Lecture 4: Astronomical Measurement 2

Outline

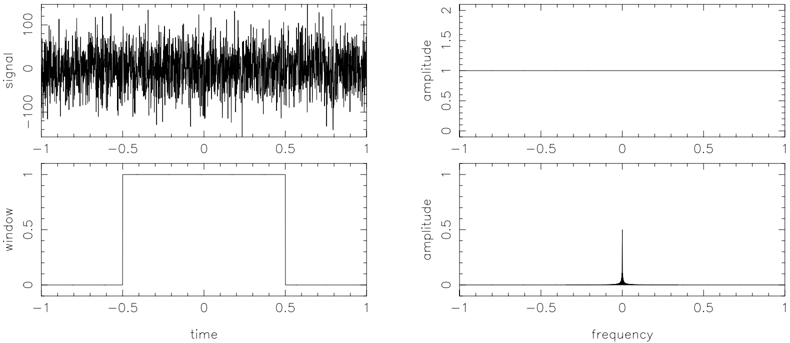
1. Time Filtering
2. First-Order Transfer Function
3. Aliasing

Time Filtering

Finite Exposure and Time Resolution

- measurement of stochastic process always takes place
  - over finite time period T, e.g. due to limited hours in one night
  - with time resolution ΔT, e.g. due to finite exposure time
- sampling interval does not have to be the same as exposure time, e.g. CCD needs time to read out image
- *duty cycle* of measurement (typically expressed in %): ratio of exposure time to sampling interval
- in the following: assume 100% duty cycle

Limited Measuring Time

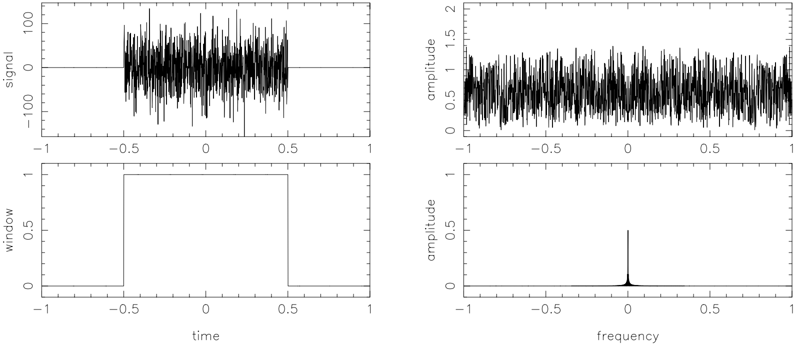


- limitation in measuring time T corresponds to *multiplication in time domain* of stochastic variable $X(t)$ with window function $\Pi(t/T)$

$$\Pi\left(\frac{t}{T}\right) \equiv 1 \quad for \quad |t| \leq \frac{1}{2}T$$

$$\Pi\left(\frac{t}{T}\right) \equiv 0 \quad for \quad |t| > \frac{1}{2}T$$

Limited Measuring Time (continued)

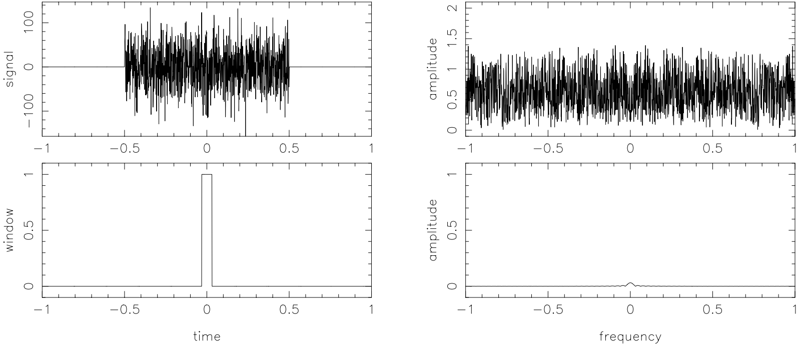


- new, time filtered, stochastic variable $Y(t)$

$$Y(t) = \Pi\left(\frac{t}{T}\right) X(t)$$

- all measurements are limited in time

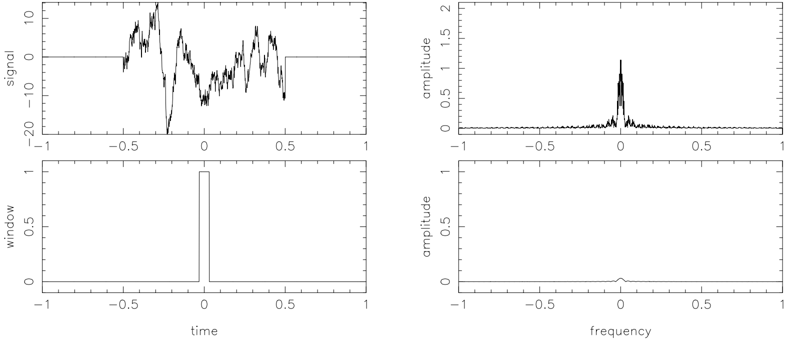
Time Resolution



- measurement at time t with temporal resolution ΔT
- is integration of stochastic variable $Y(t)$ between $t - \Delta T/2$ and $t + \Delta T/2$, divided by ΔT (running average)

$$Z(t) \equiv Y_{\Delta T}(t) = \frac{1}{\Delta T} \int_{t-\Delta T/2}^{t+\Delta T/2} Y(t')dt' = \frac{1}{\Delta T} \int_{-\infty}^{+\infty} \Pi\left(\frac{t-t'}{\Delta T}\right) Y(t')dt'$$

Time Resolution (continued)

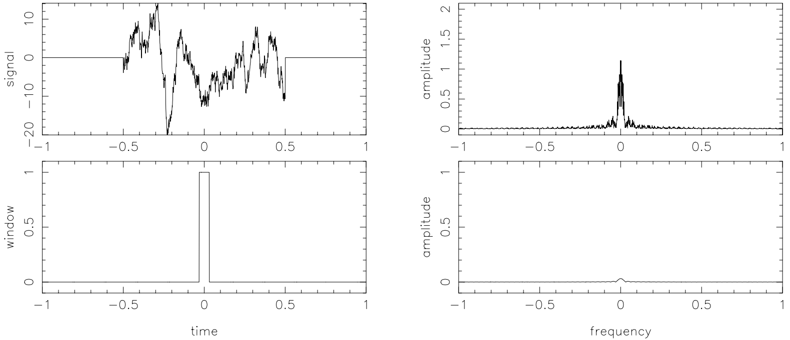


- express previous equation as convolution in time:

$$Z(t) \equiv Y_{\Delta T}(t) = \frac{1}{\Delta T} \int_{-\infty}^{+\infty} \Pi \left(\frac{t - t'}{\Delta T} \right) Y(t') dt'$$

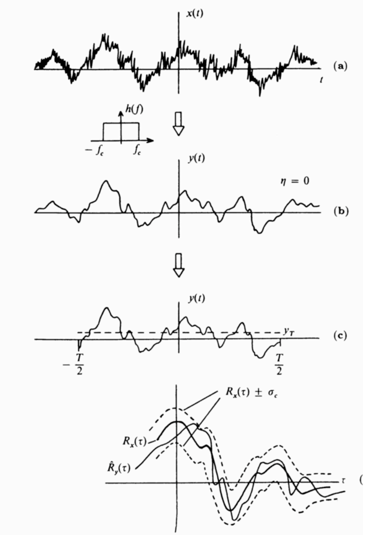
$$Z(t) = \frac{1}{\Delta T} \Pi \left(\frac{t}{\Delta T} \right) * Y(t) = \frac{1}{\Delta T} \Pi \left(\frac{t}{\Delta T} \right) * \Pi \left(\frac{t}{T} \right) X(t)$$

Time Resolution (continued)



- low-frequency (or 'low-pass') filtering of stochastic variable $Y(t)$
- limitation in time resolution always due to frequency-limited transmission characteristic of any physical measuring device
- μ_T, $R_T(\tau)$ for ergodic process obtained from finite measuring period T will slightly differ from true μ, $R(\tau)$
- error introduced by measuring sample average μ_T instead of μ

Error Assessment in Sample Average μ_T

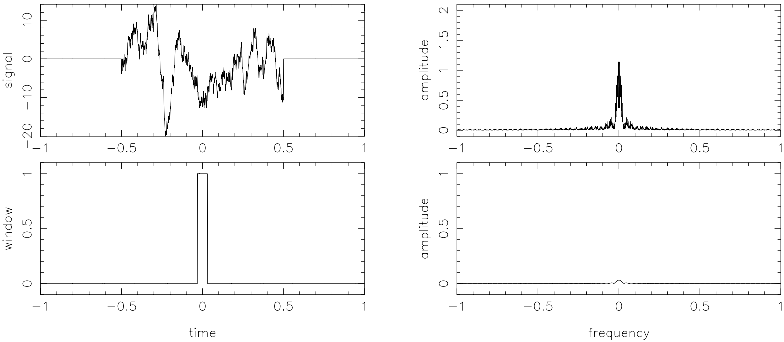


- accuracy with which approximate value μ_T approaches real value μ
- determining average corresponds to convolution in time domain with block function

$$X(t) \rightarrow \boxed{\frac{1}{T}\Pi\left(\frac{t}{T}\right)} \rightarrow X_T$$

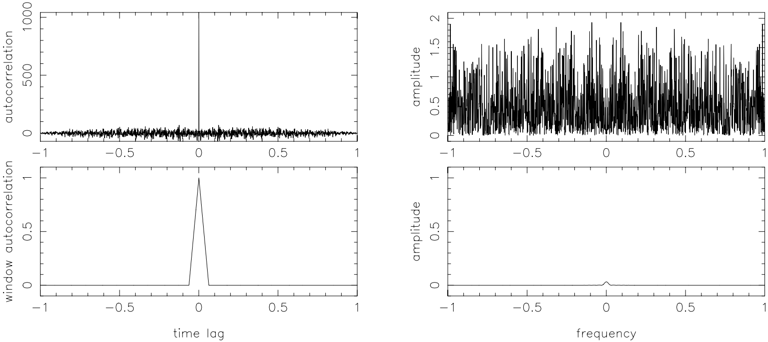
- in Fourier domain averaging corresponds to multiplication with sinc-function

Transfer Function



- influence of measuring device on signal described in Fourier domain: $Y(f) = X(f)H(f)$, $Y^*(f) = X^*(f)H^*(f)$
- $H(f)$ is *transfer function*
- therefore $|Y(f)|^2 = |X(f)|^2|H(f)|^2$
- *transfer function* used both for $H(f)$ (signal transfer function) and $|H(f)|^2$ (power transfer function)

Autocorrelation in Fourier Domain



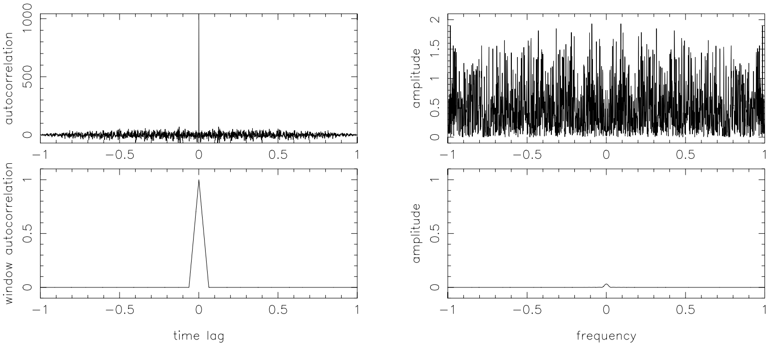
- Fourier transform of autocorrelation:

$$S_{X_T}(f) = |H(f)|^2 S_{X(t)}(f) = \mathrm{sinc}^2(Tf) \cdot S_{X(t)}(f)$$

- transforming back to time domain

$$R_{X_T}(\tau) = h(\tau) * h(\tau) * R_{X(t)}$$

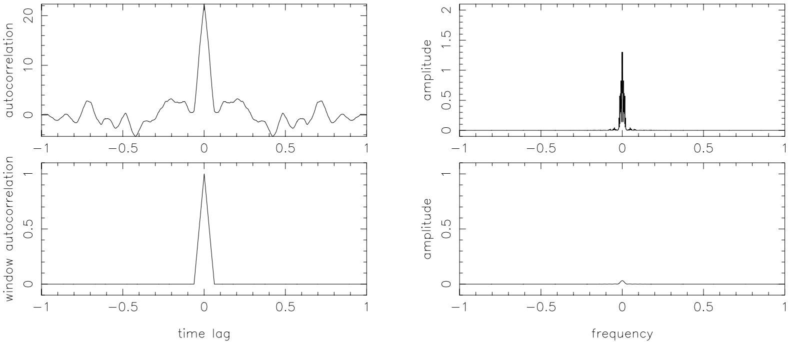
Autocorrelation in Fourier Domain



- $h \equiv (1/T)\Pi(t/T)$ is real function
- convolution of block with itself is a triangle

$$h(\tau) * h(\tau) \equiv \rho(\tau) \equiv \frac{1}{T}\Lambda\left(\frac{\tau}{T}\right)$$

Autocorrelation (continued)



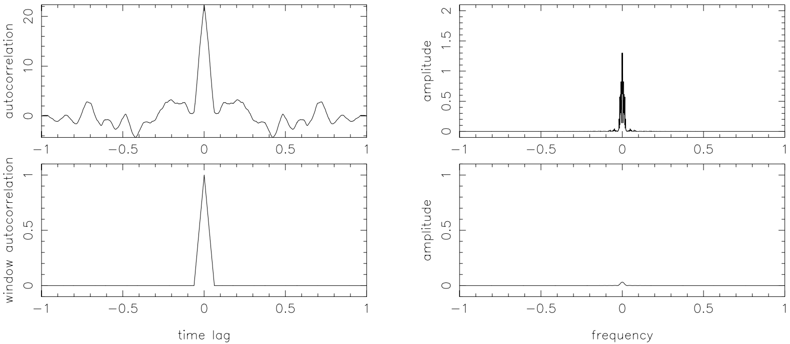
- from before

$$R_{X_T}(\tau) = h(\tau) * h(\tau) * R_{X(t)}$$

- rewrite as

$$R_{X_T}(\tau) = \frac{1}{T}\Lambda\left(\frac{\tau}{T}\right) * R_{X(t)} \equiv \frac{1}{T}\int_{-\infty}^{+\infty} \Lambda\left(\frac{\tau'}{T}\right) R_{X(t)}(\tau - \tau')d\tau'$$

Autocorrelation (continued)



- consider $\mu = 0$, i.e. $R = C$

$$C_{X_T}(\tau) = \frac{1}{T} \int_{-\infty}^{+\infty} \Lambda\left(\frac{\tau'}{T}\right) C_{X(t)}(\tau - \tau') d\tau'$$

Variance

- variance from time lag $\tau = 0$ and C even

$$C_{X_T}(0) \equiv \left[\sigma_{X_T}\right]^2 = \frac{1}{T}\int_{-\infty}^{+\infty} \Lambda\left(\frac{\tau'}{T}\right) C_{X(t)}(-\tau')d\tau'$$

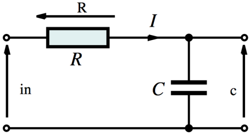
$$= \frac{1}{T}\int_{-\infty}^{+\infty} \Lambda\left(\frac{\tau'}{T}\right) C_{X(t)}(\tau')d\tau'$$

- explicitly writing triangle function Λ

$$\left[\sigma_{X_T}\right]^2 = \frac{1}{T}\int_{-T}^{+T} \left(1 - \frac{|\tau'|}{T}\right) C_{X(t)}(\tau')d\tau'$$

- integral over $\pm T$, normalization still $1/T$
- autocovariance is *always* limited in frequency domain

Example: First-Order Transfer Function



- *first-order system*: described by first-order differential equation

$$\tau_0 \frac{dY(t)}{dt} + Y(t) = X(t)$$

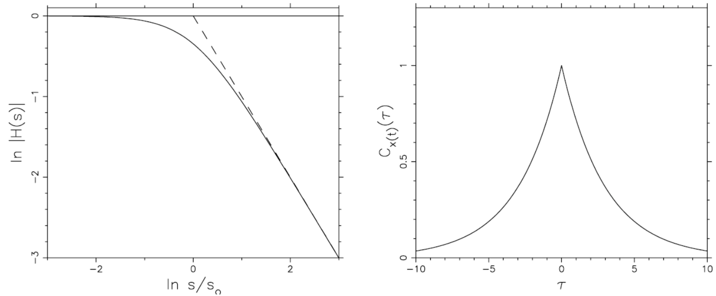
- RC circuit: $\tau_0 = RC$
- Fourier transform differential equation

$$2\pi i f \tau_0 Y(f) + Y(f) = X(f)$$

- first-order transfer function from $Y(f) = H(f) * X(f)$:

$$H(f) = \frac{1}{1 + 2\pi i f \tau_o}$$

Example: First-Order Transfer Function

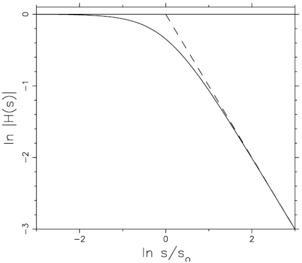


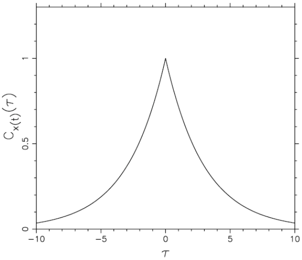
- first-order transfer function from before:

$$H(f) = \frac{1}{1 + 2\pi i f \tau_o}$$

- $f \ll 1/(2\pi\tau_o) \equiv f_o$, complete transfer, $|H(f)| = 1$
- $f \gg f_o$, transfer inversely proportional to f, $|H(f)| = f_o/f$
- cut-off frequency f_o of transfer function $H(f)$

Auto-Covariance of First-Order System





- without proof: autocovariance of first-order system drops exponentially with $|\tau|$:

$$C_{X(t)}(\tau) = C_{X(t)}(0)e^{-|\tau|/\tau_o} \qquad \text{where} \qquad \tau_o \equiv \frac{1}{2\pi f_o}$$

- $\tau \gg \tau_o \Rightarrow$ correlation virtually zero; integrate:

$$\left[\sigma_{X_T}\right]^2 = 2\left[\sigma_{X(t)}\right]^2 \frac{\tau_o}{T}\left[1 - \frac{\tau_o}{T}\left(1 - e^{-T/\tau_o}\right)\right]$$

Auto-Covariance of First-Order System (continued)

- limiting case 1: duration of measurement much longer than correlation time, $T \gg \tau_o$
- in general:

$$\left[\sigma_{X_T}\right]^2 = 2\left[\sigma_{X(t)}\right]^2 \frac{\tau_o}{T}\left[1 - \frac{\tau_o}{T}\left(1 - e^{-T/\tau_o}\right)\right]$$

- in limiting case 1

$$\left[\sigma_{X_T}\right]^2 = 2\left[\sigma_{X(t)}\right]^2 \frac{\tau_o}{T} = \frac{\left[\sigma_{X(t)}\right]^2}{\pi f_o T}$$

- variance proportional to variance of incoming signal
- variance approaches zero when duration of measurement goes to infinity
- variance approaches zero when measurement frequency goes to infinity

Auto-Covariance of First-Order System (continued)

- from before

$$\left[\sigma_{X_T}\right]^2 = 2\left[\sigma_{X(t)}\right]^2 \frac{\tau_o}{T} = \frac{\left[\sigma_{X(t)}\right]^2}{\pi f_o T}$$

- measured signal is *ergodic in the mean*
- limit can be understood by noting that $f_o T$ is number of cycles during T with a frequency f_o, i.e. it gives number of measurements
- analogous to equation for variance of average $\sigma_\mu^2 = \sigma^2/N$

Auto-Covariance of First-Order System (continued)

- limiting case 2: duration of measurement equals correlation time, $T = \tau_o$
- from before

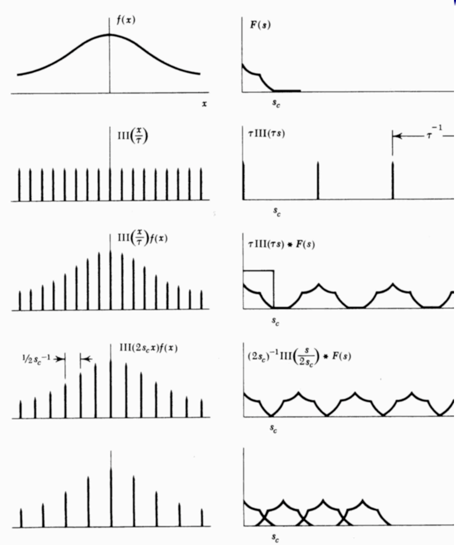
$$\left[\sigma_{X_T}\right]^2 = 2\left[\sigma_{X(t)}\right]^2 \frac{\tau_o}{T}\left[1 - \frac{\tau_o}{T}\left(1 - e^{-T/\tau_o}\right)\right]$$

- in this limiting case:

$$\sigma_{X_T}{}^2 = 2\sigma_{X(t)}{}^2 e^{-1} \simeq \sigma_{X(t)}{}^2$$

- understandable in terms of determining average in case of single measurement ($N = 1$)
- duration of measurement should be much longer than correlation time, $T \gg \tau_o$, to avoid *large* errors in estimates of average and variance
- must take into account errors in average and variance when looking for really *small* effects

Nyquist Frequency

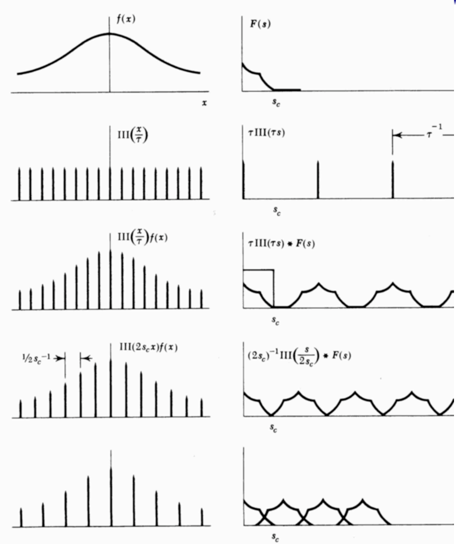


- signal $S(x)$ subject to instrument response $R(x)$
- resulting measurement

$$M(x) = S(x) * R(x)$$

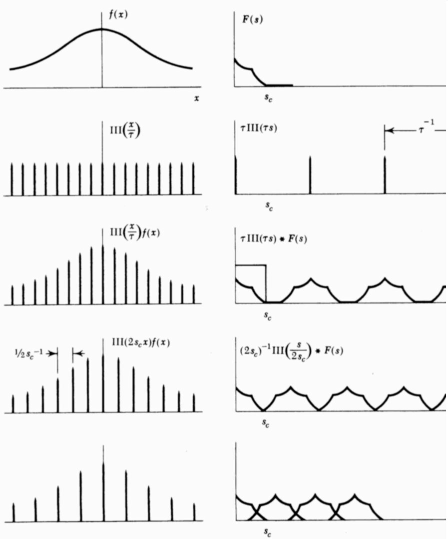
- finite frequency response of instrument $\Rightarrow M(x)$ always bandwidth limited
- Fourier transform $M(s) \Leftrightarrow M(x)$ is bandwidth-limited function
- characterized by cut-off frequency s_{max}, critical or Nyquist frequency (s_c)

Nyquist Frequency (continued)



- gaussian response: impossible because no physical system transmits frequencies to ∞
- Shannon and Nyquist established theorem for optimum sampling of band-limited observations
- theorem states that no information is lost if sampling occurs at intervals $\tau = 1/(2s_c)$

Regular Sampling

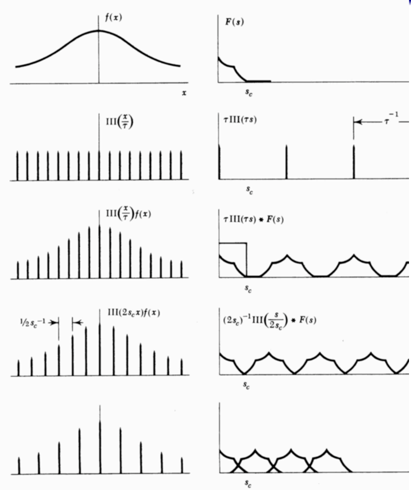


- $M(x)$ sampled at regular intervals, $M(x) \rightarrow M(n\tau)$
  - n integer
  - τ sampling interval
- describe sampling process quantitatively with Dirac comb (series of δ functions at regular distances equal to 1):

$$⊥⊥⊥(x) \equiv \sum_{n=-\infty}^{\infty} \delta(x-n)$$

- extended comb to arbitrary distances: $a⊥⊥⊥(ax) = \sum_n \delta(x - n/a)$

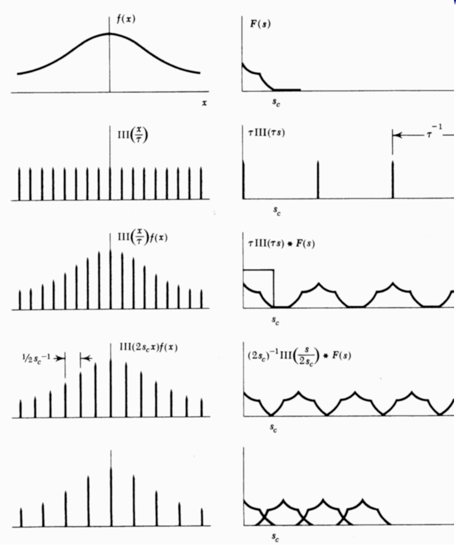
Regular Sampling



- write sampled signal $M_s(x)$ as

$$
\begin{aligned}
M_s(x) &= \sum_n M(n\tau)\delta(x - n\tau) \\
&= \frac{1}{\tau}\mathrm{III}\left(\frac{x}{\tau}\right) M(x)
\end{aligned}
$$

Regular Sampling (continued)

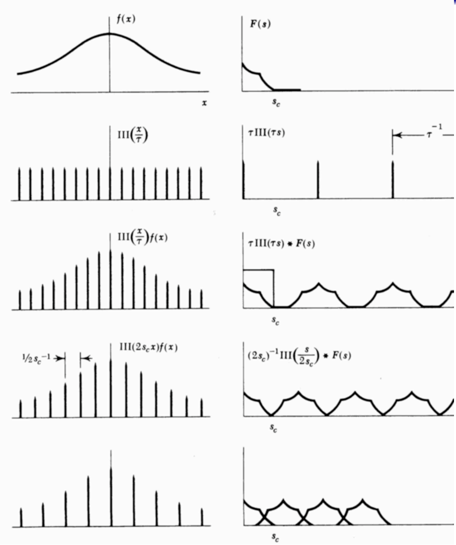


- Fourier transform pair $M_s(s) \Leftrightarrow M_s(x)$

$$
\begin{aligned}
M_s(s) &= \sqcup\!\sqcup\!\sqcup(\tau s) * M(s) \\
&= \frac{1}{\tau} \sum_n M\left(s - \frac{n}{\tau}\right)
\end{aligned}
$$

- except for factor $1/\tau$, $M_s(s)$ is series of replications of $M(s)$ at intervals $1/\tau$

Regular Sampling (continued)



- $M(s)$ bandwidth-limited function with cut-off frequency $s = s_c \Rightarrow$ fully recover single (i.e. not repeated) function $M(s)$ from series by multiplication with τ and by filtering with gate function $\Pi(s/2s_c)$:

$$\Pi\left(\frac{s}{2s_c}\right)\tau \text{Ш}(\tau s) * M(s)$$

$$\Leftrightarrow$$

$$2s_c \mathrm{sinc}\, 2s_c x * \text{Ш}\left(\frac{x}{\tau}\right) M(x)$$

Regular Sampling (continued)

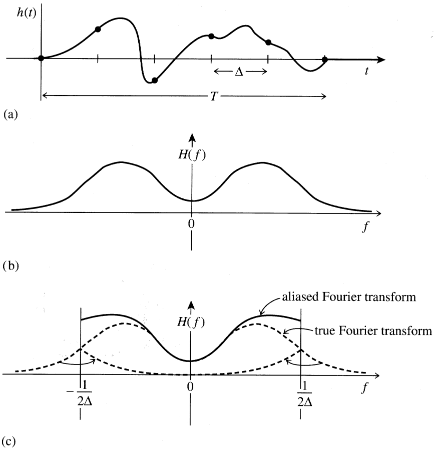
- reconstruct $M(x)$ exactly if series of $M(s)$ functions in frequency domain touch without overlap
- only possible by sampling at $\tau = 1/(2s_c)$ (optimum sampling interval)
- convolution to fully reconstruct $M(x)$:

$$\begin{aligned} M(x) &= \int_{-\infty}^{+\infty} \mathrm{sinc}\left(\frac{x-x'}{\tau}\right) \sum_n M(n\tau)\delta(x'-n\tau)dx' \\ &= \sum_n \mathrm{sinc}\left(\frac{x-n\tau}{\tau}\right) M(n\tau) \end{aligned}$$

- check result for one sampling point $x = j\tau$, with $\mathrm{sinc}(j-n) = 1$ for $j = n$ and $= 0$ for $j \neq n$:

$$M(x) = M(j\tau)$$

Aliasing



Press et al. (1992)

- function $h(t)$ shown in top panel is undersampled
- sampling interval Δ larger than $\frac{1}{2f_{max}}$
- lower panel shows that power in frequencies above $\frac{1}{2\Delta}$ is 'mirrored' with respect to this frequency
- produces aliased transform that deviates from true Fourier transform

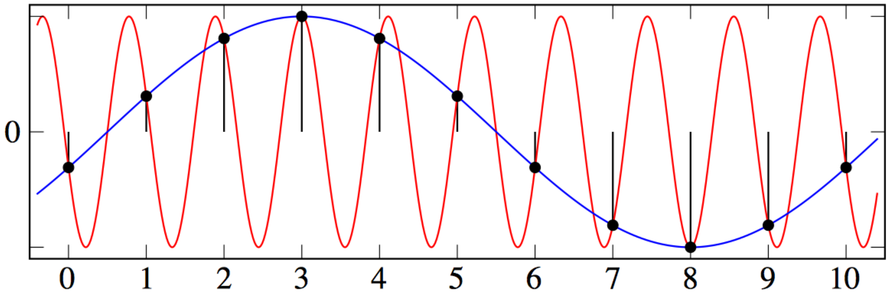
Aliasing (continued)

- calculation of intermediate points from samples does not depend on calculating Fourier transforms
- equivalent operation in x-domain is direct convolution of $2s_c \mathrm{sinc} 2s_c x$ with $⊥⊥⊥(x/\tau)M(x)$
- omission of $1/\tau$ factor ensures proper normalization in s-domain
- superposition of series of sinc-functions with weight factors $M(n\tau)$, i.e. the sample values, at intervals τ exactly reconstruct the continuous function $M(x)$
- sinc-functions provide proper interpolation between consecutive sample points
- sinc-function referred to as *interpolation function*

Aliasing (continued)

- discrete Fourier transform causes no loss of information if sampling frequency $\frac{1}{\tau}$ is twice the highest frequency in continuous input function
- maximum frequency s_{max} for given sampling interval is $\frac{1}{2\tau}$
- input signal sampled too slowly (contains frequencies higher than $\frac{1}{2\tau}$) $\Rightarrow$ source signal cannot be determined after sampling process
- loss of fine details
- must apply low-pass filter before sampling:
  - electronic low-pass filter for electrical signals
  - defocusing of telescope for imaging

Aliasing in Fourier Domain



http://en.wikipedia.org/wiki/File:AliasingSines.svg

- unresolved, high frequencies beat with measured frequencies
- produce spurious components in frequency domain below Nyquist frequency
- may give rise to major problems and uncertainties in the determination of source function