Lecture 2: Radiation Fields 2

Outline

1. Bose-Einstein Statistics
2. Fermi-Dirac Statistics
3. Blackbody Bose Gas
4. Quantum Noise and Thermal Noise
5. Radiation Field in Thermal Limit
6. Radiation Field in Quantum Limit
7. Photon Bunching

Why all this Theory?

- need to understand intrinsic noise in astronomical observations
- need to understand this in terms of photons (optical, X-ray) and electromagnetic waves (radio)
- noise distribution depends on measurement length and spectral resolution
- noise provides information on radiation source

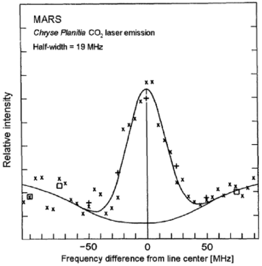
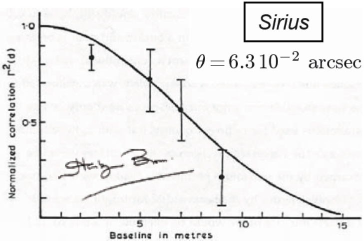


Fig. 10 Spectra of a Martian CO_2 laser emission line ($^{12}C^{16}O_2$ R8 line, $\lambda 10.33\mu m$) as a function of frequency difference from line center (in MHz). Lower curve is the total emergent intensity in the absence of laser emission; the emission peak is modeled with a Gaussian fit: Mumma et al [107]. Reprinted with permission from Science **212**, 45, © 1981 AAAS



Bose-Einstein Statistics

Summary

- $\sum_{i=1}^{\infty} \Delta n_i \left[\ln(n_i + Z_i - 1) - \ln n_i - \alpha - \beta\epsilon_i\right] \;=\; 0$ for arbitrary variations Δn_i if for each i

$$\ln(\overline{n}_i + Z_i - 1) - \ln \overline{n}_i - \alpha - \beta\epsilon_i = 0$$

- Bose-Einstein distribution $\frac{\overline{n}_i}{Z_i - 1} = \frac{1}{e^{\alpha+\beta\epsilon_i} - 1}$
- $Z_i \gg 1$: $\overline{n}_i/(Z_i - 1) \Rightarrow \overline{n}_i/Z_i$ average occupation at energy level ϵ_i
- α, β depend on total number of particles, total energy
- determine by substituting n_i in $N = \sum_{i=1}^{\infty} n_i$ and in $E = \sum_{i=1}^{\infty} n_i\epsilon_i$
- $\beta = 1/kT$, $\alpha = -\mu/kT$, μ: internal energy
- expected number of particles in energy state ϵ_i

$$\overline{n}_i = \frac{Z_i - 1}{e^{(\epsilon_i - \mu)/kT} - 1}$$

Planck Function

- photons do not collide, but reach equilibrium via interaction with atoms
- atom can absorb one photon and then emit 2 photons
- number of photons is not conserved $\Rightarrow$ drop α-term
- Planck function:

$$\frac{\overline{n}_i}{Z_i - 1} = \frac{1}{e^{\epsilon_i/kT} - 1} = \overline{n}_{\nu_k}$$

- $\overline{n}_{\nu_k}$: average occupation number at frequency ν_k

Connection to Thermodynamics

- connection to thermodynamics via entropy S

$$S \equiv k \ln W \Rightarrow \Delta S = k\Delta \ln W$$

- from derivation of Bose-Einstein distribution

$$\Delta \ln W - \alpha\Delta N - \beta\Delta E = 0$$

- therefore

$$\Delta S = k\alpha\Delta N + k\beta\Delta E$$

- for reversible processes energy change and entropy change are linked through

$$\Delta S = \frac{\Delta Q}{T}$$

- $T\Delta S = \Delta Q = -\zeta\Delta N + \Delta E \Rightarrow \beta = 1/(kT)$
- $\zeta \equiv -\alpha/\beta$: thermodynamical potential per particle

Fluctuations Around Equilibrium

- most likely distribution in equilibrium determined from

$$\Delta \ln W = \sum_{i=1}^{\infty} \frac{\partial \ln W(n_i)}{\partial n_i} \Delta n_i = 0$$

- ways to distribute $n_i + \Delta n_i$ particles (to 2nd order):

$$\ln W(n_i + \Delta n_i) = \ln W(n_i) + \Delta n_i \frac{\partial \ln W(n_i)}{\partial n_i} + \frac{\Delta n_i^2}{2} \frac{\partial^2 \ln W(n_i)}{\partial n_i^2}$$

- equilibrium $\Rightarrow$ term proportional to Δn_i is zero

$$W(n_i + \Delta n_i) = W(n_i) \mathrm{e}^{-\frac{W''(n_i)}{2} \Delta n_i^2} \text{ where } W''(n_i) \equiv -\frac{\partial^2 \ln W(n_i)}{\partial n_i^2}$$

- probability of deviation Δn_i drops exponentially with square of Δn_i
- probability of Δn_i is a Gaussian

Fluctuations Around Equilibrium (continued)

- average of Δn_i^2 by integrating over all possible Δn_i:

$$\overline{\Delta n_i^2} = \frac{\int_{-\infty}^{\infty} \Delta n_i^2 W(n_i) e^{-\frac{W''(n_i)}{2}\Delta n_i^2} d\Delta n_i}{\int_{-\infty}^{\infty} W(n_i) e^{-\frac{W''(n_i)}{2}\Delta n_i^2} d\Delta n_i} = \frac{1}{W''(n_i)}$$

- $W(n_i)$, $W''(n_i)$ do not depend on $\Delta n_i \Rightarrow$ constants in integrations
- maximum negative deviation: $\Delta n_i = -n_i$
- maximum positive deviation: $\Delta n_i = N - n_i$
- integrals to be evaluated between these values
- for large Δn_i integrand drops rapidly to zero
- extend integrals to full range $-\infty$ to $+\infty$ without changing result

Variance

- variance from second derivative of $\ln W(n_i)$ and changing sign:

$$\overline{\Delta n_i^2} = \left[W''(n_i)\right]^{-1} = \frac{\overline{n}_i(\overline{n}_i + Z_i - 1)}{Z_i - 1} = \overline{n}_i \left[1 + \frac{1}{e^{\alpha+\beta\epsilon_i} - 1}\right]$$

- $\alpha = 0$ for Planck function:

$$\overline{\Delta n_i^2} = \overline{n}_i \left[1 + \frac{1}{e^{\beta\epsilon_i} - 1}\right] = \overline{n_i}(1 + \overline{n}_{\nu_k})$$

- fluctuation in average occupation number

$$\overline{\Delta n_{\nu_k}{}^2} = \frac{\overline{\Delta n_i^2}}{Z_i} = \overline{n}_{\nu_k}(1 + \overline{n}_{\nu_k})$$

Fermi-Dirac Statistics

Distribution

- particles not allowed to share a box
- number of ways $W(n_i)$ in which n_i particles can be distributed over Z_i boxes with energies ϵ_i:

$$W(n_i) = \frac{Z_i!}{n_i!(Z_i - n_i)!}$$

- difference in $\ln W(n_i)$ between nearby numbers to first order in Δn_i:

$$\ln W(n_i + \Delta n_i) - \ln W(n_i) = -\Delta n_i \left[\ln n_i - \ln(Z_i - n_i)\right]$$

- equilibrium $\Rightarrow$ *Fermi Dirac distribution*:

$$\frac{\overline{n}_i}{Z_i} = \frac{1}{e^{\alpha+\beta\epsilon_i} + 1} = \overline{n}_k$$

Fluctuations

- average value of square of deviation

$$\overline{\Delta n_i^2} = \frac{\overline{n}_i(Z_i - \overline{n}_i)}{Z_i} = \overline{n}_i \left[1 - \frac{1}{e^{\alpha+\beta\epsilon_i} + 1} \right] = \overline{n}_i(1 - \overline{n}_k)$$

- fluctuation in average occupation number

$$\overline{\Delta n_k{}^2} = \frac{\overline{\Delta n_i^2}}{Z_i} = \overline{n}_k(1 - \overline{n}_k)$$

Blackbody Bose Gas

Introduction

- volume density of photons in blackbody Bose gas between ν, $\nu + d\nu$ from

$$\bar{N}(\nu)d\nu = g(\nu)\bar{n}_\nu d\nu$$

- $g(\nu_k)$: volume density of quantum states per unit frequency at ν_k
- stochastic variables n_{ν_k} independent $\Rightarrow$ Bose-fluctuations

$$\overline{\Delta N^2}(\nu) = \bar{N}(\nu)\left(1 + \frac{1}{\exp(h\nu/kT) - 1}\right)$$

- $\bar{N}(\nu)$ follows from specific energy density $\bar{\rho}(\nu) = \rho(\nu)^{equilibrium}$ using $\bar{N}(\nu) = \bar{\rho}(\nu)/h\nu$

$$\bar{\rho}(\nu)d\nu = \frac{8\pi h}{c^3}\frac{\nu^3}{\exp\left(\frac{h\nu}{kT}\right) - 1}d\nu$$

Radiation Detection

- detector inside blackbody radiation field at temperature T
- incident photon flux:

$$\bar{n}(\nu) = \frac{1}{2}\frac{c}{4\pi}\bar{N}(\nu)A_e\Omega$$

- factor $\frac{1}{2}$ refers to one component of polarization
- A_e is effective area of detector
- Ω is solid angle subtended by detector beam viewing radiation field
- if radiation illuminates extended surface (A_e) with various directions of the wave vector, i.e. an omnidirectional blackbody radiation field, coherence theory states that spatial coherence is limited to $A_e\Omega \approx \lambda^2$, the so-called *extent (etendue) of coherence*.
- same as size $\theta = \lambda/D$ of diffraction-limited beam ($\Omega \approx \theta^2$) for aperture diameter D: $A_e \approx D^2$

Radiation Detection (continued)

- substituting $\bar{N}(\nu)$, specific photon flux $\bar{n}(\nu)$ (in photons s^{-1} Hz^{-1}) becomes:

$$\begin{aligned} \bar{n}(\nu) &= \frac{1}{\exp(h\nu/kT) - 1} \\ \overline{\Delta n^2}(\nu) &= \bar{n}_\nu \left(1 + \frac{1}{\exp(h\nu/kT) - 1}\right) \end{aligned}$$

- $h\nu \gg kT \Rightarrow$ second term becomes much smaller than 1:

$$\overline{\Delta n^2}(\nu) = \bar{n}(\nu)$$

- Poissonian noise in sample containing $\bar{n}(\nu)$ photons
- *quantum limit of fluctuations*
- represents minimum value of intrinsic noise present in any radiation beam

Thermal Noise Limit

- $h\nu \ll kT$ noise in terms of average radiation power $\bar{P}(\nu)$ (Watt Hz^{-1})
- with $\bar{P}(\nu) = (h\nu)\bar{n}(\nu)$ and $\overline{\Delta P^2}(\nu) = (h\nu)^2\overline{\Delta n^2}(\nu)$:

$$\overline{\Delta P^2}(\nu) = \bar{P}(\nu)\left(h\nu + \frac{h\nu}{\exp(h\nu/kT) - 1}\right) = \bar{P}(\nu)(h\nu + \bar{P}(\nu))$$

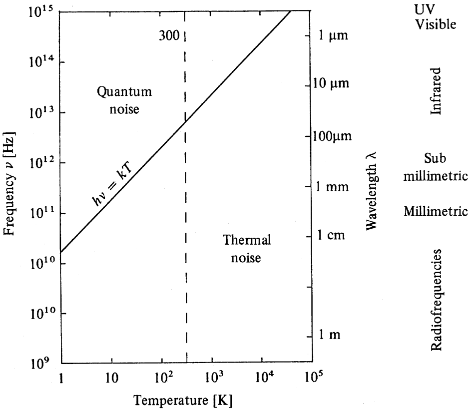
- $h\nu \ll kT$:

$$\begin{array}{rrcl} & \overline{\Delta P^2}(\nu) & = & \bar{P}^2(\nu) \\ and & \bar{P}(\nu) & = & kT \end{array}$$

- expression for classical thermal noise power per unit frequency bandwidth
- compare to Rayleigh-Jeans:

$$B_\nu(T) = 2kT\lambda^2$$

Quantum Noise and Thermal Noise



- transition between quantum limit to thermal limit at $h\nu \approx kT$
- $T \approx 300$ K $\Rightarrow \nu \approx 6$ THz, $\lambda \approx 50\ \mu$m

Quantum and Thermal Noise in Radio Astronomy

- radio observations always dominated by wave character of incoming beam $\Rightarrow$ thermal limit
- treatment of noise in radio observations very different from measurements at shorter wavelengths
- submillimeter and infrared observations aim at quantum limit
- fluctuations in average power $\bar{P}(\nu)$ for thermal limit: wave packet interference $\Rightarrow$ fluctuations have same magnitude as signal
- low frequency fluctuations due to random phase differences and beats of wavefields

Detector Outside of Blackbody Photon Gas

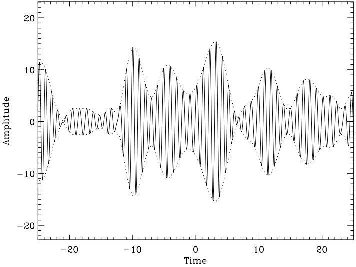
- expression for fluctuations in blackbody photon gas applies only to detector in interior of blackbody where $\lambda^2 = c^2/\nu^2 = A_e\Omega$
- if not, even in limit $h\nu \ll kT$ quantum noise may dominate
- example: blackbody star at temperature T, observed at frequency ν, where $h\nu \ll kT$, thermal noise should dominate
- star is so far away that radiation is unidirectional and $A_e\Omega \ll \lambda^2$
- photons will arrive well separated in time
- quantum noise dominates

Radiation Field in Thermal Limit

Wavepackets

- astronomy: most sources of photons have thermal origin
- observed wave is superposition of many individual wavepackets
- each wavepacket generated by independent atomic transitions at source
- wavepacket duration given by time scale of atomic transition
- frequency spread of wavepacket $\Delta\nu = 1/\Delta t$
- duration of wavepacket $\Delta t \equiv \tau_c = 1/\Delta\nu$ is *coherence time*
- typical time scale over which phase of the EM-wave can be predicted

Random Superposition of Wavepackets



- stochastic signal due to random superposition of wavepackets
- wave signal fluctuates in amplitude and frequency
- frequency fluctuations have typical bandwidth $\Delta\nu$ around average frequency $\bar{\nu}$
- quasi-monochromatic wave with a frequency stability $\bar{\nu}/\Delta\nu$

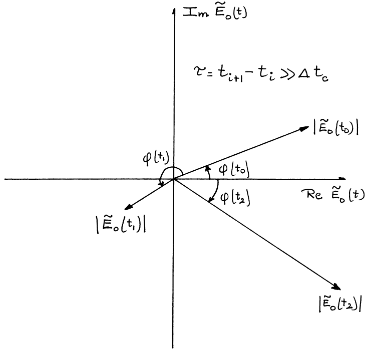
Quasi-Monochromatic Radiation Field

- description of quasi-monochromatic radiation field from thermal source:
  - complex expression for electric field $\tilde{E}(t)$
  - harmonic oscillation at average frequency $\bar{\nu}$
  - modulation by slowly varying envelope $\tilde{E}_0(t)$

$$\tilde{E}(t) = \tilde{E}_0(t) \cdot e^{i(2\pi\bar{\nu}t)}$$

- complex amplitude $\tilde{E}_0(t)$ is *phasor*
- phasor has time-dependent magnitude $|\tilde{E}_0(t)|$, phase $\phi(t)$
- ideal monochromatic plane wave: $\Delta\nu$ reduces to delta function $\delta(\nu - \bar{\nu})$
  - in time domain: infinitely long wave train
  - resolve wave train into 2 orthogonal polarization components, must have same frequency, be infinite in extent and therefore *mutually coherent*
  - perfectly monochromatic plane wave is always polarized

Polarized Light



- phasor $\tilde{E}_0(t)$ of linearly polarized plane wave:

$$\tilde{E}_0(t) = |\tilde{E}_0(t)|\, e^{i\phi(t)} = |\tilde{E}_0|\, e^{i\phi_0}$$

- amplitude $|\tilde{E}_0|$ and phase ϕ_0 of phasor are constant over short times

Thermal Radiation

- thermal radiation source: emission from extremely large number of randomly oriented atomic emitters
- each atom radiates polarized wave train for 10^{-8} or 10^{-9} (optical light from transition with natural line width $\Delta\nu$)
- time scale for molecular vibrational or rotational transitions and forbidden lines are longer
- wave propagation direction $\vec{k} \Rightarrow$ individual atomic (molecular) emissions at same frequency along that direction will combine into single polarized wave that only exists for coherence time τ_c of wave packet (optical: $10^{-8} - 10^{-9}$ s)
- wave trains continuously emitted $\Rightarrow$ magnitude, polarization direction of electric vector $\vec{E}(t)$ changes in random manner on typical time scale τ_c

Unpolarized Light

- change rate 10^8 to 10^9 s^{-1} $\Rightarrow$ single polarization state not discernable
- thermal radiation also called *natural* or *unpolarized* light
- consists of rapid succession of different polarization states
- describe random fluctuations of $\vec{E}(t)$ in scalar approach
- consider fluctuations in phasor $\tilde{E}_0(t)$: magnitude $|\tilde{E}_0(t)|$, phase $\phi(t)$
- time scales short compared to coherence time $(\Delta\nu)^{-1}$, $\tilde{E}_0(t)$ almost constant in time
- optical light with $\tau_c \approx 10^{-8}$ s contains millions of harmonic oscillations of electric vector $\vec{E}(t)$ ($\bar{\nu} \approx$ few 10^{14} Hz)
- on time scales $\tau \gg \tau_c$, $|\tilde{E}_0(t)|$ and $\phi(t)$ vary randomly

Phasor Fluctuations

- consider real, imaginary parts of $\tilde{E}_0(t)$, $\mathrm{Re}(\tilde{E}_0(t))$ and $\mathrm{Im}(\tilde{E}_0(t))$, as uncorrelated Gaussian stochastic variables with equal standard deviations
- linearly polarized waves that are *mutually incoherent*
- joint (bivariate) probability density distribution given by product of distributions:

$$\mathbf{p}\left(\mathrm{Re}\tilde{E}_0(t), \mathrm{Im}\tilde{E}_0(t)\right) d\mathrm{Re}\tilde{E}_0(t) d\mathrm{Im}\tilde{E}_0(t) = \frac{1}{2\pi\sigma^2} e^{-\frac{\mathrm{Re}^2\tilde{E}_0(t)+\mathrm{Im}^2\tilde{E}_0(t)}{2\sigma^2}} d\mathrm{Re}\tilde{E}_0(t) d\mathrm{Im}\tilde{E}_0(t)$$

- furthermore

$$|\tilde{E}_0(t)|^2 = \mathrm{Re}^2\tilde{E}_0(t) + \mathrm{Im}^2\tilde{E}_0(t)$$

$$\phi(t) = \arg(\tilde{E}_0(t)) = \arctan\frac{\mathrm{Im}\tilde{E}_0(t)}{\mathrm{Re}\tilde{E}_0(t)}$$

Phasor Fluctuations (continued)

- bivariate probability density in polar coordinates

$$\mathbf{p}\left(|\tilde{E}_0(t)|, \phi(t)\right)\; d\,|\tilde{E}_0(t)|\; d\phi(t) = \frac{|\tilde{E}_0(t)|}{2\pi\sigma^2}\, e^{-\frac{|\tilde{E}_0(t)|^2}{2\sigma^2}}\, d\,|\tilde{E}_0(t)|\; d\phi(t)$$

- integration over $|\tilde{E}_0(t)|$:

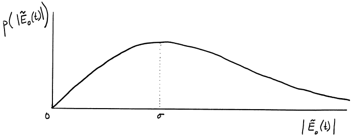
$$\mathbf{p}\left(\phi(t)\right) = \frac{1}{2\pi}$$

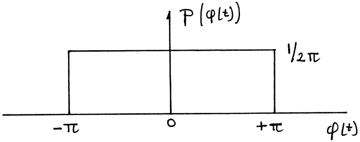
- all phase angles $\phi(t)$ are equally probable for unpolarized radiation
- integration over all phase angles $\phi(t) \Rightarrow$ amplitude distribution for an unpolarized thermal radiation beam (*Rayleigh Distribution*):

$$\mathbf{p}\left(|\tilde{E}_0(t)|\right) = \frac{|\tilde{E}_0(t)|}{\sigma^2}\, e^{-\frac{|\tilde{E}_0(t)|^2}{2\sigma^2}}$$

Rayleigh Distribution

- $\mathbf{p}\left(\left|\tilde{E}_0(t)\right|\right)$ and $\mathbf{p}\left(\phi(t)\right)$





- without proof: most probable value of $\left|\tilde{E}_0(t)\right|$ is σ, average amplitude of unpolarized beam is $\sigma\sqrt{\frac{\pi}{2}}$
- distribution of $\left|\tilde{E}_0(t)\right| \Rightarrow$ probability density of instantaneous intensity (or irradiance) $I(t)$ for thermal radiation

Power Flux Density of Electromagnetic Wave

- wave energy shared between electric and magnetic fields
- energy density of electrostatic field (in Joule/m^3)

$$\rho_{\vec{E}} = \epsilon_r \epsilon_0 |\vec{E}|^2/2$$

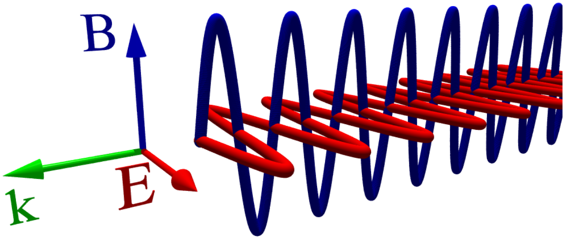
$|\vec{E}|$ magnitude of electric vector (in V/m)
ϵ_0 vacuum permittivity ($8.8543 \cdot 10^{-12}$ Asec/Vm)

- energy density of a magnetic field (in Joule/m^3)

$$\rho_{\vec{B}} = |\vec{B}|^2/(2\mu_r \mu_0)$$

$|\vec{B}|$ magnitude of magnetic vector (in Tesla = Vsec/m^2)
μ_0 vacuum permeability ($4\pi \cdot 10^{-7}$ Vsec/Am)

Electromagnetic Wave



- wave equation for a *plane electromagnetic wave* traveling along x in vacuum:

$$\frac{\partial^2 E(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E(x,t)}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 B(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 B(x,t)}{\partial t^2}$$

- magnetic field is perpendicular to electric field
- electric field and the magnetic field directions are perpendicular to direction of propagation (x)

Thermal Radiation

- plane wave

$$\tilde{E}(x,t) = E_0 e^{i \cdot 2\pi(\nu t - x/\lambda)} \quad \text{and} \quad \tilde{B}(x,t) = B_0 e^{i \cdot 2\pi(\nu t - x/\lambda)}$$

- Maxwell's equations require $\rho_{\vec{E}} = \rho_{\vec{B}}$
- $B_0 = E_0/c$
- flow of electromagnetic energy through space represented by Poynting vector $\vec{S} = (1/\mu_0)\vec{E}\,\mathbf{x}\,\vec{B}$
- direction and magnitude of the energy transport per unit time across a unit area (e.g. in units Watt m^{-2})
- vector magnitude $|\vec{S}| = |\tilde{E}||\tilde{B}|(\sin\phi)/\mu_0$ equals $|\tilde{E}||\tilde{B}|/\mu_0$, since magnetic field is perpendicular to electric field ($\phi = \pi/2$)

Poynting Vector

- *actual* wave signal by taking real part:

$$\begin{aligned} |\vec{S}| &= E_0 B_0 \cos^2 2\pi(\nu t - x/\lambda) \\ &= \epsilon_0 c E_0^2 \cos^2 2\pi(\nu t - x/\lambda) \\ &= (\epsilon_0/\mu_0)^{\frac{1}{2}} E_0^2 \cos^2 2\pi(\nu t - x/\lambda) \end{aligned}$$

- *average* power flux density for *ideal monochromatic* plane wave, $\overline{I(t)}$ equals $\overline{|\vec{S}(t)|}$:

$$\overline{I(t)} = (\epsilon_0/\mu_0)^{\frac{1}{2}} E_0^2 \overline{\cos^2 2\pi(\nu t - x/\lambda)} = (\epsilon_0/\mu_0)^{\frac{1}{2}} \frac{E_0^2}{2}$$

- ideal monochromatic plane wave represented in time domain by infinitely long wave train, *fully polarized*

Unpolarized, Quasi-Monochromatic Radiation Field

- *unpolarized, quasi-monochromatic*, radiation field from thermal source described by complex expression for electric field

$$\tilde{E}(t) = \tilde{E}_0(t) \cdot e^{i(2\pi\bar{\nu}t)}$$

- average power flux density from expectation value of $\tilde{E}(t)\tilde{E}^*(t)$:

$$\overline{I(t)} = (\epsilon_0/\mu_0)^{\frac{1}{2}} \mathbf{E}\left\{\tilde{E}(t)\tilde{E}^*(t)\right\} = 2.6544 \cdot 10^{-3}\, \mathbf{E}\left\{|\tilde{E}_0(t)|^2\right\}$$

- drop constant as we observe *relative power flux densities generated by these traveling waves within the same medium* and noise can be expressed as a relative quantity
- in practical computations, this constant should be applied

Variance

- following equalities hold:

$$I(t) = \tilde{E}(t) \cdot \tilde{E}^*(t) \; = |\,\tilde{E}(t)\,|^2 = |\,\tilde{E}_0(t)\,|^2$$

- from before:

$$\mathbf{p}\left(|\,\tilde{E}_0(t)\,|\right) = \frac{|\,\tilde{E}_0(t)\,|}{\sigma^2}\; e^{-\frac{|\tilde{E}_0(t)|^2}{2\sigma^2}}$$

- transformation of variables

$$\mathbf{p}\,(I)\;\,dI = (\bar{I})^{-1}\; e^{-I/\bar{I}}\; dI$$

with $\bar{I} = \mathbf{E}\left\{|\tilde{E}_0(t)|^2\right\} = 2\sigma^2$.

- exponential probability density distribution
- without proof: variance is $\overline{\Delta I^2} = \bar{I}^2$

Summary

- bivariate Gaussian-distributed stochastic process with zero-mean for harmonic wave components is same as fluctuation in average monochromatic radiation power (Watt Hz^{-1}) of blackbody radiation field: $\overline{\Delta P^2(\nu)} = \bar{P}^2(\nu)$
- stochastic description for unpolarized thermal radiation field using scalar treatment of complex expression for electric field:

$$\tilde{E}(t) = \tilde{E}_0(t)\; e^{i(2\pi\bar{\nu}t)} = |\tilde{E}_0(t)|\;\; e^{i\phi(t)}\; e^{i(2\pi\bar{\nu}t)} = |\tilde{E}_0(t)|\;\; e^{i(2\pi\bar{\nu}t+\phi(t))}$$

- all values of $\phi(t)$ are equally probable
- amplitude $|\tilde{E}_0(t)|$ distribution is Rayleigh distribution
- instantaneous frequency:

$$\nu = \frac{1}{2\pi}\,\frac{d}{dt}(2\pi\bar{\nu}t + \phi(t))$$

- bandwidth $\Delta\nu$ from $\nu - \bar{\nu} = \frac{d}{dt}\phi(t)$

Polarized Thermal Radiation

- radiation beam generally neither completely polarized nor completely unpolarized
- radiation should be regarded as partially polarized
- describe as superposition of specific amounts of natural and polarized light
- quantitative assessment via Stokes parameters
- easy in radio astronomy as receiver front-end is sensitive to a particular direction of polarization

Statistics of Radiation Field in Quantum Limit

Photon Generation

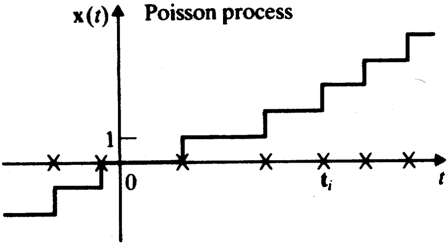
- quantum limit: radiation field fluctuations described by photon statistics
- radiation beam (wide-sense stationary, ergodic) with average flux of $\bar{n}_b$ photons per second
- generation of photons at random times t_i described by stochastic variable $X(t)$
- staircase functions with discontinuities at t_i

$$X(t) = \sum_i U(t - t_i)$$

- with $U(t)$ the unit-step function:

$$U(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

Photon Generation (continued)



- derivative of stochastic variable $X(t)$:

$$Y(t) = \frac{dX(t)}{dt} = \sum_i \delta(t - t_i)$$

represents train of Dirac impulses at random time positions t_i

Photon Detection Statistics

- photons detected during ΔT (part of total measurement time T):

$$X_{\Delta T} = \int_{t}^{t+\Delta T} \sum_i \delta(t - t_i)\, dt = k$$

- *random variable* $X_{\Delta T}$ distributed according to Poisson distribution
- probability to detect k photons if mean value is $\mu (= \bar{n}_b \Delta T)$:

$$p_P(k, \mu) = \frac{\mu^k}{k!}\, e^{-\mu}$$

- (continuous) probability density function for Poissonian statistics:

$$p(x, \mu) = \sum_{k=0}^{\infty} p_P(k, \mu)\, \delta(x - k)$$

$$\mathbf{E}\{X_{\Delta T}\} = \int^{+\infty} x\, p(x, \mu)\, dx = \mu (= \bar{n}_b \Delta T)$$

Photon Statistics (continued)

- $\mathbf{E}\{X_T\} = \mu$: average number of photons in time period T
- probability p that photon arrives in subinterval of T from $p = \mu/m$ if m equals number of subintervals within T
- probability that no photon arrives is $1 - p$
- measurement is series of m trials to find a photon, each having probability p of succeeding
- probability that in total k photons will be detected given by binomial probability function ($k < m$):

$$p_B(k, m, p) = \binom{m}{k} p^k (1 - p)^{m-k}$$

Thermal Radiation

- if subinterval is large, finite probability that more than one photon arrives in interval
- limit of trials m to go to infinity while $mp = \mu$
- binomial distribution becomes Poisson distribution:

$$p_P(k, \mu) = \frac{\mu^k}{k!} e^{-\mu}$$

- exponential factor normalizes distribution

$$\sum_{k=0}^{\infty} p_P(k, \mu) = 1$$

Autocorrelation

- autocorrelation

$$\begin{aligned} R_{X_{\Delta T}}(\tau) &= \mathbf{E}\{X_{\Delta T}(t+\tau)\cdot X_{\Delta T}(t)\} \\ &= \mu^2 + \mu\,\delta(\tau) \\ &= (\bar{n}_b\,\Delta T)^2 + (\bar{n}_b\,\Delta T)\delta(\tau) \end{aligned}$$

- $R_{X_{\Delta T}}(0) = \mu^2 + \mu$
- first term is square of average
- second term is covariance, which is variance here since covariance is 0 everywhere except for $\tau = 0$
- obvious since photon arrival times t_i are uncorrelated
- without proof: $\mathbf{E}\{X_{\Delta T}\} = \mu$, $R_{X_{\Delta T}}(0) = \mu^2 + \mu$

Signal-to-Noise Ratio

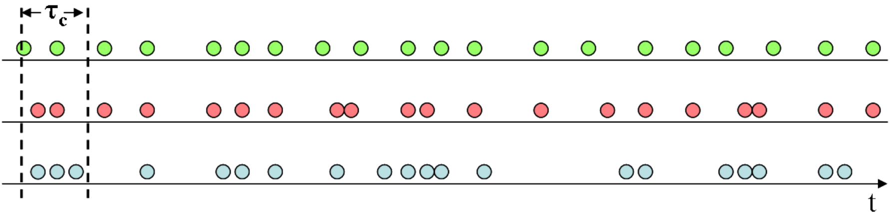
- signal-to-noise ratio *SNR* defines intrinsic limitation to accuracy of measurement due to photon noise:

$$SNR = \frac{\mathbf{E}\{X_{\Delta T}\}}{\sqrt{C_{X_{\Delta T}}(0)}} = \sqrt{\bar{n}_b \, \Delta T}$$

- intrinsic *SNR* of radiation field measurement increases with square root of mean photon flux $\bar{n}_b$ and with square root of measurement interval ΔT

Photon Bunching

Introduction



Photon detections as a function of time for a) antibunched, b) random, and c) bunched light

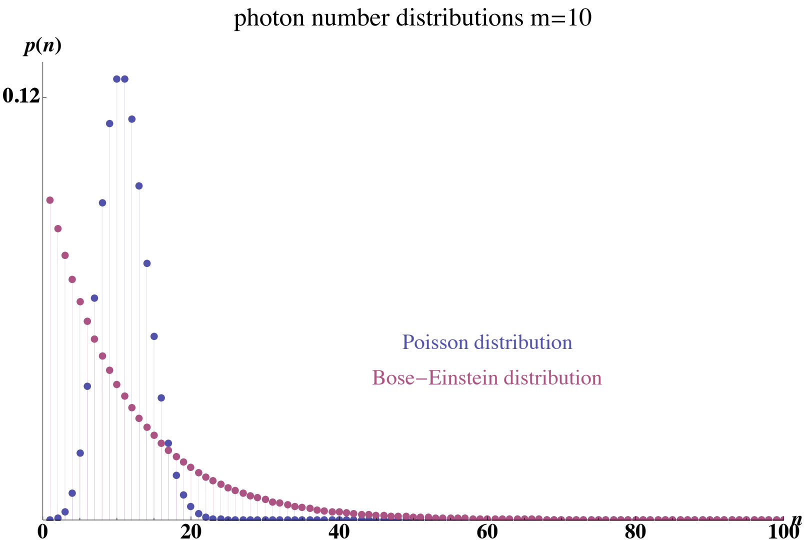
en.wikipedia.org/wiki/File:Photon_bunching.png

- photons distribute themselves in bunches rather than at random (Poisson)
- photons arrive more simultaneously (positive correlation)
- excess correlations only for $\Delta T < \tau_c$

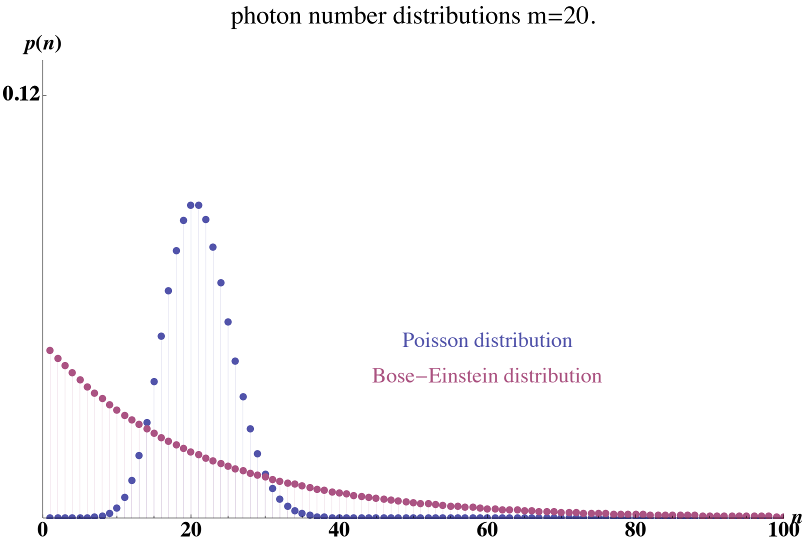
Photon Bunching (continued)

- photons should arrive according to Bose-Einstein distribution
- fluctuations are larger than for Poissonian statistics (BE: $\sigma^2 = n^2 + n$, Poisson: $\sigma^2 = n$)
- for very small, average count rates n, BE becomes Poisson
- predicted by quantum mechanics
- can be understood classically as a pure wave effect
- intensity interferometry by Hanbury-Brown and Twiss

Bose-Einstein vs. Poisson Statistics



Bose-Einstein vs. Poisson Statistics



Bose-Einstein vs. Poisson Statistics