The formation of galaxies and large-scale structure
Small problem set 1

February 11, 2014

The problem set will be discussed in the next problem class.

Problem 1 How flat was the Universe?

(a) Define $\Omega_X(t) = \rho(t)/\rho_c(t)$, where ρ is the mass density in a given component, X, and ρ_c is the critical density and we label the value of Ω today with a subscript 0 (which we only do here). Show that

$$\Omega(t) - 1 = \frac{\Omega_0 - 1}{\Omega_{R,0}\; a(t)^{-2} + \Omega_{m,0}\; a(t)^{-1} + \Omega_{\Lambda,0} a(t)^2 + (1 - \Omega_0)}, \tag{1}$$

where $\Omega = \Omega_R + \Omega_m + \Omega_\Lambda$ and a is the scale-factor as usual. What is the limit of this equation as $a \to 0$? What is the interpretation of this?

Problem 2 Expansion and horizons

As said in the lecture, we often make reference to the size of the "horizon" (see also below) at matter-radiation equality. In this problem you will calculate the size of this. In class I made use of a rather approximate number for the size of the horizon, here you will do a more careful calculation. Recall that the horizon size is given as $r_H = \int_0^t c dt'/a(t')$.

To do this you need to account for all sources of radiation energy density. At the redshifts we consider we should also include neutrinos. For this you should recall from your cosmology course that each neutrino species contribute $\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}$ of the energy density in photons to the radiation energy density.

(a) Show that the redshift of equality for three species of neutrinos is

$$z_{\rm eq} \approx 23900.9 \mathrm{h}^2 \Omega_{\rm NR}, \tag{2}$$

where $\Omega_{\rm NR}$ is the mass density in non-relativistic matter today relative to that of the critical density. As I will do throughout I have dropped the subscript 0 from the densities today.

(b) Prove that the comoving horizon at matter-radiation equality is given by

$$r_H(z = z_{\rm eq}) = \frac{2(\sqrt{2} - 1)c}{\Omega_{\rm NR}^{1/2} H_0} \frac{1}{(1 + z_{\rm eq})^{1/2}}, \tag{3}$$

Hint: you need to take both matter and radiation into account — do you need to consider curvature and vacuum energy?

(c) We expect that $r_{\rm H,eq} \sim ct_{\rm eq}(1+z_{\rm eq})$ (why?). Show that this is indeed the case. *Hint:* You can use the Friedmann equation neglecting the matter contribution to get a rough estimate for $t_{\rm eq}$.

(d) Estimate the value of $r_{\rm H,eq}$ in Mpc and the corresponding mass in $M_\odot$. You can use $\Omega_m = 0.258$, $h = 0.719$, $T_{\rm CMB,0} = 2.73$K. How does this change if you had fewer or more species of neutrinos?