Astronomical Data Analysis 2011: Exercises to Lecture on Radiation Fields 1 (Due on 21 February 2011 at 13:15)

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1 Bose Einstein Statistics

a) The equation for the stationary state of a free particle follows from the solution of the time-independent Schrödinger equation. The wave amplitude Ψ is then given by:

$$\Psi(\vec{r}) = e^{\frac{2\pi i}{h}\vec{p}\cdot\vec{r}} \tag{1}$$

When considering a finite volume, the allowed values for the momentum $\vec{p}$ must meet the specific boundary conditions, which causes quantization of the allowed momentum and energy levels.

Show that the number of occupation levels dZ_k for values of the momentum between p_k and $p_k + dp_k$ *per unit of volume* is given by:

$$dZ_k = \frac{4\pi p_k^2}{h^3} dp_k \tag{2}$$

b) Consider a black body cavity with temperature T. The radiation field can be described as a *boson gas*. Calculate the number of occupation levels per cm^3 for the *average* photon frequency that corresponds to a temperature of $T = 5000K$, integrated over the frequency interval of the natural line width. Assume a coherence time of $\Delta t_c = 10^{-8}$ s.

c) Derive the *energy density per Hz* of the black body radiation field, using the result you found above and the mean occupation number for a photon gas.

2 Stochastic processes

a) Consider a stochastic process $X(t) = S(t) + N(t)$.
Component $S(t)$ represents the periodic light curve of a stellar object:

$$S(t) = S_0(t)\left[1 + m\sin\left(\frac{2\pi t}{P} + \phi\right)\right], \tag{3}$$

with P the period and m the so-called modulation index. The stochastic signal $S(t)$ is wide sense stationary with $\overline{S(t)} = \overline{S_0(t)} = S_{av}$ and variance $\overline{(S_0 - S_{av})^2} = \overline{\triangle S_0^2} = \sigma_{S_0}^2$.
Component $N(t)$ represents the sky noise (wide sense stationary), with $\overline{N(t)} = N_{av}$ and $\overline{(N - N_{av})^2} = \overline{\triangle N^2} = \sigma_N^2$.
The periodic signal is submerged in the sky noise and the signal noise: $\sigma_N^2 = 3\sigma_{S_0}^2$ and the modulation index $m = 0.2$.
The periodic component in the stochastic process $X(t)$ can be extracted by using the autocorrelation function of $X(t)$.

Prove that the autocovariance of $X(t)$ is given by:

$$C_X(\tau) = C_N(\tau) + C_{S_0}(\tau)\left[1 + \frac{1}{2}m^2\cos\left(\frac{2\pi\tau}{P}\right)\right] + \frac{1}{2}m^2 S_{av}^2\cos\left(\frac{2\pi\tau}{P}\right) \tag{4}$$

b) Plot the autocovariance function $C_X(\tau)$ for positive values of τ, assuming that $C_N(\tau) \sim \exp(-\tau/\tau_N)$ with $\tau_N^{-1} = 2\pi/(3P)$, and $C_{S_0}(\tau) \sim \exp(-\tau/\tau_{S_0})$ with $\tau_{S_0}^{-1} = 2\pi/P$.

c) Can the periodic component of $X(t)$ be completely reconstructed by the measurement of the autocovariance $C_X(\tau)$. Explain your answer!