Lecture 3: Astronomical Measurement 1

Outline

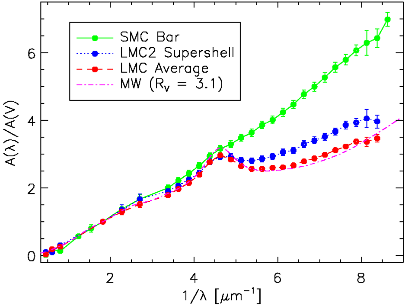
1. Tales of a Photon
2. Measurement System
3. Filtering
4. Aliasing

The Source

- stellar spectrum
- unpolarized

Interstellar Travel

- interstellar dust modifies spectrum (extinction)
- aligned grains introduce linear polarization



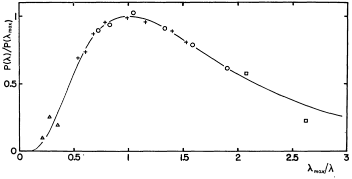
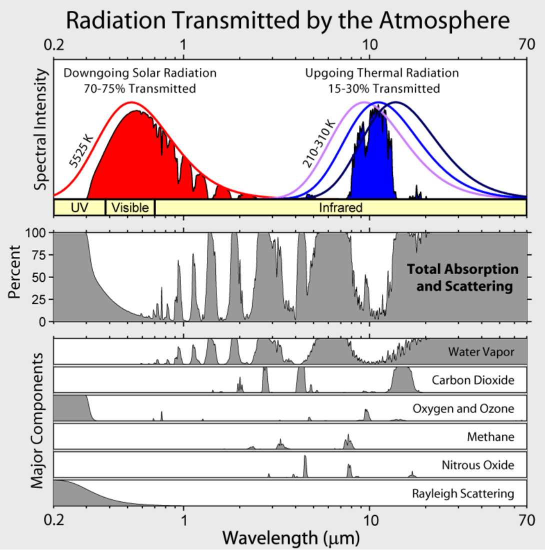


Fig. 5. Normalized wavelength dependence of interstellar polarization averaged for 6 stars in Perseus-Cepheus (crosses) for which $\lambda_{max} \cong 0.52\ \mu$ and for 5 stars in Scorpius (open circles) for which $\lambda_{max} \cong 0.70\ \mu$. Ultraviolet balloon observations for ζ Ophiuchi at $\lambda = 0.225\ \mu$ and $0.286\ \mu$ by Gehrels (1973; squares) and the infrared observations of HD 183143 at $\lambda = 1.6\ \mu$ and VI Cyg *12 at $\lambda = 1.6\ \mu$ and $\lambda = 2.2\ \mu$ by Dyck (1973; triangles) are also plotted. The solid line is calculated from Equation (1).

Serkowski (1973)

http://en.wikipedia.org/wiki/File:Interstellar_extinction_ave_curves_local_group.png

Earth Atmosphere – Spectral Absorption



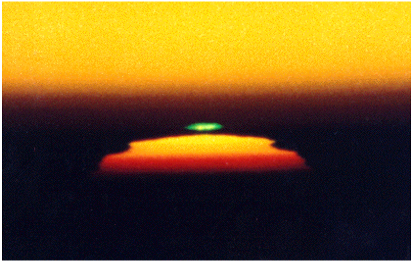
absorption depends on

- location
- altitude
- elevation of object
- pressure
- water vapor content
- cirrus and clouds

Earth Atmosphere – Refraction and Dispersion

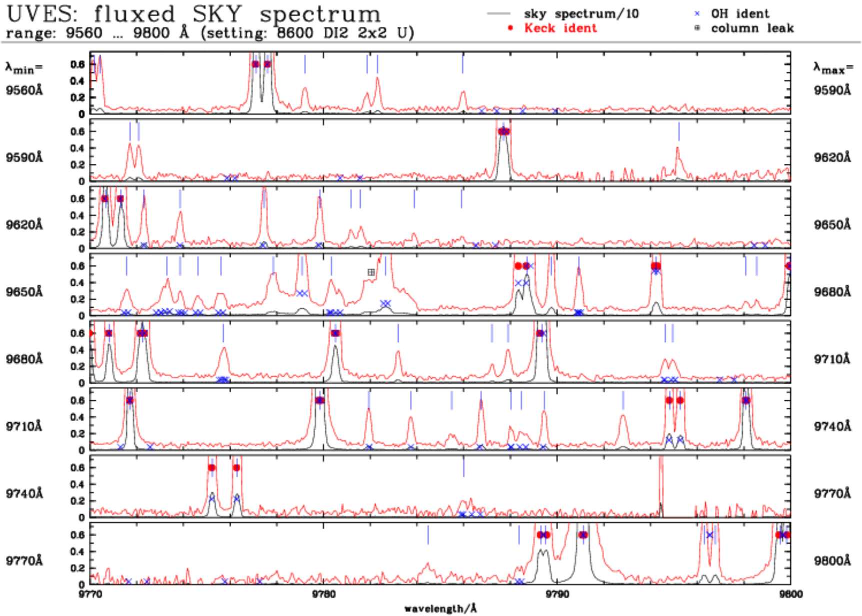
- refraction: from vacuum ($n = 1$) into air ($n > 1$)
- dispersion due to wavelength-dependence of index of refraction of air

http://cseligman.com/text/sky/atmosphericdispersion.htm



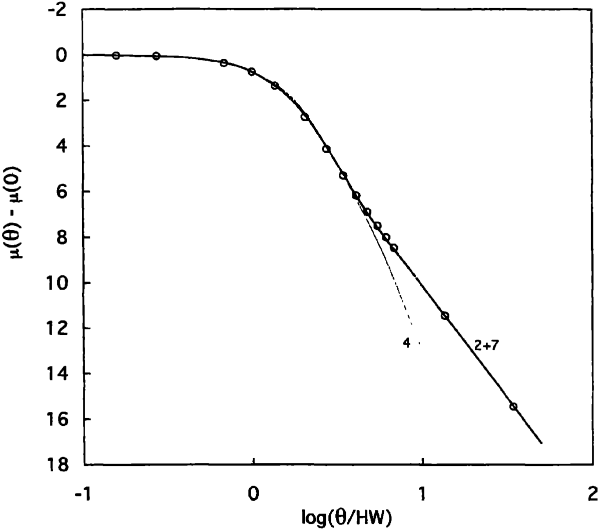
http://www.isc.tamu.edu/ astro/research/sandiego.html

Earth Atmosphere – Sky Emission



Earth Atmosphere – Angular

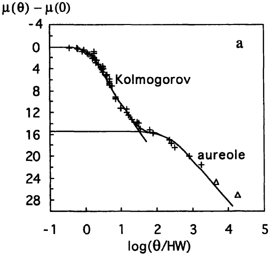
- seeing, scattering, point-spread function



Racine (1996)

Telescope Optics

- point-spread function depends on wavelength
- optical aberrations
- scattering
- polarization



Racine (1996)

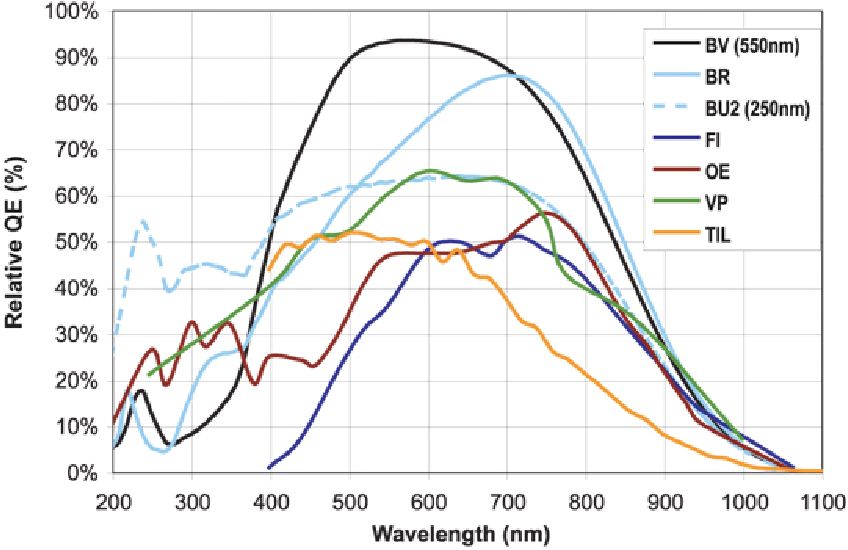
Spectrograph Slit

- spatial sampling

Spectrograph

- spectral smearing due to finite slit, finite grating
- wavelength to position translation

Detector – Quantum Efficiency: Photons to Electrons



www.andor.com

Detector – Other Effects

- spatial and temporal sampling
- additive electronic bias
- additive dark current
- additive cosmic rays
- bad pixels
- gain: from photo-electrons to ADUs (Arbitrary Digital Units)

Analog to Digital Conversion

- readout noise
- discretization noise

Data Reduction

- dark and flat correction
- bad pixel and cosmic ray removal
- wavelength calibration

Astronomer

- interpretation

Integral response function for astronomical measurements

- response of astronomical measuring process to incoming radiation characterised by filtering process
- filtering arising from individual elements making up measurement system
- stochastic process described by monochromatic intensity $I(\nu, \vec{\Omega}, t)$
- time-dependent output of system described by

$$X(t) = S(t) + N(t)$$

$S(t)$ outcome of filtering of signal source
$N(t)$ sum of all (filtered) noise components

Noise Sources in Astronomical Observations

- background radiation
  - background sources
  - foreground sources
  - sky emission
  - warm optics emission
- disturbances arising from operational environment
  - mechanical vibration
  - induction of electrical signals
- intrinsic noise in detection system
  - dark current
  - readout noise

Integral Response Function for Astronomical Measurements

- consider measuring process of source signal $S(t)$ as series of consecutive convolutions
- convolution kernels are angular and spectral response functions of measurement system

$$S(t) = \int_{\Delta\nu} \left[R(\nu) * \int_{\Delta\vec{\Omega}_{FOV}} \left[I(\vec{\Omega}, \nu, t) * P(\vec{\Omega}, \nu) \right] d\vec{\Omega} \right] d\nu$$

- $P(\vec{\Omega}, \nu)$: collecting power of telescope, depends on frequency
  - function of telescope off-axis angle in field of view $\vec{\Omega}_{FOV}$
  - contains point spread function (PSF) $H(\vec{\Omega}, \nu)$, quantitatively describes angular resolution (field position dependent)
- $\Delta\vec{\Omega}_{FOV}$ is solid angle over which convolution $I(\vec{\Omega}, \nu, t) * P(\vec{\Omega}, \nu)$ is integrated

Integral Response Function (continued)

- from before

$$S(t) = \int\limits_{\Delta\nu} \left[R(\nu) * \int\limits_{\Delta\vec{\Omega}_{FOV}} \left[I(\vec{\Omega}, \nu, t) * P(\vec{\Omega}, \nu) \right] d\vec{\Omega} \right] d\nu$$

- choice of $\Delta\vec{\Omega}_{FOV}$ depends on
  - number of available pixels
  - science goal of observation
- integral over whole field of view $\vec{\Omega}_{FOV}$
- may cover large part of sky
- may just be one pixel

Integral Response Function (continued)

- from before

$$S(t) = \int\limits_{\Delta\nu} \left[R(\nu) * \int\limits_{\Delta\vec{\Omega}_{FOV}} \left[I(\vec{\Omega}, \nu, t) * P(\vec{\Omega}, \nu) \right] d\vec{\Omega} \right] d\nu$$

- integration signal after second convolution with $R(\nu)$ covers spectral range of interest $\Delta\nu$, which is part of total bandwidth ν
- from very narrow range (e.g. measuring the line profile of a single spectral line) to a very broad range (in case of photometry)
- number of frequency elements can therefore range from 1 (e.g. in the case of a bolometric detector) to approximately 10^6 in a high-resolution spectrograph

Fourier Frequencies

- term *frequency* covers 3 types of Fourier pairs:
  1. $I(\vec{\Omega}) \Leftrightarrow I(\vec{\zeta})$ refers to spatial resolution, frequency $\vec{\zeta}$ in Fourier domain is a *spatial frequency*; structures in image
  2. $I(\nu) \Leftrightarrow I(s)$ refers to spectral resolution, frequency s is Fourier frequency related to *spectral frequency*; spectrum containing large number of sharp features (narrow emission and absorption lines) has much power in high spectral frequencies; featureless continuum contains only low spectral frequencies
  3. The pair $I(t) \Leftrightarrow I(f)$ refers to time resolution, frequency f relates to *temporal frequency*.
- every measurement or observation implies bandwidth limitations on each of these frequencies

Modulation Transfer Function

- normalised value of the Fourier transform of particular instrument response function, e.g. $R(s)$ or $H(\vec{\zeta})$, is called the *Modulation Transfer Function* (MTF) and
- MTF describes frequency-dependent filtering of source signal in Fourier domain
- MTF refers either to amplitude/phase transfer function of signal or to power transfer function
- in practice this will be explicitly clear from the specific context in which the MTF is employed.

Finite Exposure and Time Resolution

- measurement or registration of a stochastic process always takes place
  - over a finite period T
  - with a certain resolution ΔT, i.e. the minimum time bin for a data point
- limitation in measuring time T corresponds to *multiplication in the time domain* of a stochastic variable $X(t)$ with a window function $\Pi(t/T)$

$$
\begin{aligned}
\Pi\left(\frac{t}{T}\right) &\equiv 1 \quad for \quad |t| \leq \frac{1}{2}T \\
\Pi\left(\frac{t}{T}\right) &\equiv 0 \quad for \quad |t| > \frac{1}{2}T
\end{aligned}
$$

Time Resolution (continued)

- new, time filtered, stochastic variable $Y(t)$

$$Y(t) = \Pi\left(\frac{t}{T}\right) X(t)$$

- limitation in time resolution always arises in practice due to frequency-limited transmission characteristic of any physical measuring device

Example

- measurement taken at time t within measuring period T with temporal resolution ΔT
- corresponds to integration of stochastic variable $Y(t)$ between $t - \Delta T/2$ and $t + \Delta T/2$, divided by ΔT (running average)
- as an equation

$$Z(t) \equiv Y_{\Delta T}(t) = \frac{1}{\Delta T} \int_{t-\Delta T/2}^{t+\Delta T/2} Y(t')dt' = \frac{1}{\Delta T} \int_{-\infty}^{+\infty} \Pi\left(\frac{t-t'}{\Delta T}\right) Y(t')dt'$$

Example (continued)

- from before

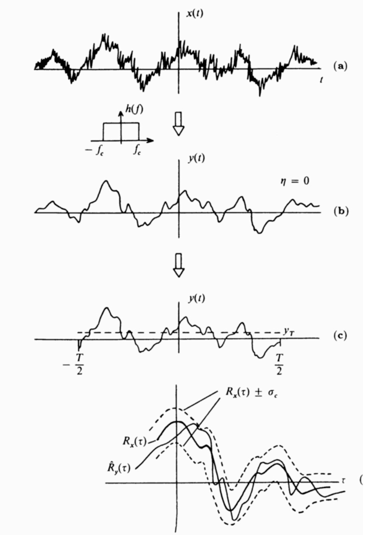
$$Z(t) \equiv Y_{\Delta T}(t) = \frac{1}{\Delta T} \int_{t-\Delta T/2}^{t+\Delta T/2} Y(t')dt' = \frac{1}{\Delta T} \int_{-\infty}^{+\infty} \Pi\left(\frac{t-t'}{\Delta T}\right) Y(t')dt'$$

- express in terms of convolution in time domain:

$$Z(t) = \frac{1}{\Delta T}\Pi\left(\frac{t}{\Delta T}\right) * Y(t) = \frac{1}{\Delta T}\Pi\left(\frac{t}{\Delta T}\right) * \Pi\left(\frac{t}{T}\right) X(t)$$

- low-frequency (or 'low-pass') filtering of stochastic variable $Y(t)$
- values μ_T and $R_T(\tau)$ for an ergodic process obtained from finite measuring period T will therefore slightly differ from the true values μ and $R(\tau)$
- error introduced by measuring sample average μ_T rather than true average μ

Error Assessment in Sample Average μ_T



- accuracy with which approximate value μ_T approaches real value μ
- determining average corresponds to convolution in time domain with block function

$$X(t) \rightarrow \boxed{\frac{1}{T}\Pi\left(\frac{t}{T}\right)} \rightarrow X_T$$

- in Fourier domain averaging corresponds to multiplication with sinc-function

Sample Average (continued)

- influence of measuring device on signal in Fourier domain:

$$Y(f) = X(f)H(f)\,,\; Y^*(f) = X^*(f)H^*(f)$$

- $H(f)$ is the *transfer function*
- therefore $|Y(f)|^2 = |X(f)|^2|H(f)|^2$
- *transfer function* used both for $H(f)$ (signal transfer function) and $|H(f)|^2$ (power transfer function)

Autocorrelation in Fourier Domain

- Fourier transform of autocorrelation:

$$S_{X_T}(f) = |H(f)|^2 S_{X(t)}(f) = \mathrm{sinc}^2(Tf) \cdot S_{X(t)}(f)$$

- transforming back to time domain

$$R_{X_T}(\tau) = h(\tau) * h(\tau) * R_{X(t)}$$

- $h \equiv (1/T)\Pi(t/T)$ is real function
- convolution of block with itself is a triangle

$$h(\tau) * h(\tau) \equiv \rho(\tau) \equiv \frac{1}{T}\Lambda\left(\frac{\tau}{T}\right)$$

Autocorrelation (continued)

- from before

$$R_{X_T}(\tau) = h(\tau) * h(\tau) * R_{X(t)}$$

- rewrite as

$$R_{X_T}(\tau) = \frac{1}{T}\Lambda\left(\frac{\tau}{T}\right) * R_{X(t)} \equiv \frac{1}{T}\int_{-\infty}^{+\infty} \Lambda\left(\frac{\tau'}{T}\right) R_{X(t)}(\tau - \tau')d\tau'$$

- consider $\mu = 0$, i.e. $R = C$

$$C_{X_T}(\tau) = \frac{1}{T}\int_{-\infty}^{+\infty} \Lambda\left(\frac{\tau'}{T}\right) C_{X(t)}(\tau - \tau')d\tau'$$

Variance

- variance from $\tau = 0$

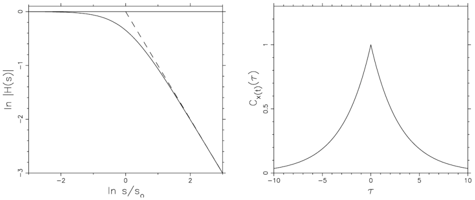
$$C_{X_T}(0) \equiv \left[\sigma_{X_T}\right]^2 = \frac{1}{T}\int_{-\infty}^{+\infty} \Lambda\left(\frac{\tau'}{T}\right) C_{X(t)}(-\tau')d\tau' = \frac{1}{T}\int_{-\infty}^{+\infty} \Lambda\left(\frac{\tau'}{T}\right) C_{X(t)}$$

- used fact that C is even
- Explicitly writing Λ we finally obtain

$$\left[\sigma_{X_T}\right]^2 = \frac{1}{T}\int_{-T}^{+T} \left(1 - \frac{|\tau'|}{T}\right) C_{X(t)}(\tau')d\tau'$$

- integral ranges from $-T$ to $+T$, i.e. over a range with length $2T$, but nonetheless the normalization factor is $1/T$
- autocovariance is *always* limited in frequency domain

Example: First-Order Transfer Function



- first-order transfer function:

$$H(f) = \frac{1}{1 + 2\pi i f \tau_o}$$

- $f \ll 1/(2\pi\tau_o) \equiv f_o$, complete transfer, $|H(f)| = 1$
- $f \gg f_o$, transfer inversely proportional to temporal frequency, $|H(f)| = f_o/f$
- cut-off frequency f_o of transfer function $H(f)$
- autocovariance drops exponentially with $|\tau|$ (right)

Auto-Covariance of First-Order System

- without proof: autocovariance of first-order system drops exponentially with (the absolute value of) the time difference τ:

$$C_{X(t)}(\tau) = C_{X(t)}(0)e^{-|\tau|/\tau_o} \qquad \text{where} \qquad \tau_o \equiv \frac{1}{2\pi f_o}$$

- at times $\tau \gg \tau_o$ the correlation is virtually zero
- performing the integration, we get

$$\left[\sigma_{X_T}\right]^2 = 2\left[\sigma_{X(t)}\right]^2 \frac{\tau_o}{T}\left[1 - \frac{\tau_o}{T}\left(1 - e^{-T/\tau_o}\right)\right]$$

Auto-Covariance of First-Order System (continued)

- first limiting case: duration of measurement much longer than correlation time, $T \gg \tau_o$
- in general:

$$\left[\sigma_{X_T}\right]^2 = 2\left[\sigma_{X(t)}\right]^2 \frac{\tau_o}{T}\left[1 - \frac{\tau_o}{T}\left(1 - e^{-T/\tau_o}\right)\right]$$

- becomes in this limiting case

$$\left[\sigma_{X_T}\right]^2 = 2\left[\sigma_{X(t)}\right]^2 \frac{\tau_o}{T} = \frac{\left[\sigma_{X(t)}\right]^2}{\pi f_o T}$$

- variance of measured signal proportional to variance of incoming signal
- variance approaches zero when duration of measurement goes to infinity
- variance approaches zero when number of frequencies over which one measures goes to infinity

Auto-Covariance of First-Order System (continued)

- measured signal is *ergodic in the mean*.
- limit can be understood by noting that $f_o T$ is the number of cycles during T with a frequency f_o, i.e. it gives the number of measurements
- analogous to equation which gives variance of average $\sigma_\mu^2 = \sigma^2/N$

Auto-Covariance of First-Order System (continued)

- Second limiting case: duration of measurement equals correlation time, $T = \tau_o$
- from before

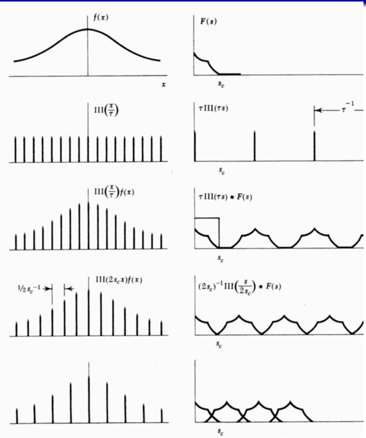
$$\left[\sigma_{X_T}\right]^2 = 2\left[\sigma_{X(t)}\right]^2 \frac{\tau_o}{T}\left[1 - \frac{\tau_o}{T}\left(1 - e^{-T/\tau_o}\right)\right]$$

- in this limiting case:

$$\sigma_{X_T}{}^2 = 2\sigma_{X(t)}{}^2 e^{-1} \simeq \sigma_{X(t)}{}^2$$

- understandable in terms of determining average in case of single measurement ($N = 1$)
- duration of measurement should be much longer than correlation time, $T \gg \tau_o$, to avoid *large* errors in estimates of average and variance
- must take into account errors in average and variance when looking for really *small* effects

Nyquist Frequency

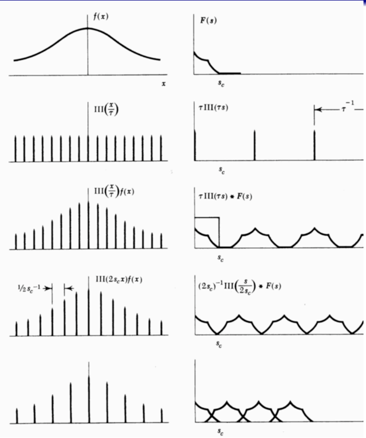


- general case of signal $S(x)$ subject to instrument response $R(x)$
- resulting measurement

$$M(x) = S(x) * R(x)$$

- finite frequency response of instrument $\Rightarrow M(x)$ is always limited in bandwidth
- Fourier transform $M(s) \Leftrightarrow M(x)$ is bandwidth-limited function
- function is characterised by maximum cut-off frequency s_{max}, also called the critical or Nyquist frequency (s_c)

Nyquist Frequency (continued)



- gaussian response: frequencies will never be distributed purely gaussian, since no physical system can transmit the tail frequencies up to ∞
- . Shannon (and Nyquist) established a theorem for optimum sampling of band limited observations.
- theorem states that no information is lost if sampling occurs at intervals $\tau = 1/(2s_c)$

Regular Sampling

- $M(x)$ is sampled at regular intervals, $M(x) \rightarrow M(n\tau)$ with n an integer and τ the sampling interval
- describe sampling process quantitatively with Dirac comb (series of δ functions at regular distances equal to 1):

$$\text{Ш}(x) \equiv \sum_{n=-\infty}^{\infty} \delta(x-n)$$

- Dirac comb function can be extended to arbitrary distances by noting $a\text{Ш}(ax) = \sum_n \delta(x - n/a)$.
- sampled signal $M_s(x)$ can now be expressed as

$$M_s(x) = \sum_n M(n\tau)\delta(x - n\tau) = \frac{1}{\tau}\text{Ш}\left(\frac{x}{\tau}\right) M(x)$$

Regular Sampling (continued)

- Fourier transform $M_s(s) \Leftrightarrow M_s(x)$ equals

$$M_s(s) = \perp\!\perp\!\perp(\tau s) * M(s) = \frac{1}{\tau}\sum_n M\left(s - \frac{n}{\tau}\right)$$

- except for a proportionality factor $1/\tau$, $M_s(s)$ represents a series of replications of $M(s)$ at intervals $1/\tau$
- $M(s)$ bandwidth-limited function with cut-off frequency $s = s_c \Rightarrow$ fully recover single (i.e. not repeated) function $M(s)$ from series by multiplication with τ and by filtering with gate function $\Pi(s/2s_c)$:

$$\Pi\left(\frac{s}{2s_c}\right)\tau\perp\!\perp\!\perp(\tau s) * M(s) \Leftrightarrow 2s_c\mathrm{sinc}2s_c x * \perp\!\perp\!\perp\left(\frac{x}{\tau}\right)M(x)$$

Regular Sampling (continued)

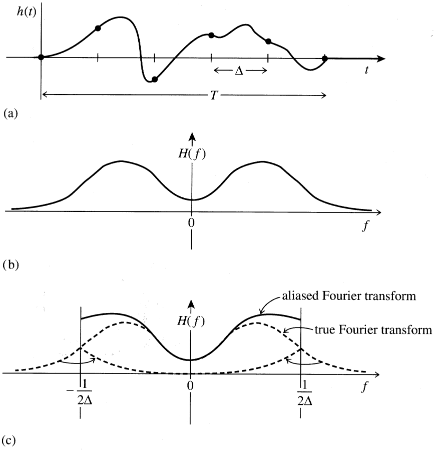
- $M(x)$ can be reconstructed exactly if series of $M(s)$ functions in frequency domain touch without overlap
- only possible if we sample at $\tau = 1/(2s_c)$, which therefore is optimum sample interval
- convolution to fully reconstruct $M(x)$:

$$\begin{aligned} M(x) &= \int_{-\infty}^{+\infty} \mathrm{sinc}\left(\frac{x-x'}{\tau}\right) \sum_n M(n\tau)\delta(x'-n\tau)dx' \\ &= \sum_n \mathrm{sinc}\left(\frac{x-n\tau}{\tau}\right) M(n\tau) \end{aligned}$$

- check result for one sampling point $x = j\tau$, with $\mathrm{sinc}(j-n) = 1$ for $j = n$ and $= 0$ for $j \neq n$:

$$M(x) = M(j\tau)$$

Aliasing



Press et al. (1992)

- function $h(t)$ shown in top panel is undersampled
- sampling interval Δ larger than $\frac{1}{2f_{max}}$
- lower panel shows that power in frequencies above $\frac{1}{2\Delta}$ is 'mirrored' with respect to this frequency
- produces aliased transform that deviates from true Fourier transform

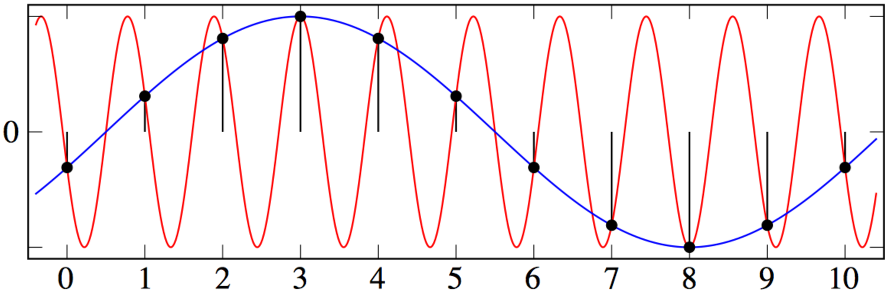
Aliasing (continued)

- calculation of intermediate points from samples does not depend on calculating Fourier transforms
- equivalent operation in x-domain is direct convolution of $2s_c \mathrm{sinc} 2s_c x$ with $\perp\!\!\perp\!\!\perp(x/\tau)M(x)$
- omission of $1/\tau$ factor ensures proper normalization in s-domain
- superposition of series of sinc-functions with weight factors $M(n\tau)$, i.e. the sample values, at intervals τ exactly reconstruct the continuous function $M(x)$
- sinc-functions provide proper interpolation between consecutive sample points
- sinc-function referred to as *interpolation function*

Aliasing (continued)

- discrete Fourier transform causes no loss of information if sampling frequency $\frac{1}{\tau}$ is twice the highest frequency in continuous input function
- maximum frequency s_{max} for given sampling interval is $\frac{1}{2\tau}$
- input signal sampled too slowly (contains frequencies higher than $\frac{1}{2\tau}$) $\Rightarrow$ source signal cannot be determined after sampling process
- loss of fine details
- must apply low-pass filter before sampling:
    - electronic low-pass filter for electrical signals
    - defocusing of telescope for imaging

Aliasing in Fourier Domain



http://en.wikipedia.org/wiki/File:AliasingSines.svg

- unresolved, high frequencies beat with measured frequencies
- produce spurious components in frequency domain below Nyquist frequency
- may give rise to major problems and uncertainties in the determination of source function