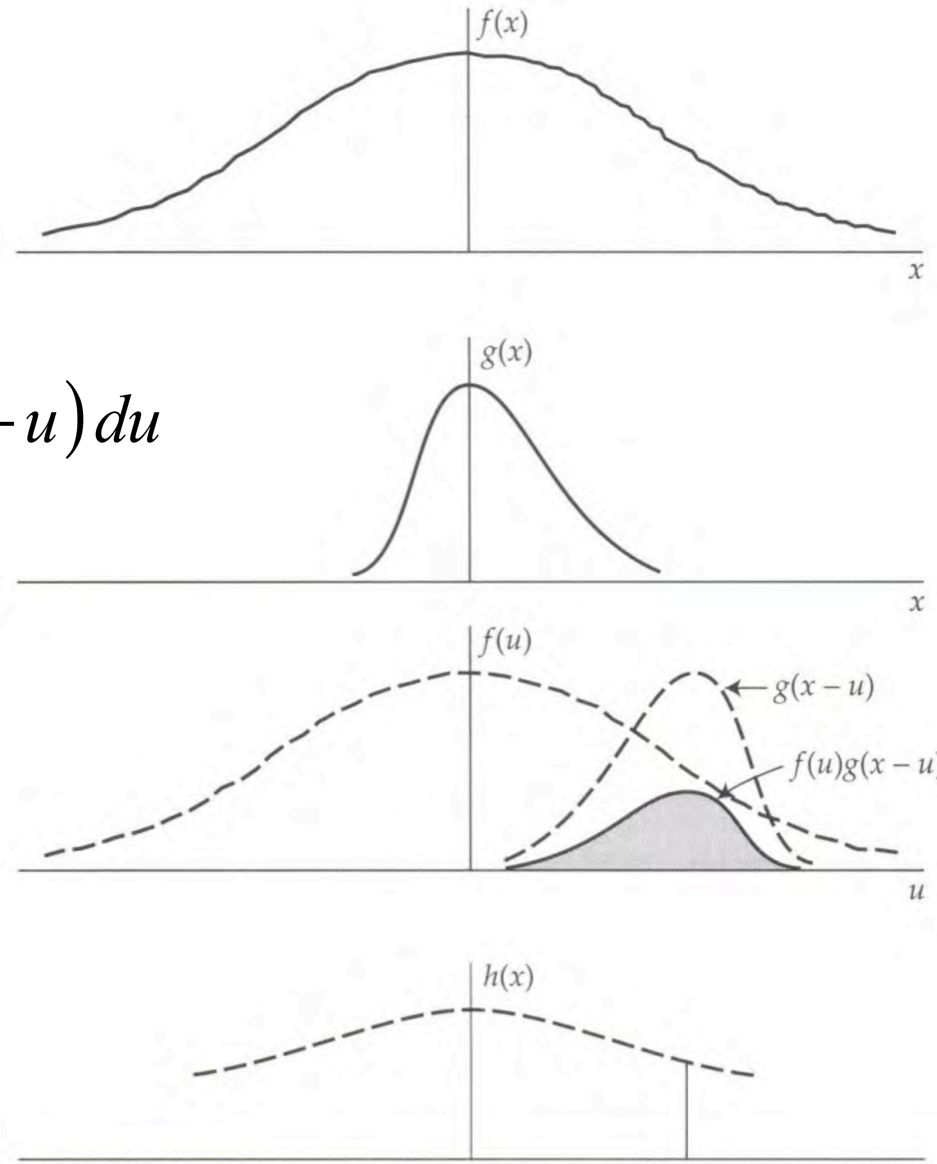
Example 5:
Star cluster observed with HST/NICMOS



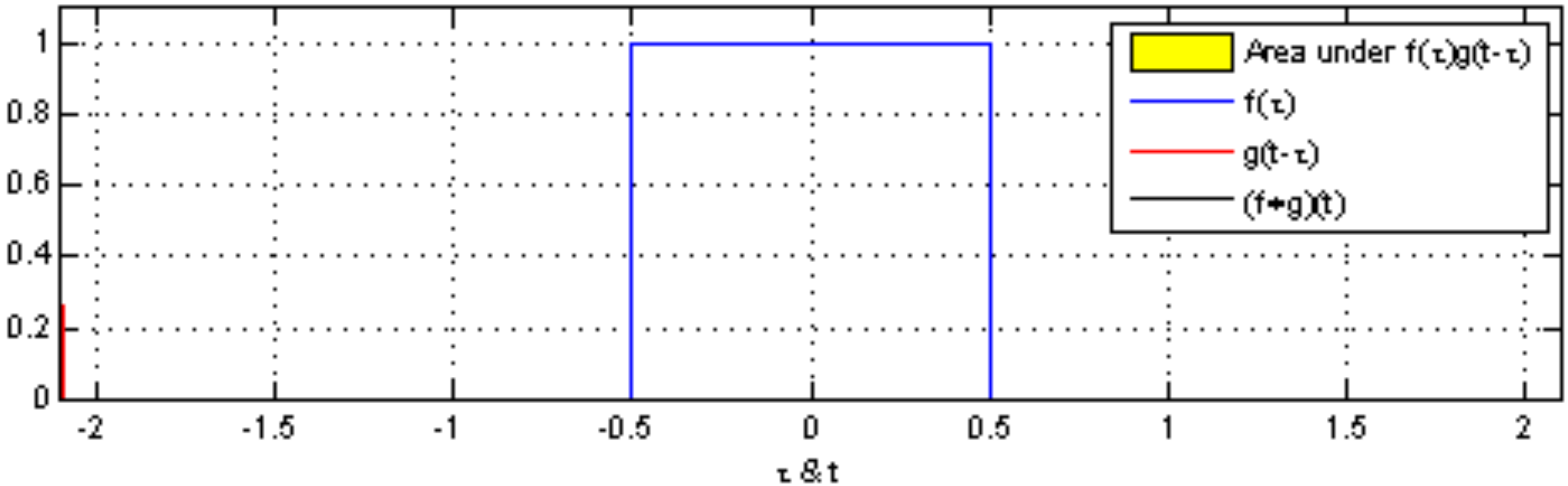
Convolution Theorem (1)

Convolution of two functions, $f*g$, is integral of product of functions after one is reversed and shifted:

$$h(x) = f(x) * g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x-u)\, du$$



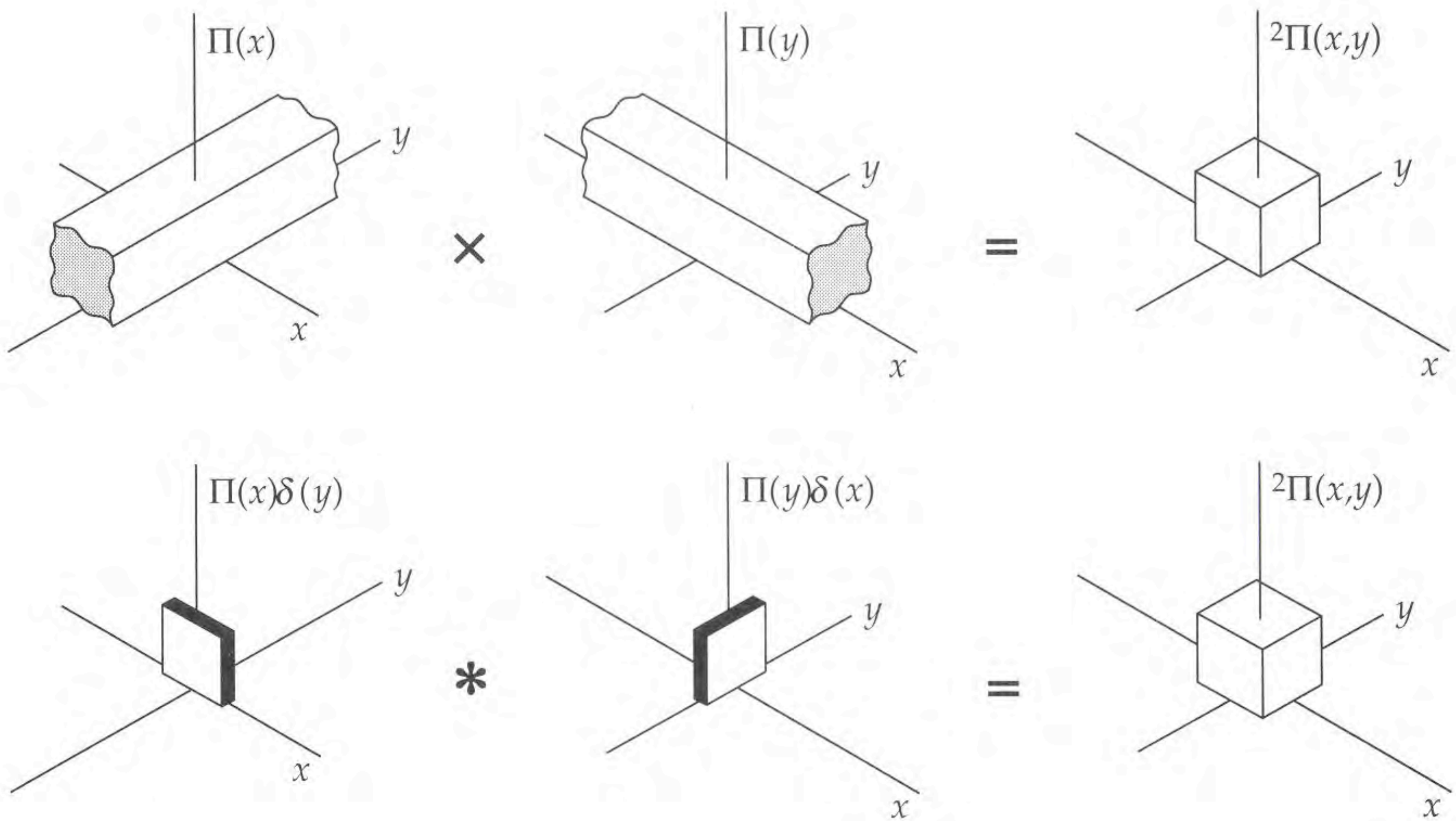
Convolution Theorem (2)



Convolution Theorem (3)

Convolution of two functions (distributions) is equivalent to product of their Fourier transforms:

$$\left.\begin{array}{l} f(x) \Leftrightarrow F(s) \\ g(x) \Leftrightarrow G(s) \end{array}\right. \rightarrow h(x) = f(x) * g(x) \Leftrightarrow F(s) \cdot G(s) = H(s)$$



Convolution Theorem (4)

<u>Example:</u>

f(x) : star

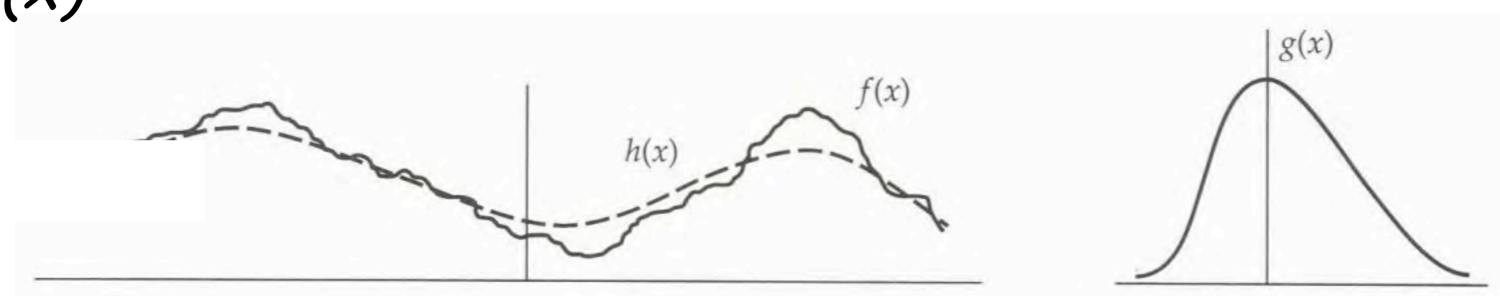
g(x) : telescope transfer function

$$f(x) * g(x) = h(x)$$

Then *h(x)* is the point spread function (PSF) of the system

<u>Example:</u>

Convolution of *f(x)* with a smooth kernel g(x) can be used to smoothen *f(x)*



<u>Example:</u>

The inverse step (deconvolution) can be used to "disentangle" two components, e.g., removing the spherical aberration of a telescope.

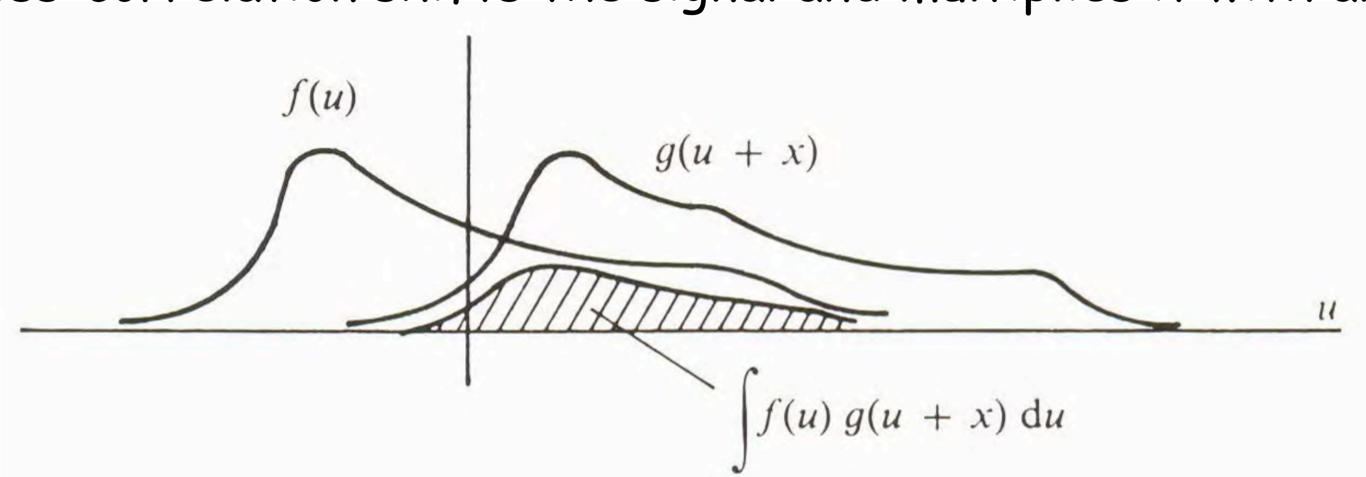
Cross-Correlation Theorem

Cross-correlation (or covariance) is measure of similarity of two waveforms as function of time-lag between them.

$$k(x) = f(x) \otimes g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x+u)\, du$$

The difference between cross-correlation and convolution is:

- Convolution reverses the signal ('-' sign)
- Cross-correlation shifts the signal and multiplies it with another



Interpretation: By how much (x) must *g(u)* be shifted to match *f(u)*? The answer is given by the maximum of *k(x)*

Convolution and Cross-Correlation

The cross-correlation is a measure of similarity of two waveforms as a function of an offset (e.g., a time-lag) between them.

$$k(x) = f(x) \otimes g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x+u)\, du$$

Example: search a long duration signal for a shorter, known feature.

The convolution is similar in nature to the cross-correlation but the convolution first reverses the signal ("mirrors the function") prior to calculating the overlap.

$$h(x) = f(x) * g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x-u)\, du$$

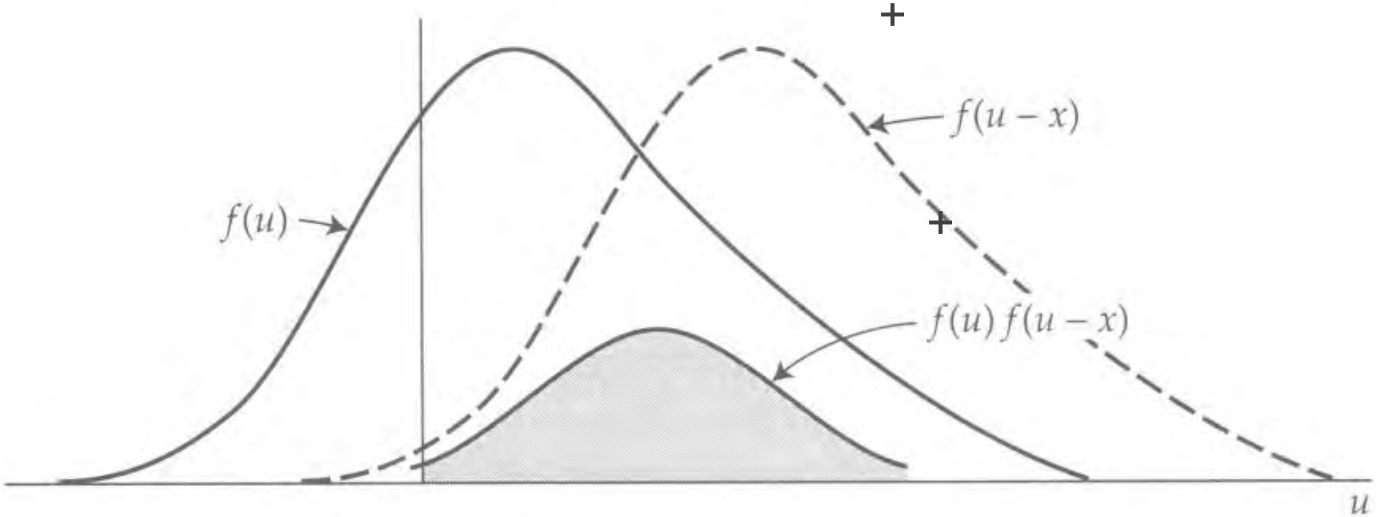
Example: the measured signal is the intrinsic signal convolved with the response function

Whereas convolution involves reversing a signal, then shifting it and multiplying by another signal, correlation only involves shifting it and multiplying (no reversing).

Auto-Correlation Theorem

Auto-correlation is cross-correlation of function with itself:

$$k(x) = f(x) \otimes f(x) = \int_{-\infty}^{+\infty} f(u) \cdot f(x+u)\, du$$



The autocorrelation function represented by an area (shown shaded).

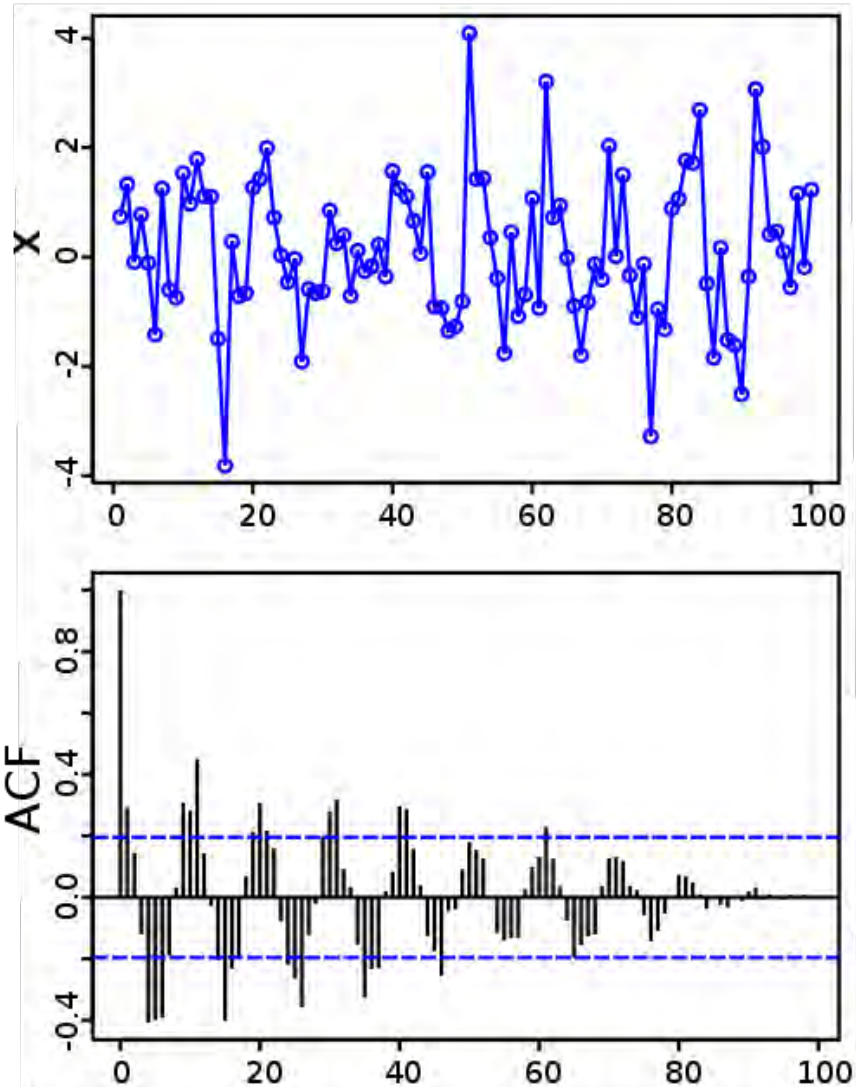
$$f(x) \otimes f(x) \Leftrightarrow F(s) F^*(s) = |F(s)|^2$$

Auto-Correlation (2)

Auto-correlation is cross-correlation of function with itself:

Wikipedia: *Auto-correlation yields the similarity between observations as a function of the time separation between them.*

Auto-correlation is a mathematical tool for finding repeating patterns, such as the presence of a periodic signal which has been buried under noise.



Power Spectrum

Power Spectrum S_f of $f(x)$ (or the Power Spectral Density, PSD) describes how the power of a signal is distributed with frequency.

Power is often defined as squared value of signal:

$$S_f(s) = |F(s)|^2$$

Power spectrum is Fourier transform of autocorrelation and indicates what frequencies carry most of the energy.

Total energy of a signal is: $$\int_{-\infty}^{+\infty} S_f(s)\,ds$$

<u>Applications:</u> spectrum analyzers, calorimeters of light

sources, …

Parseval's Theorem

Parseval's theorem (or Rayleigh's Energy Theorem) states that the sum of the square of a function is the same as the sum of the square of the Fourier transform:

$$\int_{-\infty}^{+\infty} |f(x)|^2 \, dx = \int_{-\infty}^{+\infty} |F(s)|^2 \, ds$$

<u>Interpretation:</u> Total energy contained in signal *f(x)*, summed over all x is equal to total energy of signal's Fourier transform *F(s)* summed over all frequencies *s*.

Wiener-Khinchin Theorem

Wiener-Khinchin theorem states that the power spectral density S_f of a function *f(x)* is the Fourier transform of its auto-correlation function:

$$\left|F(s)\right|^2 = FT\{f(x) \otimes f(x)\}$$

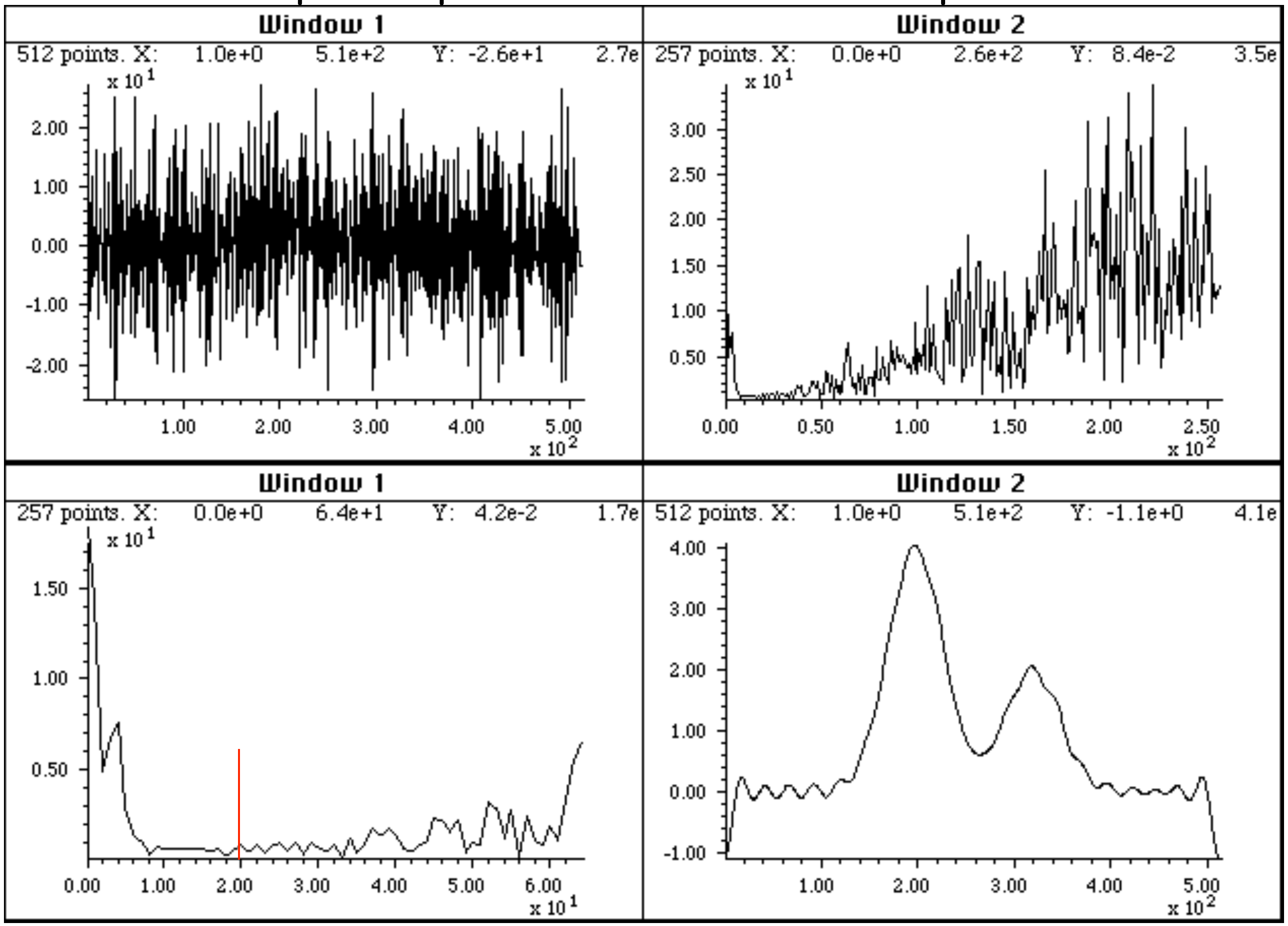
$$\updownarrow$$

$$F(s) \cdot F^*(s)$$

Applications: E.g. in the analysis of linear time-invariant systems, when the inputs and outputs are not square integrable, i.e. their Fourier transforms do not exist.

Fourier Filtering – an Example

Example taken from http://terpconnect.umd.edu/~toh/spectrum/FourierFilter.html



Top left: signal – is I just random noise?
Top right: power spectrum: high-frequency components dominate the signal
Bottom left: power spectrum expanded in X and Y to emphasize the low-frequency region.
Then: use Fourier filter function to delete all harmonics higher than 20
Bottom right: reconstructed signal → signal contains two bands at x=200 and x=300.

Overview

Convolution	$h(x) = f(x) * g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x-u)\, du$
Cross-correlation	$k(x) = f(x) \otimes g(x) = \int_{-\infty}^{+\infty} f(u) \cdot g(x+u)\, du$
Auto-correlation	$k(x) = f(x) \otimes f(x) = \int_{-\infty}^{+\infty} f(u) \cdot f(x+u)\, du$
Power spectrum	$S_f(s) = \lvert F(s) \rvert^2$
Parseval's theorem	$\int_{-\infty}^{+\infty} \lvert f(x) \rvert^2 dx = \int_{-\infty}^{+\infty} \lvert F(s) \rvert^2 ds$
Wiener-Khinchin theorem	$\lvert F(s) \rvert^2 = FT\{f(x) \otimes f(x)\}$
	$\updownarrow$ $F(s) \cdot F^*(s)$