The formation of galaxies and large-scale structure
Hand-in problem set 1

March 16, 2014

This is the first obligatory problem set to hand in. Please hand your solutions in by placing them in Jarle's pigeon hole or in the lecture. The deadline for this problem set is April 1. The second problem set will be handed out at the end of April/early May. The weight of each question is indicated in bold face at the end of each entry. Please write your solutions out in full.

The FRW metric can in general be written

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right], \tag{1}$$

with $k = -1, 0$ or 1 depending on the geometry of the Universe.

Problem 1 Life as a future cosmologist

In this problem you will look at how the Universe will look for a cosmologist in the future. We will adopt $\Omega_M = 0.3$ and $\Omega_\Lambda = 0.7$ in this problem and all times are counted in units of H_0.

To fix the notation, taken an object observed today, at $t = t_0$, whose light was emitted at $t = t_{\rm em}$. Now assume that this object will continue shining and that at a time $t = t_0' > t_0$, a future observer sees the same object but now the light that was emitted at $t = t'_{\rm em}$ (in other words the same source was $t'_{\rm em} - t_{\rm em}$ older than that observed today.

(a) Show that

$$\int_{t_0}^{t_0'} \frac{dt}{a(t)} = \int_{t_{\rm em}}^{t'_{\rm em}} \frac{dt}{a(t)}, \tag{2}$$

where $a(t)$ is the scale factor in the FRW metric. We can also introduce the conformal time, $\eta(t) = \int_0^t \frac{dt}{a(t)}$, so equation (2) can also be written

$$\eta(t_0') - \eta(t_0) = \eta(t'_{\rm em}) - \eta(t_{\rm em}). \tag{3}$$

Hint: Start with the FRW metric **[2]**

(b) For a flat Universe where we ignore the contribution of radiation, so that $\Omega_M + \Omega_\Lambda = 1$, show that

$$a(t) = \left(\frac{\Omega_M}{1-\Omega_M}\right)^{1/3}\left(\sinh\left(\frac{3}{2}\sqrt{1-\Omega_M}H_0 t\right)\right)^{2/3} \tag{4}$$

Hint: You might find the following integral useful:

$$\int \frac{\sqrt{x}dx}{\sqrt{1+Kx^3}} = \frac{2}{3}\sinh^{-1}(\sqrt{K}x^{3/2})/\sqrt{K}$$

[3]

(c) The exact expression for $a(t)$ derived above is a bit unwieldy and does not lead to an expression for $\eta(t)$ in elementary functions. Instead we will approximate $a(t)$ with its limits at large and small t. Show that

$$a(t) \approx \left(\frac{\Omega_M}{1-\Omega_M}\right)^{1/3} \left(\frac{3}{2}\Omega_\Lambda^{1/2} H_0 t\right)^{2/3} = a^-(t) \qquad \text{for } H_0 t \ll 1 \tag{5}$$

$$a(t) \approx \left(\frac{\Omega_M}{1-\Omega_M}\right)^{1/3} 2^{-2/3}\, e^{H_0 t\sqrt{1-\Omega_M}} = a^+(t) \qquad \text{for } H_0 t \gg 1 \tag{6}$$

$$\tag{7}$$

We can then approximate $a(t)$ as

$$a(t) = \begin{cases} a^-(t) & \text{for } t \le t_{\text{cut}} \\ a^+(t) & \text{for } t > t_{\text{cut}} \end{cases} \tag{8}$$

where t_{cut} can be chosen in various ways. Explain why $H_0 t = \frac{2}{3}\Omega_\Lambda^{-1/2}$ would be the natural physical choice for t_{cut}. For analytic convenience in the following you should take $t_{\text{cut}} = t_0$ however. **[2+2]**

(d) Show that for a given t_{em} there is a maximum t'_{em} and calculate its value for a galaxy seen today at $z = 5$. What is your interpretation of this result? **[6]**

Problem 2 Large-scale structure — decaying neutrinos

In the 1990s the so-called τCDM model received serious consideration. In this model the neutrino is short-lived, massive and decouples while it is still relativistic. As the Universe cools, the neutrinos become non-relativistic and consequently their energy decreases less rapidly with time than that of relativistic particles.

The non-relativistic neutrino dominates the energy density of the Universe from a time $t_{\gamma\nu}$, but they decay into relativistic products ("radiation") at time $t_{\nu\gamma} \gg t_{\gamma\nu}$. As in the current standard scenario, cold dark matter dominates the energy density from $t_{\text{eq}} \gg t_{\nu\gamma}$ (we will ignore the vacuum dominated epoch here).

(a) Let $\delta_\lambda(t_i)$ be the amplitude of a matter density perturbation on the co-moving length scale λ at some early time $t_i \ll t_{\gamma\nu}$. Let t_{ent} be the time a perturbation enters the horizon, $t_{\text{ent}}(\lambda) = t(r_H = \lambda)$, ie. it is the time the co-moving size of the perturbation equals that of the co-moving horizon. We will only consider perturbations such that $t_i \ll t_{\text{ent}}(\lambda)$, ie. that all scales of interest are outside the horizon at t_i.

Now assume that the primordial power spectrum $P(t_i, k) \propto k^n$ and derive the scaling of the linear power spectrum $P(t, k)$ with co-moving wavenumber k in the matter dominated epoch ($t \gg t_{\text{eq}}$) for the following five regimes:

1 $M < M_{\rm FS}$, where $M_{\rm FS}$ is the free streaming mass, i.e., the mass corresponding to the free streaming length.

2 $M_{\rm FS} < M < M_{\gamma\nu}$, where $M_{\gamma\nu}$ is the horizon mass at the time non-relativistic neutrinos start to dominate.

3 $M_{\gamma\nu} < M < M_{\nu\gamma}$, where $M_{\nu\gamma}$ is the horizon mass at the time the neutrinos decay into relativistic products.

4 $M_{\nu\gamma} < M < M_{\rm eq}$, where $M_{\rm eq}$ is the horizon mass at final matter-radiation equality.

5 $M > M_{\rm eq}$.

Hint: To avoid confusion, first draw a time line, indicating clearly the order of the different time-scales and what dominates the energy density during the different epochs.

[**6**]

(b) In a standard CDM model with a Harrison-Zeldovich spectrum ($n = 1$), the variance of the density contrast $\sigma^2 \propto k^3 P(k) \propto k^3 \delta_k^2$, is independent of the scale of the perturbations at the time the fluctuations enter the horizon. In other words $\sigma(t_{\rm ent}(M)) = \sigma_{\rm ent}$ is constant. Does this hold in the τCDM scenario? Please prove your answer. [**4**]

Problem 3 Press-Schechter theory

As mentioned above, use $n = -2.35$ as the effective power-law slope here as well, when relevant.

(a) $M_*(z)$ is the mass of a 1-sigma fluctuation collapsing at redshift z. What fraction of the mass has collapsed into objects with mass greater than $M_*(z)$? What is M_* at redshifts 0, 3 and 9?

[**6+2**]

(b) In the lectures we found that the Press-Schechter (PS) mass function is given by

$$f(M,z) = n(M,z)dM = \sqrt{\frac{2}{\pi}}\frac{\rho_b}{M}\left|\frac{d\ln\sigma(M)}{dM}\right|\frac{\delta_c(z)}{\sigma(M)}\exp\left(-\frac{\delta_c^2(z)}{2\sigma^2(M)}\right). \qquad (9)$$

Estimate the co-moving number density of halos with mass M between $10^{12}h^{-1}M_\odot$ and $2\times10^{12}h^{-1}M_\odot$ at redshifts 0 and 3, respectively. What is the answer for $\sigma_8 = 0.9$? [**6**].

(c) Compute the ratio of $Mf(M,z)$ for 3-sigma fluctuations collapsing at $z = 3$ and $z = 1$ respectively. In other words, compute $(Mf)(\nu = 3, z = 3)$ divided by $(Mf)(\nu = 3, z = 1)$. [**4**]

(d) Consider an object that collapsed as a high sigma fluctuation at redshift 1. Do you expect the progenitors of this object to have a mass function at redshift 6 that is similar to the PS mass function at that redshift? Let M be the characteristic mass of the progenitors at $z = 6$. Do you expect the progenitors to cluster like other objects with mass M?

Hint: You do not need the extended Press-Schechter theory for this! [**2+2**].

(e) We assumed matter domination, but if $\Omega_\Lambda = 1 - \Omega_m = 0.7$ as observations suggest, then we have been living in a vacuum energy dominated universe from $z = (\Omega_\Lambda/\Omega_m)^{1/3} - 1 \approx 0.3$. How will this change your answers to problems 3. *qualitatively*.

[**8**]