Astronomical Data Analysis 2011: Exercises to Lecture on Radiation Fields 2 (Due on 21 February 2011 at 13:15)

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1 Statistics of Unpolarized Radiation

a) Show $X(t)$ that the probability density function for the intensity of an unpolarised signal from a thermal source is given by:

$$p(I) \;=\; \bar{I}^{-1}\, e^{-I/\bar{I}}, \tag{1}$$

where $\bar{I} \;=\; 2\sigma^2$. The variance σ^2 here relates to the (uncorrelated) gaussian stochastic variables $\mathrm{Re}\Big(\tilde{E}_0(t)\Big)$ and $\mathrm{Im}\Big(\tilde{E}_0(t)\Big)$, that give a scalar description of the fluctuating behaviour of the *phasor* $\tilde{E}_0(t)$.

b) Proof that for the variance of the intensity I of an unpolarised signal:

$$\overline{\Delta I^2} = \left(\bar{I}\right)^2 \tag{2}$$

2 Poisson Distribution

a) If the expected value $\mathbf{E}\{X_T\}$ equals μ for the average number of photons in time period T, the probability p that a photon arrives in a subinterval of T can be equated from $p = \mu/m$ if m equals the number of subintervals within T. The probability that no photon arrives is $1 - p$. The measurement can thus be considered as a series of m trials to find a photon, each having a probability of p of succeeding. The probability that in total k photons will be detected is therefore given by the binomial probability function $(k < m)$:

$$p_B(k, m, p) = \begin{pmatrix} m \\ k \end{pmatrix}\; p^k\; (1-p)^{m-k}. \tag{3}$$

However, if the subinterval is large, there is a finite probability that more than one photon will arrive within this interval, hence the limit should be taken for the number of trials m to go to infinity while maintaining $mp = \mu$ constant. In this limit $m \to \infty$, $p \to 0$, the binomial distribution changes to the Poisson distribution:

$$p_P(k, \mu) = \frac{\mu^k}{k!}\; e^{-\mu}. \tag{4}$$

Derive this distribution transformation and show that the Poisson distribution is normalized.

b) Show that for a *Poisson distributed* stochastic process the autocorrelation of the random variable $X_{\Delta T}$ (number of particles or photons per interval of time ΔT) is given by:

$$R_{X_{\Delta T}} \;=\; \mu^2 + \mu \quad \text{for } \tau = 0 \tag{5}$$

$$R_{X_{\Delta T}}(\tau) \;=\; \mu^2 + \mu\delta(\tau) \quad \text{in the general case} \tag{6}$$