The formation of galaxies and large-scale structure
Problem set 4

Weeks of March 3 & 10, 2014

Problem 1 Size of the horizon

How far away do you need to hold your thumb for its width to subtend an angle equal to the angular scale of the horizon at the time of matter-radiation equality?

Hint: Approach this as a research question. Take your solutions with you to the lecture on March 11 where we will collect them anonymously, compare results and discuss the problem (briefly).

Problem 2 Spherical collapse

If you take the matter and the curvature terms into account, the Friedman equation becomes

$$\frac{1}{H_0^2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Omega_m}{(a/a_0)^3} + \frac{1-\Omega_m}{(a/a_0)^2}. \tag{1}$$

(a) Prove that equation **??** has the parametric solutions

$$\frac{a}{a_0} = (1-\cos\theta)\frac{\Omega_m}{2(\Omega_m - 1)} \tag{2}$$

$$H_0 t = (\theta - \sin\theta)\frac{\Omega_m}{2(\Omega_m - 1)^{3/2}} \tag{3}$$

We know (e.g. Birkhoff's theorem), that the evolution of a spherically symmetric perturbation in a smooth background can be treated as a separate universe. Hence it's evolution is given by the Friedmann equation.

Let $\delta = (\rho - \rho_b)/\rho_b$ be, as usual, the density contrast of the perturbation relative to the background universe which has matter density ρ_b. For a sufficiently early time, t_i, we have $\delta_i \ll 1$ and we may assume that the perturbation expands with the Hubble flow at that time. Thus the critical densities for the background and the perturbation are approximately identical at time $t = t_i$, and the density parameter for the perturbation becomes

$$\Omega_m(t = t_i) = \left.\frac{\rho}{\rho_c}\right|_{t=t_i} = \left.\frac{\rho_b}{\rho_c}(1+\delta)\right|_{t=t_i} = \Omega_{m,i}(1+\delta_i), \tag{4}$$

where $\Omega_{m,i}$ is the density parameter of the background universe evaluated at time $t = t_i$. For simplicity we will assume that $\Omega_m = 1$ and that $\Omega_\Lambda = \Omega_R = 0$ for the background universe. In that case we have $\Omega_m(t=t_i) = (1+\delta_i)$ and by setting $t_0 = t_i$, we can write the parametric solution of the radius of a spherically symmetric perturbation with mea n internal density contrast, δ, as

$$\begin{aligned} \frac{a}{a_i} &= (1-\cos\theta)\frac{1+\delta_i}{2\delta_i} && (5)\\ H_i t &= (\theta - \sin\theta)\frac{1+\delta_i}{2\delta_i^{3/2}}. && (6) \end{aligned}$$

(b) Prove that the density evolution of the perturbation is given by

$$1+\delta = \frac{9}{2}\frac{(\theta-\sin\theta)^2}{(1-\cos\theta)^3}. \tag{7}$$

Do you expect this relation to hold for $\theta = 2\pi$?

(c) In linear theory (subscript L), we have (you do not need to prove this but feel free to do so):

$$\delta_L = \frac{3}{5}\delta_i\left(\frac{t}{t_i}\right)^{2/3}. \tag{8}$$

(d) Assume that $\delta_i > 0$. What are the true and linear theory density contrasts at turn-around (i.e. the time when expansion turns into collapse)?

(e) Demonstrate that the virial radius $r_{\rm vir} = r_{\rm ta}/2$, where $r_{\rm ta}$ is the radius at turn-around. **[4]**

(f) Assume that virialization occurs when $\theta = 2\pi$. Show that after virialization, the true density contrast is $1+\delta_{\rm vir} = 18\pi^2 \approx 178$, while the linear theory density contrast $\delta_L \approx 1.69$.

We will assume that spherical pertrubations will virialize at $\rho = 18\pi^2\rho_b(z)$ when $\delta_L = 1.69$, even if $\Omega_m \neq 1$. It turns out that this is a very good approximation when $\Omega_m(1+z)^3 \gg (1-\Omega_m-\Omega_\Lambda)(1+z)^2 + \Omega_\Lambda$, i.e. when matter dominates.

Problem 3 Using the Press-Schechter mass function to constrain cosmology

The P-S mass function has its weaknesses, but it has proven very useful for fitting to the abundance of clusters. In this problem you will find out how this works.

The best known galaxy clusters in the Universe are known as Abell clusters. A typical Abell cluster has mass in a fairly narrow range, and it is customary to define this within the Abell radius of $r_A = 1.5\mathrm{h}^{-1}\mathrm{Mpc}$. This mass is

$$M_A = 7.8\times 10^{14} m_A \mathrm{h}^{-1} M_\odot. \tag{9}$$

This mass also defines a natural density within r_A and we can then calculate the overdensity within r_A which also defines the mass parameter, m_A:

$$\bar{\delta}(r_A) = \frac{\rho-\bar{\rho}}{\bar{\rho}} = \frac{200 m_A}{\Omega_{m,0}}. \tag{10}$$

(a) To connect the expressions above to the PS mass function we need the total mass of the halo, out to the virial radius. Let us define a radius, commonly written as r_{200}, which encloses the region 200 times overdense relative to the background. Show that the total mass of the halo within this radius is

$$M_{\rm halo} = \frac{4\pi}{3} r_{200}^3 \bar{\rho} \left[1 + \delta(r_{200})\right]. \tag{11}$$

(b) Assume that the halo mass profile is similar to an isothermal sphere over the radii in question, so that $\rho \propto r^{-2}$. Show that we then have

$$r_{200} = m_A^{1/2} r_A \tag{12}$$
$$M_{\rm halo} = m_A^{1/2} M_A \tag{13}$$

(c) Show that the radius of the halo, defined through $M = 4\pi\bar{\rho}/3$ can be written as

$$R \approx 1.1 m_A^{1/2} \Omega_m^{-1/3} \frac{r}{8{\rm h}^{-1}{\rm Mpc}} = 1.1 m_A^{1/2} \Omega_m^{-1/3} r_8. \tag{14}$$

This shows that Abell clusters have radii fairly close to r_8 since m_A varies relatively little.

(d) Explain why over a small range in R can write

$$\sigma(R) = \sigma_8 \left(\frac{R}{r_8}\right)^{-\beta}, \tag{15}$$

and show that for the Press-Schechter mass function you can write the number of clusters with mass in excess of M_A as

$$N(> M_A) \approx \frac{2}{\sqrt{\pi}} \left(\frac{\delta_c}{\sqrt{2}\sigma_8}\right)^{3/\beta} \int_{y_{\rm min}}^{\infty} y^{-3/\beta} e^{-y^2} dy, \tag{16}$$

and find an expression for $y_{\rm min}$ and thereby show that number counts of clusters can be used to constrain σ_8 and Ω_m.

The preceding analysis gives reasonably good results, but with today's high-quality data it is necessary to move past the PS formalism. A major step can be had by moving to ellipsoidal collapse models, but the best studies in the literature (e.g. Vikhlinin et al 2009, ApJ 692, 1060) use mass functions inferred/calibrated from numerical simulations rather than semi-analytic studies like the PS mass-function but the principle is essentially the same.