The formation of galaxies and large-scale structure
Small problem set 2

February 19, 2013

The aim of this problem set is to familiarise you a bit more with two concepts — namely the Jeans length in the Universe and the free-streaming scale of neutrinos. The first problem is really short (hopefully!) while the second is more complex and could have made up part of an exam.

Problem 1 The Jeans length

(a) In this and the next step you can assume a cosmology that has $\Omega_b = 1 = \Omega$, ie. a baryonic, flat universe, and that the baryons are all hydrogen. Focus on the time after recombination.

Derive an expression for the co-moving Jeans wavenumber in this case and show that k_J is independent of the scale-factor a.

(b) Derive corresponding expressions for the proper Jeans length and Jeans mass and show explicitly the dependence on a.

(c) We will now consider a mixture of baryons, radiation and dark matter. We will also only consider adiabatic perturbations of the baryon-photon fluid. In that case we have

$$\frac{d\rho_b}{\rho_b} = \frac{3}{4}\frac{d\rho_\gamma}{\rho_\gamma}, \tag{1}$$

where a subscript γ corresponds to photons and b to baryons. Show that in this case we can write the perturbation equation for baryons as

$$\frac{\partial^2 \delta_b(k)}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_b(k)}{\partial t} = 4\pi G\left[\bar{\rho}_{\mathrm{DM}}\delta_{\mathrm{DM}}(k) + (\bar{\rho}_b + \frac{4}{3}\bar{\rho}_\gamma)\delta_b(k)\right] - \frac{c_s^2}{a^2}k^2\delta_b(k), \tag{2}$$

where the time-dependence is suppressed and the k-dependence is given as function argument instead of the subscript used in the lecture notes.

Problem 2 The free-streaming scale for neutrinos

Here we will calculate the free-streaming scale for a particle with a given mass.

(a) One of the key reasons a pure hot dark matter (HDM) cosmology is ruled out is the effect of free streaming. Any particle with significant velocity will wipe out any perturbations in that particle species (any any other closely coupled species) smaller than the free streaming scale and if neutrinos were the only dark matter particle they would wipe out too much structure. On larger scales the particle will behave like any other dark matter particle.

The proper distance a free streaming particle can travel until a time t is given by:

$$l_{\text{FS}} = a \int_0^t \frac{v(t)}{a} dt, \tag{3}$$

where a is the scale factor and v is the velocity of the particle. Assume that the particles become non-relativistic at a time $t_{\text{NR}} < t_{\text{EQ}}$ and show that the at late times when $t \gg t_{\text{EQ}}$ we have

$$l_{\text{FS}} = a \frac{2ct_{\text{NR}}}{a_{\text{NR}}} \left[\frac{5}{2} + \ln \frac{a_{\text{EQ}}}{a_{\text{NR}}} \right], \tag{4}$$

where subscript NR indicates the time the particle went non-relativistic and EQ refers to the time of matter-radiation equality. You may use that when a particle is non-relativistic, its velocity declines as $v \propto 1/a$.

Hint: Split the integration into three regimes, before NR, before EQ and later.

(b) We will now estimate **the comoving free-streaming length** $\lambda_{\text{FS}} = l_{\text{FS}}/a$ and the mass enclosed within this. We will consider massive neutrinos, with mass m_ν. Define the time they go non-relativisitic, t_{NR} to be when $m_\nu c^2 \approx 3kT_\nu$. In addition you can take the mass density in radiation to be $\Omega_{r,0} = 4.2 \times 10^{-5} h^{-2}$, $z_{\text{eq}} = 3200$ and $h = 0.72$ (**note that strictly speaking these values are not consistent with the neutrino dominated universe we are interested in here, ignore that for this problem**). Show that this gives

$$\lambda_{\text{FS}} \approx 42.1 \left(\frac{m_\nu}{30 \text{ eV}} \right)^{-1} \text{ Mpc}. \tag{5}$$

To convert this to a mass in neutrinos we need the mass density in neutrinos. This comes from the thermodynamics of the big bang (see section 3.3.5 in Mo, van den Bosch & White, for instance, in particular their equation 3.172). This then is

$$\Omega_\nu h^2 \approx 0.32 \frac{m_\nu}{30 \text{ eV}}, \tag{6}$$

show that this gives

$$M_{\text{FS}} \approx 3.5 \times 10^{15} \left(\frac{m_\nu}{30\text{eV}} \right)^{-2} M_\odot, \tag{7}$$

where I have assumed that λ_{FS} is the diameter of the region.

While we have made some approximations here, the result captures the essential aspects. It is important to note that a heavier particle goes non-relativistic earlier and thus the free-streaming mass decreases.